The Dual Code of Some Evaluation Codes Arising from Complete Bipartite Graphs

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Abstract

In this paper we compute the dual code of the evaluation codes arising from the incidence matrix of a complete bipartite graph.

Mathematics Subject Classification: 94B27, 94B60

Keywords: Dual codes, Evaluation codes, Complete bipartite graphs.

1 Introduction

Many researchers have studied the dual code of some evaluation codes (cf. [1], [2], [3], [7], [8]).

In this work we will describe the dual code of the evaluation codes arising from the incidence matrix of a complete bipartite graph. In [5] we found the main parameters of this kind of codes. The length, dimension and minimum distance were computed. The notation used in [5] will be useful later on.

In order to analyze the main result we need to introduce some concepts. In the next three sections we will show the preliminary results.
2 Evaluation Codes

Let $K$ be a finite field with $q$ elements, let $\mathbb{P}_K^l$ be the $l$–projective space over $K$ and $X = \{P_1, \ldots, P_s\}$ be a subset of $\mathbb{P}_K^l$. We always use the standard representation for the points in $\mathbb{P}_K^l$, i.e., $P = (0, 0, \ldots, 0, 1, a_1, \ldots, a_l)$. Let $\mathcal{L}$ be a finite dimensional $K$–linear space of functions which are defined on the set $X$ and take values on $K$. Thus the evaluation map

$$ev : \mathcal{L} \to K^s,$$

$$ev (f) = (f(P_1), \ldots, f(P_s))$$

defines a $K$–linear code: $C_X = ev(\mathcal{L})$.

Let $S = K[X_0, \ldots, X_l] = \bigoplus_{d \geq 0} S_d$ be the polynomial ring over the finite field $K$ with the natural graduation. If $\mathcal{L} = S_d$ is the $d$–graded homogeneous component of the polynomial ring $S$, the corresponding linear code $C_X(d) := ev(S_d)$ will be called the evaluation linear code over the set $X$, which is isomorphic to $S_d/I_X(d)$, where $I_X = \bigoplus_{d \geq 0} I_X(d)$ is the graded vanishing ideal of $X$. The dimension of these codes is given by the Hilbert function of $S/I_X$, i.e., $\dim_K C_X(d) = H_X(d)$.

3 $a$–Invariant

Let $I_X = \bigoplus_{d=\gamma}^\infty I_X(d)$ with $I_X(\gamma) \neq 0$, so that $\gamma$ is the lowest degree of a nontrivial homogeneous component of the ideal $I_X$. There is an integer $a_X$ called the $a$–invariant of $S/I_X$ (or the $a$–invariant of the ideal $I_X$ or even the $a$–invariant of $X$) such that

1. $H_X(d) = \dim_K S_d = \binom{d+l}{l}$ if and only if $d < \gamma$;
2. $H_X(d) < H_X(d+1) < s$ for $0 \leq d < a_X$;
3. $H_X(d) = s$ for $d > a_X$.

The number $a_X + 1$ is called the regularity index of $S/I_X$ or $I_X$. Moreover, the vanishing ideal $I_X$ is given by

$$I_X = \langle I_X(\gamma), I_X(\gamma + 1), \ldots, I_X(a_X + 2) \rangle$$

4 Complete Bipartite Graphs

Let $K_{m,n}$ be a complete bipartite graph (cf. [6]). The incidence matrix associated to $K_{m,n}$ is the $(m + n) \times (mn)$ matrix $B = (b_{ij})$ with $b_{ij} = 1$ if the vertex $v_i$ and the edge $a_j$ are incident and $b_{ij} = 0$ otherwise.

In the general case, the toric variety $X$ associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is given by (the way that we associate an evaluation code from a matrix is explained in [4])
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Let

\[ X = \{(t_1t_{m+1}, t_1t_{m+2}, \ldots, t_1t_{m+n}, t_2t_{m+1}, t_2t_{m+2}, \ldots, t_2t_{m+n}, \ldots, t_mt_{m+n+1}, t_mt_{m+n+2}, \ldots, t_mt_{m+n} : t_i \in K^* \text{ for all } i = 1, \ldots, m+n\} \]

And in fact, it can be written as

\[ X = \{ (1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \beta_1, \alpha_1\beta_1, \alpha_2\beta_1, \ldots, \alpha_{n-1}\beta_1, \\
\beta_2, \alpha_1\beta_2, \alpha_2\beta_2, \ldots, \alpha_{n-1}\beta_2, \beta_1, \beta_2, \alpha_1\beta_1\beta_2, \ldots, \beta_{m-1}, \alpha_1\beta_{m-1}, \alpha_2\beta_{m-1}, \ldots, \alpha_{n-1}\beta_{m-1}) : \alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{m-1} \in K^* \} \]

Let

\[ X_1 = \{(1, \alpha_1, \ldots, \alpha_{n-1}) : \alpha_1, \ldots, \alpha_{n-1} \in K^* \} \]

and

\[ X_2 = \{(1, \beta_1, \ldots, \beta_{m-1}) : \beta_1, \ldots, \beta_{m-1} \in K^* \} \]

Let

\[ \theta_1 : K[X_0, \ldots, X_{n-1}] \to K^{s_1} \]
\[ \theta_1(g) = (g(Q_1), \ldots, g(Q_{s_1})) \]

where \( s_1 = (q-1)^{n-1} \) and \( X_1 = \{ Q_1, \ldots, Q_{s_1} \} \). Then \( C_{X_1}(d) \) (the generalized Reed-Solomon code of order \( d \) associated to \( X_1 \)) is the image of the last map. In the same way, we define

\[ \theta_2 : K[Y_0, \ldots, Y_{m-1}] \to K^{s_2} \]
\[ \theta_2(h) = (h(R_1), \ldots, h(R_{s_2})) \]

where \( s_2 = (q-1)^{m-1} \) and \( X_2 = \{ R_1, \ldots, R_{s_2} \} \). Then \( C_{X_2}(d) \) (the generalized Reed-Solomon code of order \( d \) associated to \( X_2 \)) is the image of the last map. Let \( s = \#(X) \) and consider the following evaluation map

\[ \theta : K[Z_{00}, \ldots, Z_{(m-1)(n-1)}] \to K^s \]
\[ \theta(f) = (f(P_{11}, \ldots, f(P_{s_1s_2})) \]

where \( X = \{ P_{11}, \ldots, P_{s_1s_2} \} \).

In this case, the evaluation code of order \( d \), \( C_X(d) \), associated to the incidence matrix of the complete bipartite graph \( K_{m,n} \) is the image of the last evaluation map.

Remark 4.1 In [5] we described the main parameters of the evaluation codes \( C_X(d) \) defined above. The results that we obtained are the following

- **Length**: \( s = (q-1)^{m+n-2} \)
- **Dimension**: \( H_X(d) = H_{X_1}(d) \cdot H_{X_2}(d) \)
- **a-invariant**: \( a_X = \max \{(n-1)(q-1) - n, (m-1)(q-1) - m\} \)
- **Minimum distance**: \( \delta_X(d) = \delta_{X_1}(d) \cdot \delta_{X_2}(d) \)
5 Main Result

In this section we give the main result: the dual code of the evaluation codes arising from the incidence matrix of a complete bipartite graph. We will use the notation introduced in the last section.

In order to prove the main theorem we need two previous results. These results correspond to complete intersections and due to the fact that $X_1$ and $X_2$ are complete intersections, they will be useful. We recall that a set of points $X \subseteq P^l_K$ is called a (zero-dimensional ideal-theoretic) complete intersection if the ideal $I_X$ is generated by a regular sequence of $l$ elements, i.e., there exists a set of $l$ generators $F_1, \ldots, F_l$ of the vanishing ideal $I_X$ such that the class of $F_i$ in the ring $S/\langle F_1, \ldots, F_l \rangle$ is not a zero divisor, for $i = 1, \ldots, l - 1$.

Lemma 5.1 If $X$ is a complete intersection then

$$H_X(d) + H_X(a_X - d) = \#X$$

Proof cf. [1]. □

Lemma 5.2 If $X$ is a complete intersection and $d$ is an integer such that $1 \leq d \leq a_X$ then $C_X(a_X - d)$ and $C_X^\perp(d)$ are equivalent codes. In fact

$$C_X^\perp(d) = (b_1, \ldots, b_s)C_X(a_X - d)$$

where $C_X^\perp(a_X) = \langle (b_1, \ldots, b_s) \rangle$

Proof cf. [2]. □

With this information we can prove the main result

Theorem 5.3 The dual code of the evaluation code of order $d$ $(1 \leq d \leq \min(a_{X_1}, a_{X_2}))$, $C_X(d)$, arising from the incidence matrix of a complete bipartite graph, and which is denoted by $C_X^\perp(d)$, is given by

$$[C_X^\perp_1(d) \otimes_K C_X(d) \oplus [C_X^\perp_1_1(d) \otimes_K C_X^\perp_2(d)] \oplus [C_X^\perp_1_2(d) \otimes_K C_X^\perp_2(d)]]$$

Proof Let $W$ be the last direct sum. Then

$$\dim W = H_{X_1}(a_{X_1} - d) \cdot H_{X_2}(d) + H_{X_1}(d) \cdot H_{X_2}(a_{X_2} - d) + H_{X_1}(a_{X_1} - d) \cdot H_{X_2}(a_{X_2} - d)$$

$$= [H_{X_1}(d) + H_{X_1}(a_{X_1} - d)] \cdot [H_{X_2}(d) + H_{X_2}(a_{X_2} - d)] - H_{X_1}(d) \cdot H_{X_2}(d) = (\#X_1) \cdot (\#X_2) - H_X(d) = \#X - H_X(d)$$

In order to finish the proof is enough to show that $W \subseteq C_X^\perp(d)$. We will work only with the first term of $W$ (the other terms work in a similar way). Let $U = C_X^\perp_1(d) \otimes_K C_X(d)$ and $\Upsilon \in C_X(d)$ given by
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\[ (f(X_0Y_0, \ldots , X_{n-1}Y_{m-1})(Q_1, R_1), \ldots , f(X_0Y_0, \ldots , X_{n-1}Y_{m-1})(Q_{s_1}, R_{s_2})). \]

Without loss of generality we can work with the following elements of \( U \):

\[ \Lambda = (b_1g(Q_1)h(R_1), \ldots , b_{s_1}g(Q_{s_1})h(R_{s_2})) \]

where \( \deg g = a_{X_1} - d \), \( \deg h = d \), \( C_{X_1}^\perp (a_{X_1}) = \langle (b_1, \ldots , b_{s_1}) \rangle \). Then

\[ \Upsilon \cdot \Lambda = b_1(f_{R_1}g)(Q_1)h(R_1) + \cdots + b_{s_1}(f_{R_{s_2}}g)(Q_{s_1})h(R_{s_2}) \]

where \( f_{R_j} \) means the polynomial \( f(X_0Y_0, \ldots , X_{n-1}Y_{m-1})(R_j) \) and therefore \( f_{R_j} \in K[X_0, \ldots , X_{n-1}]d \).

Moreover, \( \deg(f_{R_j}g) = a_{X_1} \) and due to the fact that \( C_{X_1}^\perp (a_{X_1}) = \langle (b_1, \ldots , b_{s_1}) \rangle \), we conclude that \( \Upsilon \cdot \Lambda = 0 \). Therefore \( U \subseteq C_{X}^\perp(d) \), \( W \subseteq C_{X}^\perp(d) \) and the claim follows. ■

References


Received: April 16, 2008