**Strongly Principal Ideals of Rings**

**with Involution**

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**Abstract**

The notion of strongly principal ideal groups for associative rings was introduced in [3] and its several properties were studied in [2] by using cyclic groups. Motivated by these concepts, we introduce here *-cyclic groups and strongly principal *-ideal ring groups for rings with involution and investigate their structural properties by attaching involution on their corresponding ground groups.

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1. Introduction

Let $P$ be a ring property. A group $G$ is said to be $P$–group, in the sense of Feigelstock and Schlussel [3] (see also [2]), if there exists a $P$–ring $R$ such that $G = R^+$. A group $G$ is said to be nil, again in the sense of the above cited references, in case the only ring $R$ with $G = R^+$ is the zero ring. If $G$ is not nil and every ring $R$, with $R^2 \neq 0$ and $G = R$ is a $P$–ring, then $G$ is said to be strongly $P$–group. By using these concepts, strongly principal ideals are thoroughly investigated. For rings with involution, we introduce here the notion of strongly principal *-ideal ring groups and study their structural properties. To achieve this goal and to make the calculations simpler we have introduced *-cyclic groups, which is generated by an element $a$ and its involutary image $a^*$ in the group. Moreover, we study some structural properties of *-cyclic groups as well. In particular a formula (Corollary 3.2) for computing
the number of involutions for abelian groups is obtained. Their direct sum and
direct summand properties are also outlined. We have used *-cyclic groups
to obtain various properties and classifications of (strongly) principal *-ideal
ring groups. In particular, it is noticed in Theorem 4.7 that there exists no
mixed strongly principal *-ideal ring groups.

Throughout we assume that all groups are additive abelian and all rings
are associative. If \( R \) is a ring, then its underlying additive group is denoted by
\( G = R^+ \). If \( x \in R \), then \( \langle x \rangle \) (respectively \( (x) \)) means the ideal of \( R \) (respectively
the subgroup of \( G \)) generated by \( x \).

A ring \( R \) (respectively a group \( G \)) together with a unary operation
\( * \) is said
to be a ring (respectively, a group) with involution in case for all \( a, b \in R \),
\[(a^*)^* = a, \quad (a + b)^* = a^* + b^*, \quad \text{and} \quad (ab)^* = b^*a^*\]
(respectively, for all \( a, b \in G \), \( (a^*)^* = a \), \( \text{and} \quad (a + b)^* = a^* + b^* \)). Thus the
involution on \( R \) is an antiisomorphism of order two.

For commutative rings, the identity mapping is clearly an involution. Never-
theless, every group has at least one involution, namely, the unary operation
of taking inverse; that is \( g^* = -g \) for every \( g \in G \).

Let a group \( G \) be decomposed into its subgroups as \( G = H \oplus K \). If \( G \) has
an involution \( * \), then \( * \) is said to be changeless involution in case \( g^* = (h^*, k^*) \),
\( \forall \ g = (h, k) \in H \oplus K \) (see[1]).

A group \( G \) is said to be \(*\)-cyclic if for some \( a \in G \), \( G = \langle a \rangle + \langle a^* \rangle \), which
indeed one may rewrite as \( G = \langle a \rangle^* = \langle a, a^* \rangle \). Clearly, every cyclic group is
\(*\)-cyclic, but the converse is not true in general (see an example in Section 3).

A nonzero ideal \( I \) of an involution ring \( R \) (a nonzero subgroup \( H \) of an
involution group \( G \)) which is closed under involution is termed as a \( *\)-ideal
\((I \triangleleft^* R)\) (respectively a \(*\)-subgroup \((H \leq^* G)\)); that is

\[I^* = \{a^* \in R \mid a \in I \} \subseteq I \; .\]

A subring \( A \) of \( R \) is said to be a biideal of \( R \) if \( ARA \subseteq A \) and a \(*\)-biideal
if, in addition, it is closed under the involution of \( R \). \( A \) is called a principal
\(*\)-biideal (see [7]), if

\[A = \langle a \rangle_{bi}^* = Za + Za^* + aRa + a^*Ra + aRa^* + a^*Ra^* .\]

On the same ground a principal \(*\)-ideal is defined. A principal \(*\)-ideal \( I \) is
a \(*\)-ideal generated by a single element. This means that, for some \( a \in R \), one
may write:

\[I = \langle a \rangle^* = Za + Za^* + aR + Ra + RaR + a^*R + Ra^* + Ra^*R .\]
Thus, it can easily be deduced that

\[ I = \langle a \rangle + \langle a^* \rangle = \langle a, a^* \rangle. \]

A ring with involution \(*\) is said to be principal *-ideal ring if each *-ideal is a principal *-ideal.

A group \(G\) is called strongly principal *-ideal ring group, if \(G\) is not nil and every ring \(R\) with involution satisfying \(R^2 \neq 0\) and \(G = R^+\), is a principal *-ideal ring.

Let \(f : A \rightarrow B\) be a group or a ring homomorphism. If \(A\) and \(B\) are equipped with some involutions \(*_A\) and \(*_B\) such that \(f(a^*_A) = [f(a)]^*_B\), then we say that \(f\) is an involution preserved homomorphism. If \(f\) is an involution preserved isomorphism, then we will write \(A \cong B\). It is clear that *-subgroups and *-ideals are preserved under such isomorphisms. Moreover, if \(A \cong B\), as a group or a ring, then every involution on \(A\) induced an involution on \(B\).

In sections 2 and 3, we give some elementary properties for *-cyclic groups. Furthermore, in section 4, (strongly) principal *-ideal ring groups are widely studied.

2. Some Elementary Properties

**Lemma 2.1.** Let \(G\) be a group with involution \(*\). Then the following subgroups of \(G\) are closed under the involution \(*\).

(a) \(nG, \forall n \in \mathbb{Z}\).
(b) The torsion subgroup \(G_t\) of \(G\).
(c) For any prime \(p\), every \(p\)-primary subgroup \(G_p\) of \(G\).
(d) The maximal divisible subgroup of \(G\).
(e) The subgroup \(G[m] = \{g \in G \mid mg = 0\}\) of \(G\), for some integer \(m\).

**Proof:**

(a) Let \(x \in nG\). Then \(x = ng\) for some \(g \in G\), hence \(x^* = ng^*\) and \(g^* \in G\). So \(x^* \in nG\) and \(nG\) is closed under involution.

(b) Let \(x \in G_t\). Then there exists a positive integer \(n\) such that \(nx = 0\). Hence \(nx^* = 0\) and \(x^* \in G\) follows.

(c) Let \(x \in G_p\). Then \(|x| = p^n\) for some positive integer \(n\) and \(p^n x = 0\), implies \(p^n x^* = 0\). Hence \(x^* \in G_p\).

(d) Let \(D\) be the maximal divisible subgroup of \(G\). If \(x \in D\), then there exists \(y \in D\) such that \(x = ny\) for any positive integer \(n\), whence \(x^* = ny^*\). Since \(D\) is the maximal divisible subgroup, \(x^*, y^* \in D\), therefore \(D\) is closed under involution.

(e) Let \(x \in G[m]\), then \(mx = 0\), whence \(mx^* = 0\) and \(x^* \in G[m]\) follows.

**Corollary 2.2.** In every involution ring \(R\), \(nR\), \(R[n]\), \(R_t\), \(R_p\) and the maximal divisible ideal \(D\) are *-ideals.
Proof: It is clear that $G = R^+$ has involution; the same involution of $R$. So from Lemma 2.1, $nR$, $R[n]$, $R_t$, $R_p$ and the maximal divisible subgroup $D$ are $*$-subgroups of $G$. Since $nR$, $R[n]$, $R_t$, $R_p$ and $D$ are ideals in $R$ (see [5]), hence $nR$, $R[n]$, $R_t$, $R_p$ and $D$ are $*$-ideals in every involution ring $R$. ■

Lemma 2.3. (a) Every direct sum of involution groups is an involution group.
(b) Every direct summand of a group with a changeless involution is an involution group.
(c) If a direct summand of a group has an involution, then the group has an involution.

Proof: (a) Let $G = H \oplus K$, where $H$ and $K$ are groups with involutions $*_H$ and $*_K$, respectively. Then for every $g = (h, k) \in G$, where $h \in H$ and $k \in K$, define the involution $*_G$ on $G$ by

$$g^*_G = (h^*_H, k^*_K).$$

Because of the unique representation of each element, $*_G$ becomes a unary operation on $G$. Further,

$$(g^*_G)^*_G = ((h^*_H)^*_H, (k^*_K)^*_K)) = (h, k) = g.$$ Assume that $g_i \in G$, with $g_i = (h_i, k_i)$, where $h_i \in H$ and $k_i \in K$. Then

$$(g_1 + g_2)^*_G = ((h_1 + h_2)^*_H, (k_1 + k_2)^*_K) = ((h_1^*_H + h_2^*_H), (k_1^*_K + k_2^*_K))$$

$$= (h_1^*_H, k_1^*_K) + (h_2^*_H, k_2^*_K) = g_1^*_G + g_2^*_G.$$ Hence $*_G$ is an involution on $G$; it is in fact the changeless involution on $G$.

The proof can analogously be extended to finite as well as to arbitrary direct sums.

(b) Let $G = H \oplus K$. Set $H' = H \oplus 0$ and $K' = 0 \oplus K$. Clearly, $H'$ and $K'$ are direct summands and subgroups of $G$. Assume that $*$ is the changeless involution on $G$. Then $*|_{H'}$ (involution on $G$ restricted to $H'$) is an involution on $H'$ and $*|_{K'}$ is an involution on $K'$. Also, $H \cong H'$ and $K \cong K'$. Hence (b) is proved.

(c) Let $G = H \oplus K$ and $H$ be a group with an involution $*_H$. Then for every $g = (h, k) \in G$, where $h \in H$ and $k \in K$, define an operation $*_G$ on $G$ by

$$g^*_G = (h^*_H, k).$$

Clearly, $*_G$ is the changeless involution on $G$.■
Parts \((a)\) and \((b)\) of Lemma 2.3 can easily be extended to rings, subrings and ideals. But for part \((c)\) we need the following modification.

**Corollary 2.4.** Let \(R = A \oplus B\), where \(A\) and \(B\) are rings in which \(B\) is commutative. Then \(R\) has a (changeless) involution if and only if \(G\) has an involution.

Proof: One way is clear from Lemma 2.3-(b). Assume that \(A\) has an involution \(\ast_A\). Define \(\ast_R\) on \(R\) by

\[
r^{\ast_R} = (a^{\ast_A}, b)
\]

Then, \(\ast_R\) is a unary operation on \(R\) and for \(r_1 = (a_1, b_1), r_2 = (a_2, b_2) \in R,\)

\[
(r_1 r_2)^{\ast_R} = ((a_1 a_2)^{\ast_A}, b_1 b_2) = ((a_2^{\ast_A} a_1^{\ast_A}), b_2 b_1) = r_2^{\ast_R} r_1^{\ast_R}.
\]

The rest is as in Lemma 2.3-(c). \(\blacksquare\)

3. Cyclic Groups with Involution

Let \(G\) be an infinite cyclic group. Following [8], there are two involutions on \(G\), the identity involution and the involution \(a^* = -a\) of taking inverse. If \(G\) is a finite cyclic group of order \(n\), then \(\text{Aut}(G)\) consists of all automorphisms, \(\alpha_k : a \rightarrow ka\), where \(1 \leq k \leq n\) and \((k, n) = 1\). Moreover,

\[
\text{Aut}(G) \cong U(\mathbb{Z}_n)
\]

(the multiplicative group of units of the ring \(\mathbb{Z}_n\)). Since

\[
\alpha_{n-1} : a \rightarrow (n - 1) a
\]

is the only automorphism of order 2, \(\text{Aut}(G)\) has only two automorphisms of order two; the identity mapping and \(\alpha_{n-1}\) (of taking inverse), so \(G\) has only two involutions.

From this introduction, we note that every cyclic group has two involutions; namely the identity mapping and the mapping of taking inverse. Moreover, every subgroup of a cyclic group is closed under these involutions.

**Proposition 3.1.** Let \(G\) be an additive abelian group, \(G = H \oplus K\), and let \(H\) and \(K\) be cyclic subgroups of \(G\). If \((|H|, |K|) \neq 1\), then

\((a)\) \(G\) has exactly four involutions, namely:

\[
g^* = (h, k), \quad g^* = (-h, k), \quad g^* = (h, -k), \quad g^* = (-h, -k)\text{ and } g^* = (h, k).
\]
(b) Every subgroup of \( G \) is closed under involution.

Proof: (a) By Lemma 2.3, \( H \) and \( K \) are \(*\)-subgroups. Since \( H \) and \( K \) are cyclic, \( H \) and \( K \), each, has two involutions; the identity involution and \(* : a \rightarrow -a\). Hence again by Lemma 2.3, \( G \) has exactly the given four involutions.

(b) By Theorem 8.1 in [4], any subgroup \( H \) of \( G \) is a direct sum of two cyclic subgroups, or it is cyclic. Hence by (a), \( H \) is a \(*\)-subgroup. ■

The following immediate result gives the number of involutions of abelian groups.

**Corollary 3.2.** Let \( G \) be an additive abelian group. If \( G = \bigoplus_{i=1}^{n} H_i \), where each \( H_i \) is a cyclic subgroup of \( G \) such that \(|H_i|, |H_j| \neq 1, 1 \leq i, j \leq n\), then \( G \) has \( 2^n \) involutions.

**Proposition 3.3.** Let \( R \) be a ring with involution such that \( R^+ = G \). Then \( R \) has only the identity involution in case any one of the following holds:

1. \( G \) is a cyclic group.
2. \( G \) is a direct sum of cyclic subgroups.

Proof: (1) Let \( G \) be cyclic. Since \( R \) is an involution ring, \( G \) has either the identity involution or the involution \(* : a \rightarrow -a\). However, \(-ab \neq -b(-a)\), for all \( a, b \in R \). Hence \( R \) has the identity involution only.

(2) If \( G = H \oplus K \), and \( H \) and \( K \) are cyclic subgroups of \( G \), then by Proposition 3.1, \( G \) has four involutions. But then again by (1), \( G \) has only one involution. ■

**Definition 3.4.** By a \(*\)-cyclic group \( H \), we mean a \(*\)-group generated by one element.

This means that,

\[ H = (a)^* = (a, a^*) = (a) + (a^*). \]

Let \( G \) be a cyclic group, then \( G = (a) = (a^*) \) and \( G = (a) + (a^*) \), so \( G \) is a \(*\)-cyclic group. The converse of this fact is not always true.

For example in the group

\[ (M_{2\times2}(\mathbb{Z}_3), +) \]

with the transposed involution, let \( a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), whence \( a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \).

Obviously, the \(*\)-cyclic group \( H = (a) + (a^*) \) is not a cyclic group.
Proposition 3.5. Let $G = H \oplus K$. If $H$ and $K$ are *-cyclic groups such that $H = (a)^*$, $K = (b)^*$, $(|a|, |b|) = 1$. Then $G$ is *-cyclic.

Proof: The given condition $(|a|, |b|) = 1$ implies that $(a) \oplus (b)$ is a cyclic group generated by $(a, b)$ and $(a^*) \oplus (b^*)$ is a cyclic group generated by $(a^*, b^*)$. But,

$$G = H \oplus K = (a) + (a^*) \oplus (b) + (b^*) = (a) \oplus (b) + (a^*) \oplus (b^*).$$

Hence $G$ is *-cyclic with $G = ((a, b))^*$. ■

Proposition 3.6. If $G$ is a *-cyclic group, then any *-subgroup of $G$ is a *-cyclic subgroup.

Proof: A *-cyclic group is either torsion or torsion free. First assume that $G$ is torsion free and let

$$G = (a)^* = (a, a^*).$$

If

$$(a) \cap (a^*) \neq 0,$$

then $na^* = ma \neq 0$, for some integers $m$ and $n$. This implies $na = ma^*$. So, $na - na^* = ma^* - ma$, from which $n(a - a^*) = m(a^* - a) = -m(a - a^*)$ and so $(n + m)(a - a^*) = 0$. Since $G$ is torsion free, $a - a^* = 0$ implies $a = a^*$, whence $(a) \cap (a^*) = 0$ and $G = (a) \oplus (a^*)$.

Secondly assume that $G$ is torsion, $G = (a) + (a^*)$, and $|a| = |a^*| = k$. Let $g \in G$, $g = ma + na^*$, for some integers $m, n$. Since $k(ma + na^*) = 0$, it follows that $|g| \leq k$, and $a, a^*$ have maximal orders. Hence $G = (a) \oplus (a^*)$, from [6], page 81. Thus in both cases, $G = (a) \oplus (a^*)$.

If $H \leq^* G$, then $H = (b) \oplus (c)$, where $(b) \leq (a)$ and $(c) \leq (a^*)$. Then $(b) = (ma)$ and $(c) = (na^*)$, whence

$$H = (ma) \oplus (na^*).$$

Since $H$ is *-subgroup, $ma^* + na \in H$. But $ma^* \in (na^*)$, so $m > n$ and $na \in (ma)$, so $n > m$. Therefore $n = m$ and

$$H = (na) \oplus (na^*).$$

Hence $H = (na)^*$ and $H$ is a *-cyclic subgroup of $G$. ■
Proposition 3.7. Let $x_1$ and $x_2$ be elements of a group $G$ such that a prime $p | |x_1|, |x_2|$. If $G = (x_1)^* \oplus (x_2)^*$, then there exist $y_1, y_2 \in G$ such that $(y_1)^* \leq (x_1)^*$ and $(y_2)^* \leq (x_2)^*$.

Proof: Let $G = (x_1) + (x_1^*) \oplus (x_2) + (x_2^*)$.

If $p$ is a prime such that $p | |x_1|$, then there exists $y_1 \in (x_1)$ such that, $p | |y_1|$ and $|y_1|$ divides $|x_1|$. Consequently,

$$(y_1) \leq (x_1) \text{ and } (y_1^*) \leq (x_1^*).$$

Similarly there is $y_2 \in (x_2)$ such that

$$(y_2) \leq (x_2) \text{ and } (y_2^*) \leq (x_2^*).$$

Hence, it is concluded that

$$(y_1) + (y_1^*) \leq (x_1) + (x_1^*)$$

and

$$(y_2) + (y_2^*) \leq (x_2) + (x_2^*),$$

that is, $(y_1)^* \leq (x_1)^*$ and $(y_2)^* \leq (x_2)^*$. ■

4. Strongly principal *-Ideal Ring Groups.

As it is mentioned before that the notion of strongly principal ideal groups for associative rings was introduced and investigated thoroughly in [2] and [3]. Motivated by these concepts, we introduce here strongly principal *-ideal ring groups for rings with involution and study their structural properties by attaching involution on their corresponding ground groups.

Definitions 4.1. Let $R$ be a ring with involution. For some $a \in R$, one may write:

$$I = \langle a \rangle^* = Za + Za^* + aR + Ra + RaR + a^* R + Ra^* + Ra^* R.$$  

Clearly $I$ is an ideal of $R$ closed under involution and is called the principal *-ideal generated by $a$.

One may deduce that

$$I = \langle a \rangle + \langle a^* \rangle = \langle a, a^* \rangle.$$
A ring with involution $\ast$ is a principal $\ast$-ideal ring if each $\ast$-ideal is a principal $\ast$-ideal. Moreover, we say that a group $G$ is strongly principal $\ast$-ideal ring group, if $G$ is not nil, and every ring $R$ with involution satisfying $R^2 \neq 0$, and $G = R^+$, is a principal $\ast$-ideal ring.

**Lemma 4.2.** Let $G = H \oplus K$, $H \neq 0$, $K \neq 0$, be a strongly principal $\ast$-ideal ring group. Then $H$ and $K$ are either both $\ast$-cyclic or both nil.

Proof: Suppose that $H$ is not nil. Let $S$ be a $\ast$-ring with $S^+ = H$ and $S^2 \neq 0$ and let $T$ be the zero ring on $K$. By Corollary 2.5, the ring direct sum $R = S \oplus T$ is a ring with involution satisfying $R^+ = G$ and $R^2 \neq 0$. Since $T$ is a $\ast$-ideal in $R$, $T = \langle x \rangle^\ast$. Clearly $K = T^+ = (x)^\ast$. Therefore $K$ is not nil. Interchanging the roles of $H$ and $K$ yields that $H$ is $\ast$-cyclic. ■

**Corollary 4.3.** Let $G = H \oplus K$, $H \neq 0$, $K \neq 0$, be a strongly principal $\ast$-ideal ring group. Then $H$ and $K$ are $\ast$-cyclic.

Proof: It suffices to negate that $H$ and $K$ are both nil. Let $R = (G, \cdot)$ be a ring with involution satisfying $R^2 \neq 0$.

1) Suppose that $R^2 \subseteq K$. There exist $h_0 \in H$, $k_0 \in K$, such that $R = \langle h_0 + k_0 \rangle^\ast$.

Let $h \in H$, since $h \in R$, there exist integers $n$ and $m$, and $x \in R^2$ such that

$$h = n(h_0 + k_0) + m(h_0 + k_0)^\ast + x.$$  

However, $x \in K$, so

$$h = nh_0 + mh_0^\ast$$

and $H$ is $\ast$-cyclic, contradicting the fact that $H$ is nil.

2) Suppose that $R^2 \not\subseteq K$. For all $g_1, g_2 \in G$, define $g_1 \circ g_2 = \pi_H(g_1 \cdot g_2)$, where $\pi_H$ is the natural projection of $G$ onto $H$. Since

$$(g_1 \circ g_2)^\ast = (\pi_H(g_1 \cdot g_2))^\ast = \pi_H(g_1 \cdot g_2)^\ast = \pi_H(g_2 \cdot g_1^\ast) = g_2^\ast \circ g_1^\ast,$$

hence $S = (G, \circ)$ is a ring with involution satisfying $S^2 \subseteq H$. The argument employed in (1) yields that $K$ is $\ast$-cyclic which contradicts the fact that $K$ is nil. ■

**Theorem 4.4.** Let $G \neq 0$ be a torsion group. If $G$ is cyclic or $G \cong (x_1) \oplus (x_2)$, where $x_1 \neq x_2$, $|x_1| = |x_2| = p$ is a prime, then $G$ is strongly principal $\ast$-ideal ring group.

Proof: Assume that either $G$ is cyclic or as given in the hypothesis. Then by Proposition 3.3, any ring $R$ with $R^+ = G$ has only the identity involution. By [2, Theorem 4.2.3] $G$ is strongly principal $\ast$-ideal ring group. ■
Theorem 4.5. Let $G$ be a torsion strongly principal *-ideal ring group. Then $G$ is *-cyclic or $G = (x_1)^* \oplus (x_2)^*$, with $|x_i| = p$, a prime, where $i = 1, 2$.

Proof: Suppose that $G$ is a strongly principal *-ideal ring group. Let $G$ be indecomposable. Then by [2, Corollary 1.1.5], $G \cong \mathbb{Z}_{p^n}$, $p$ a prime, $1 \leq n \leq \infty$. If $n = \infty$, then $G$ is divisible by [2, Proposition 1.1.3] and so $G$ is nil, by [2, Theorem 2.1.1], which is a contradiction. Hence $G$ is cyclic and so *-cyclic.

Next, suppose that $G = H \oplus K$, $H \neq 0, K \neq 0$, by Lemma 4.2, either $H$ and $K$ are both *-cyclic or both nil. If $H$ and $K$ are nil, then they are both divisible, so $G$ is nil, by [2, Theorem 2.1.1] which is again a contradiction. Therefore $G = (x_1)^* \oplus (x_2)^*$, with $|x_i| = n_i$, $i = 1, 2$. If $(n_1, n_2) = 1$, then $G$ is *-cyclic, by Proposition 3.5. Otherwise, let $p$ be a prime divisor of $(n_1, n_2)$. Then by Proposition 3.7,

$$G = (y_1)^* \oplus (y_2)^* \oplus H$$

with

$$|y_i| = p^{m_i}, i = 1, 2,$$

and $1 \leq m_1 \leq m_2$. Since $(y_1)^* \oplus (y_2)^*$ is neither * cyclic nor nil, $H = 0$ by Lemma 4.2. The products

$$y_i y_j = p^{m_j - 1} y_2, y_i y_j^* = 0$$

where $i, j = 1, 2$,

induce a *-ring structure $R$ on $G$ with $R^2 \neq 0$. Therefore, $R = \langle s_1 y_1 + s_2 y_2 \rangle^*$, $s_1$ and $s_2$ are integers. Every element $x \in R$ has the form:

$$x = k_x s_1 y_1 + (k_x s_2 + m_x p^{m_x - 1}) y_2 + k'_x s_1 y_1^* + (k'_x s_2 + m'_x p^{m_x - 1}) y_2^*$$

where $k_x, m_x, k'_x$, and $m'_x$ are integers. In particular,

$$y_1 = k_{y_1} s_1 y_1, \text{ and } y_2 = (k_{y_2} s_2 + m_{y_2} p^{m_2 - 1}) y_2.$$

Hence if $m_2 > 1$, then

$$k_{y_1} s_1 \equiv 1 \pmod{p}, \text{ and } k_{y_2} s_2 + m_{y_2} p^{m_2 - 1} \equiv 1 \pmod{p},$$

which imply that $p \nmid k_{y_1}$ and $p \nmid s_2$. However

$$k_{y_1} s_2 + m_{y_1} p^{m_2 - 1} \equiv 0 \pmod{p}.$$

So either $p|k_{y_1}$ or $p|s_2$ which is a contradiction. Therefore $m_1 = m_2 = 1.$
Theorem 4.6. Let $G$ be a torsion group which is either *-cyclic or $G \cong (x_1)^* \oplus (x_2)^*$ with $|x_i| = p$, a prime, where $i = 1, 2$. Then for any *-ring $R$ with $R^+ = G$, $R$ is a principal *-ideal ring.

Proof: By Proposition. 3.6, non trivial *-cyclic groups are clearly principal *-ideal ring groups. Let

$$G = (x_1)^* \oplus (x_2)^*$$

with

$$|x_i| = p, i = 1, 2$$

and let $R$ be a *-ring with $R^+ = G$ and $R^2 \neq 0$. If $I$ is a proper *-ideal in $R$, then $|I| = 1$, $p$, or $p^2$, and so $I$ is a *-ideal generated by one element, we may assume that $R \neq <x_i>^*, i = 1, 2$. Hence

$$<x_i>^{*+} = (x_j)^*$$

for $i = 1, 2$. This implies the following three relations:

$$x_i x_j = k_i x_i, \quad 0 \leq k_i < p, \text{ if } i = j, \quad i, j = 1, 2, \text{ either } k_1 \neq 0 \text{ or } k_2 \neq 0,$$

$$x_i x_j = 0, \quad \text{ if } i \neq j, \quad i, j = 1, 2,$$

and

$$x_i^* x_j = 0, \quad \text{ for all } i, j = 1, 2.$$

Put

$$I = (x_1 + x_2)^*.$$

Suppose that $k_1 \neq 0$. Let $r, s$ be integers such that $rk_1 + sp = 1$. Then

$$rx_1(x_1 + x_2) = rk_1 x_1 = (1 - sp)x_1 = x_1$$

and

$$r(x_1 + x_2)^* x_1^* = rk_1 x_1^* = (1 - sp)x_1^* = x_1^*.$$

Hence $x_1 \in I$ and $x_1^* \in I$, so

$$(x_1 + x_2) - x_1 = x_2 \in I$$

and

$$(x_1 + x_2)^* - x_1^* = x_2^* \in I.$$
Therefore $I = R$. If $k_2 \neq 0$, then the above argument, reversing the roles of the indices 1, 2 again yields $I = R$. ■

**Theorem 4.7.** There are no mixed strongly principal $*$-ideal ring groups.

Proof: Let $G$ be a mixed strongly principal $*$-ideal ring group. $G$ is decomposable by [2, Corollary 1.1.5], so by Lemma 4.2, $G = H \oplus K$, $H \neq 0, K \neq 0$, with $H$ and $K$ both $*$-cyclic, or both nil.

1) Suppose that $H$ and $K$ are both nil. There are no mixed nil groups by [2, Theorem 2.1.1]. So, we may assume that $H$ is a torsion group, and that $K$ is torsion free. Let $R$ be a $*$-ring with $R^+ = G$ and $R^2 \neq 0$. Clearly $H$ is a $*$-ideal in $R$ and so $H = \langle h \rangle^*$. Let $|h| = m$, then $mH = 0$. By [2, Theorem 2.1.1], $H$ is divisible, and therefore not bounded, a contradiction.

2) Suppose that $H = (x)^*$ and $K = (e)^*$ with $|x| = n$, and $|e| = \infty$. The products

$$x^2 = xe = ex = e^*x = ex^* = x^*e = 0 \text{ and } e^2 = ne$$

induce a $*$-ring structure $R$ on $G$ satisfying $R^2 \neq 0$. Therefore there exist integers $s$ and $t$ such that $R = \langle sx + te \rangle^*$. Every $y \in R$ is of the form

$$y = m_ysx + (m_y + u_yn)te + m'_ysx^* + (m'_y + u'_yn)te^*,$$

with $m_y, m'_y, u_y$, and $u'_y$ integers. In particular, $(m_e + u_en)t = 1$. Hence $t = \pm 1$. Therefore, $m_x + u_xn = 0$ and so $n|m_x$. However, $x = m_xsx = 0$, is a contradiction. ■

**Theorem 4.8.** Let $G$ be a torsion free strongly principal $*$-ideal ring group. Then $G$ is either indecomposable, or is the direct sum of two nil groups.

Proof: By Lemma 4.2, it suffices to negate that

$$G = (x_1)^* \oplus (x_2)^*, \ x_i \neq 0, \ i = 1, 2.$$

Suppose this is so, the products:

$$x_ix_j = 3x_i \text{ and } x_i^*x_j = 0 \text{ for } i = j = 1, 2,$$

$$x_ix_j = x_i^*x_j = 0, \text{ for } i \neq j$$

induce a ring structure $R$ on $G$ with involution $*$ satisfying $R^2 \neq 0$. Therefore there exist nonzero integers $k_1, k_2$ such that

$$R = \langle k_1x_1 + k_2x_2 \rangle^*.$$
Every $x \in R$ is of the form:

$$x = (r_x + 3s_x)k_1x_1 + (r_x + 3t_x)k_2x_2 + (r'_x + 3s'_x)k_1x'_1 + (r'_x + 3t'_x)k_2x'_2$$

where $r_x$, $r'_x$, $s_x$, $s'_x$, $t_x$, $t'_x$ are integers. From

$$r_{x_1} + 3s_{x_1} = \pm 1,$$

it follows that

$$r_{x_1} \equiv \pm 1 (\text{mod } 3).$$

However,

$$r_{x_1} + 3t_{x_1} = 0$$

implies

$$r_{x_1} \equiv 0 (\text{mod } 3),$$

which is a contradiction. ■

**Lemma 4.9.** Let $G$ and $H$ be torsion free groups with $G \cong H$. Then $G$ is a strongly principal $*$-ideal ring group if and only if $H$ is.

**Proof:** Let $f : H \to G$ be a $*$-isomorphism such that $G$ is a strongly principal $*$-ideal ring group. Let $R = (H, \cdot)$ be a ring with involution $*$ such that $R^2 \neq 0$. The product

$$g_1 \circ g_2 = f(h_1 \cdot h_2),$$

where, $g_1 = f(h_1)$ and $g_2 = f(h_2)$, for all $g_1, g_2 \in G$, induces a ring structure $S = (G, \circ)$ with $S^2 \neq 0$. Then $G$ is a group with involution $\nabla$. Since

$$(g_1 \circ g_2)^\nabla = [f(h_1 \cdot h_2)]^\nabla = f(h_1 \cdot h_2)^* = f(h_2^* \cdot h_1^*) = g_2^\nabla \circ g_1^\nabla.$$

Hence $S$ has an involution. Let $I \triangleleft R$, then $f(I) \triangleleft \nabla S$ and there exists $g \in G$ such that $f(I) = \langle g \rangle^\nabla$, with $g = f(h)$, $h \in H$. We claim that $I = \langle h \rangle^*$. Clearly $\langle h \rangle^* \subseteq I$. Let $x \in I$, then $f(x) \in \langle g \rangle^\nabla$ and so

$$f(x) = ng^\nabla + mg + g \circ y_1 + g^\nabla \circ y_2 + z_1 \circ g + z_2 \circ g^\nabla$$

where $m, n$ are integers and $y_1 = f(h_1^*), y_2 = f(h_2^*), z_1 = f(h_1^*)$ and $z_2 = f(h_2^*) \in G$. Thus

$$f(x) = nf(h^*) + mf(h) + f(h \cdot h_1^*) + f(h^* \cdot h_2) + f(h_1^* \cdot h) + f(h_2^* \cdot h^*)$$

$$= f(nh^* + mh + h \cdot h_1^* + h^* \cdot h_2 + h_1^* \cdot h + h_2^* \cdot h^*)$$
which concludes that $x \in \langle h \rangle^*$. Hence $I = \langle h \rangle^*$. ■

**Theorem 4.10:** Let $G$ be a torsion group. Then the following are equivalent:

1. $G$ is bounded
2. $G$ is a principal $*$-ideal ring group.

Proof: $(1) \Rightarrow (2)$: Suppose that $nG = 0$ and $n$ is a positive integer. Then

$$G = \bigoplus_{p|n} \left( \bigoplus_{\alpha_k} \mathbb{Z}_{p^k} \right)$$

where $p$ is a prime with $p^k | n$ and $\alpha_k$ a cardinal number, by [2, Proposition 1.1.9]. For each $p^k | n$, put

$$H_{p^k} = \bigoplus_{\alpha_k} \mathbb{Z}_{p^k}.$$

Then

$$G = \bigoplus_{p^k | n} H_{p^k},$$

and there exists a commutative principal ideal ring $R_{p^k}$ with unity and

$$R_{p^k}^+ = H_{p^k}$$

for all $p^k | n$, by [5, Lemma 122.3]. The ring direct sum

$$R = \bigoplus_{p^k | n} R_{p^k}$$

is a principal $*$-ideal ring with the identity involution satisfying $R^+ = G$ and $R^2 \neq 0$.

$(2) \Rightarrow (1)$: Let $R$ be a principal $*$-ideal ring with $R^+ = G$. Then $R = \langle x \rangle^*$ and $n = |x|$. So $nG = 0$. ■

**Theorem 4.11:** Let $G$ be a mixed group. Then

1. If $G$ is a principal $*$-ideal ring group, then $G_t$ is bounded and $G/G_t$ is a principal $*$-ideal ring group.

2. Conversely, if $G_t$ is bounded and if there exists a unital principal $*$-ideal ring with additive group $G/G_t$, then $G$ is a principal $*$-ideal ring group.

Proof: $(1)$ Let $R$ be a principal $*$-ideal ring with $R^+ = G$. Since $G_t$ is a $*$-ideal in $R$, $G_t = \langle x \rangle^*$ and $nG_t = 0, n = |x|$. Now $G = G_t \oplus H$ and $H \cong G/G_t$ by [2, Proposition 1.1.2]. Now

$$R = \langle a + y \rangle^*, a \in G_t, 0 \neq y \in H.$$
Suppose that $R^2 \subseteq G_t$ and let $h \in H$. Then there exist integers $k_n$, $k'_n$ such that,

$$h = k_n y + k'_n y^* + b,$$

with $b \in R^2$. Since $R^2 \subseteq G_t$, $b = 0$, and

$$h = k_n y + k'_n y^*.$$

Therefore $H = (y)^*$. By Proposition 3.6, $H$ is a principal *-ideal ring group. If $R^2 \not\subseteq G_t$, then $\bar{R} = R/G_t$ is a principal *-ideal ring with $\bar{R}^+ \cong G/G_t$, and $\bar{R}^2 \neq 0$.

(2) Conversely, suppose that $G_t$ is bounded, and that there exists a unital principal *-ideal ring $T$ with $T^+ = G/G_t$. Hence

$$G \cong G_t \oplus G/G_t,$$

by [2, Proposition. 1.1.2]. There exists a principal *-ideal ring $S$ with unity and * is the identity involution such that $S^+ = G_t$, from [5, Lemma 122.3]. Let

$$R = S \oplus T$$

with $e, f$ the unities of $S$ and $T$, respectively. Then $R$ is a ring with involution *, by Corollary 2.5. Let $I$ be a *-ideal in $R$, then

$$I = (I \cap S) \oplus (I \cap T).$$

Now, $I \cap S \leq^* S$ and so

$$I \cap S = \langle x \rangle^*.$$

Similarly

$$I \cap T = < y >^*.$$

Clearly,

$$\langle x + y \rangle^* \subseteq I.$$

However,

$$x = e(x + y) \in \langle x + y \rangle^*, \quad x^* = e(x + y)^* \in \langle x + y \rangle^*$$

and

$$y = f(x + y) \in \langle x + y \rangle^*, \quad y^* = f(x + y)^* \in \langle x + y \rangle^*.$$

Hence we conclude that $I = < x + y >^*$.
References


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