Some Remarks on the Prime Spectrum of a Noncommutative Ring

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Abstract. Let $R$ be an associative noncommutative ring with identity. Let $Spec_l(R)$ be all prime left ideals of $R$. In the present study, we will give some new characterizations of $Spec_l(R)$ with the Zariski topology.

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1. Introduction

Throughout this paper, we assume that $R$ is an associative ring (not necessarily commutative) with unity.

Let $R$ be a ring. A proper left ideal $P$ of $R$ is said to be a prime ideal if, for any elements $a$ and $b$ in $R$ such that $aRb \subseteq P$ either $a \in P$ or $b \in P$. A proper (2-sided) ideal $P$ of $R$ is called prime if, for any elements $a$ and $b$ in $R$ such that $aRb \subseteq P$, either $a \in P$ or $b \in P$. Clearly, prime ideals are prime left ideals. If $I$ is a prime left ideal, then

$$(I : R) = \{ r \in R : rR \subseteq I \}$$
is a two-sided prime ideal. For this study, we need the following well known lemma in the literature.

**Lemma 1.1.** Let $R$ be a ring and $I$ be a left ideal of $R$. Then $((I : R) : R) = (I : R)$.

We write $\text{Spec}_l(R)$ for the set of all prime left ideals and $\text{Spec}(R)$ the set of all prime ideals. The map $\psi : \text{Spec}_l(R) \to \text{Spec}(R)$ is defined by $\psi(P) = (P : R)$ for every $P \in X_l$ will be called the natural map of $\text{Spec}_l(R)$.

In this study, we introduce a topology called the Zariski topology on $\text{Spec}_l(R)$, in which closed sets are varieties

$$V_l(I) = \{ P \in \text{Spec}_l(R) : (I : R) \subseteq (P : R) \}$$

of all left ideals $I$ of $R$. If $I$ is a an ideal of $R$, we can say that $I \leq R$ to indicate that $I$ is an ideal of $R$. If $I$ is a left ideal of $R$, we can say that $I \leq_l R$ to indicate that $I$ is a left ideal of $R$. Now, we give some important properties of $V_l$.

**Lemma 1.2.** Let $R$ be a ring. For left ideals $I, J$ and $I_j (j \in \Lambda)$ of $R$, we have the followings:

1. $V_l(0) = \text{Spec}_l(R)$.
2. $V_l(R)$ is empty.
3. $V_l(I) \cup V_l(J) = V_l(I \cap J)$.
4. $\bigcap_{j \in \Lambda} V_l(I_j) = V_l\left(\sum_{j \in \Lambda} (I_j : R)\right)$.

*Proof.* Clear from by Lemma 1.1. □

The text by Hungerford [1] is the general references for notions of rings not defined in this work.

2. **The Results**

As we mentioned in introduction, $V_l(I) = \{ P \in \text{Spec}_l(R) : (I : R) \subseteq (P : R) \}$. By this vein, we have

$$V_l^*(I) = \{ P \in \text{Spec}_l(R) : I \subseteq P \}$$

for any left ideal $I$ of $R$. Hence;

**Lemma 2.1.** Let $R$ be a ring. For left ideals $I, J$ and $I_j (j \in \Lambda)$ of $R$, we have the followings:

1. $V_l^*(0) = \text{Spec}_l(R)$.
2. $V_l^*(R)$ is empty.
3. $V_l^*(I) \cup V_l^*(J) \subseteq V_l^*(I \cap J)$.
4. $\bigcap_{j \in \Lambda} V_l^*(I_j) = V_l^*\left(\sum_{j \in \Lambda} I_j\right)$.
Proof. By [2].

**Proposition 2.2.** Let $R$ be a ring. For left ideals $I, J$ of $R$, we have the followings:

(1) Suppose that $I$ and $J$ are prime ideals. Then $(I : R) = (J : R)$ if and only if $V_I(I) = V_J(J)$.

(2) $V_I(I) = V_I((I : R)) = V_I^*(I : R))$. In particular, $V_I(JR) = V_I^*(JR)$ for every left ideal $J$ of $R$.

Proof. (1). Clear.

(2). $P \in V_I(I) \iff (I : R) \subseteq (P : R) \iff ((I : R) : R) \subseteq ((P : R) : R) = (P : R) \iff P \in V_I((I : R))$.

Let $P \in V_I((I : R))$. Then $((I : R) : R) \subseteq (P : R) \iff (I : R) \subseteq P \iff P \in V_I^*((I : R))$. For converse, let $P \in V_I(JR)$. Then $(JR : R) = (JR : R) \subseteq (P : R) \subseteq P \iff JR \subseteq P \iff P \in V_I^*(JR)$.

We consider the following sets;

$$
\Gamma(R) = \{V_I(I) : I \leq_I R\}
$$

$$
\Gamma^*(R) = \{V_I^*(I) : I \leq_I R\}
$$

and

$$
\Gamma^{**}(R) = \{V_I^{**}(IR) : I \leq_I R\}.
$$

**Theorem 2.3.** For a ring $R$, we have the followings.

(1) $\Gamma(R) = \Gamma^{**}(R) \subseteq \Gamma^*(R)$.

(2) The Zariski topology $\Gamma$ on $\text{Spec}(R)$ is identical with $\Gamma^{**}$.

Proof. Clear from definitions and Proposition 2.2.

For each subset $E$ of $R$, let $V^R(E) = \{P \in \text{Spec}(R) : E \subseteq P\}$.

**Proposition 2.4.** Let $R$ be a ring. Then the surjective map $f : \text{Spec}_1(R) \to \text{Spec}(R)$, defined by $f(P) = (P : R)$, is continuous.

Proof. Let $X$ be any closed set in $\text{Spec}(R)$. Then $X = V^R(I)$ for some ideal $I$ of $R$. For any $P \in \text{Spec}_1(R)$, we have $f(P) = (P : R) \in V^R(I) \iff I = (I : R) \subseteq ((P : R) : R) = (P : R) \iff P \in V_I(I)$. This implies that $f^{-1}(X) = f^{-1}(V^R(I)) = V_I(I)$, and so $f$ is continuous.

Let $\Lambda^R = \{V^R(E) : E \subseteq R\}$. Then $\Lambda^R$ is a closed set of a unique topology on $\text{Spec}(R)$. The topology $\Lambda^R$ on $\text{Spec}(R)$ is called Zariski topology. Any open subset of $\text{Spec}(R)$ is of the form $\text{Spec}(R) - V^R(E)$ for some $E \subseteq R$. It is clear that $V^R(f) = V^R(RrR)$ for any $r \in R$. For $r \in R$, if we define $\text{Spec}(R)^r = \text{Spec}(R) - V^R(RrR)$. Clearly, $\{\text{Spec}(R)^r : r \in R\}$ is a basis for the Zariski topology.
Theorem 2.5. For a ring $R$, let $f : \text{Spec}_l(R) \to \text{Spec}(R)$ be the surjective map of $\text{Spec}_l(R)$. Then the following statements hold.

1. $f$ is closed and open.
2. $\text{Spec}_l(R)$ is connected if and only if $\text{Spec}(R)$ is connected.

Proof. (1) By Proposition 2.4, for every ideal $I$ of $R$, $f$ is a continuous map and $f^{-1}(V^R(I)) = V_l(I)$. By Proposition 2.2, $f^{-1}(V^R((I : R))) = V_l(I)$ for every left ideal $I$ of $R$. But $f(V_l(I)) = V^R((I : R))$ because of the surjectivity of $f$. By the similar technic, we can see that $f(\text{Spec}(R) - V_l(I)) = \text{Spec}(R) - V^R((I : R))$.

(2) We assume that $\text{Spec}(R)$ is connected. For contradiction, we suppose that $\text{Spec}_l(R)$ is not connected. Then $\text{Spec}_l(R)$ must contain a non-empty proper subset $X$ that is open and closed. By Proposition 2.4, $f(X)$ is non-empty subset of $\text{Spec}(R)$ that is both open and closed. Since $X$ is open, $X = \text{Spec}_l(R) - V_l(I)$ for some $I \leq_l R$ whence $f(X) = \text{Spec}(R) - V^R((I : R))$ by Proposition 2.4. This implies that, if $f(X) = \text{Spec}(R)$ then $V^R((I : R))$ is empty. Hence $(I : R) = R$, i.e, $I = R$. Therefore $X = \text{Spec}_l(R) - V_l(I) = \text{Spec}_l(R) - V_l(R) = \text{Spec}_l(R)$. It is a contradiction. That is $(X)$ is a proper subset of $\text{Spec}(R)$.

Converse is clear by (1). \qed

Let $R$ be a division ring and $S$ a ring of $n \times n$ matrices over $R$. Then $\text{Spec}(S) = \{0\}$ is connected. By Theorem 2.5, $\text{Spec}_l(S)$ is also connected.

References


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