Finite Groups in which $(S-)$Semipermutability is a Transitive Relation

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Abstract

Let $G$ be a finite group. A subgroup $H$ of $G$ is called semipermutable (or $S$-semipermutable) in $G$ if $H$ permutes with every subgroup $K$ (or every Sylow $p$-subgroup) of $G$ with $(|K|, |H|) = 1$ (or $(p, |H|) = 1$, where $p \in \pi(G)$). A group $G$ is called an $BT$–group (or $SBT$-group) if semipermutability ($S$-semipermutability) is a transitive relation in $G$. In this paper, we determine the structure of $BT$–group (or $SBT$-group) and classify the minimal non-$BT$–groups (or minimal non-$SBT$-group).

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1 Introduction

All groups considered in this paper are finite. Our notations and notion are standard and taken mainly from [4]. Throughout this article, $G$ stands for a finite group and $\pi(G)$ denotes the primes dividing $|G|$ and $G_p$, a Sylow $p$-subgroup of $G$ for some prime $p \in \pi(G)$.

Two subgroups $H$ and $K$ of $G$ are said to be permutable if $\langle H, K \rangle = HK = KH$. A subgroup $H$ of $G$ is called permutable if $H$ permutes with every subgroup of $G$. Permutable subgroups were initially studied by Ore[8], who called them quasinormal, in 1939. A subgroup $H$ of $G$ is called $S$-permutable if $H$ permutes with every Sylow subgroup of $G$. This concept was introduced by Kegel in [7], who called them $S$-quasinormal. As the generalizations of permutability and $S$-permutability, Zhongmu Chen in [2] introduced the following: A subgroup $H$ of $G$ is called semipermutable in $G$ if $H$ permutes with every subgroup $K$ of $G$ with $(|H|, |K|) = 1$, and $S$-semipermutable if $H$ permutes with every Sylow $p$-subgroup of $G$ with $(p, |H|) = 1$, who called them seminormal, $S$-seminormal subgroup of $G$, respectively. It is easy to see that a permutable (resp. $S$-permutable) subgroup of $G$ is a semipermutable (resp. $S$-semipermutable) subgroup of $G$. The converse is not true in general. Usually $(S)$-seminormality cannot imply subnormality while $S$-permutability can by a result of Kegel [7]. For example, a Sylow 2-subgroup of the symmetric group $S_3$ of degree 3 is semipermutable in $S_3$ but not $S$-permutable in $S_3$. Many authors studied the influence of $(S-)sempemutability of some subgroups on the structure of groups. For example, Chen, in [2], proved that $G$ is supersolvable if every maximal subgroup of any Sylow subgroup of $G$ is semipermutable in $G$. Zhang and Wang [14] proved that $G$ is supersolvable if one of following holds: (1) every maximal subgroup of any Sylow subgroup of $G$ is $S$-semipermutable in $G$; (2) every minimal subgroup and cyclic subgroup of order 4 of $G$ is $S$-semipermutable in $G$. Zhang and Wang in [14] classified the finite nonabelian simple groups which contain a nontrivial semipermutable (s-semipermutable) subgroup.

Recall that $G$ is said to be a $T$-group (resp. $PT$-group, $PST$-group), if normality (resp. permutability, $S$-permutability) is a transitive relation in $G$. From 1950s, especially in recent years, due to the efforts of many leading mathematicians, such as Gaschütz, Robinson, Cossey, Ballester-Bolinches, etc, many characterizations of $T$-group, $PT$-group, $PST$-group were discovered. The study of these classes of groups has undoubtedly constituted a fruitful topic in group theory. The structure of solvable $T$-groups was determined by Gaschütz [5] in 1957, he showed that these are exactly the groups with an abelian normal Hall subgroup $L$ of odd order such that $G/L$ is a Dedekind group and the elements of $G$ induce power automorphisms in $L$. The corresponding theorem for solvable $PT$-group (resp. $PST$-group) is due to Zacher.
Stimulated by results in above, we naturally consider the parallelism questions about semipermutability and $S$-semipermutability. Semipermutability and $S$-semipermutability, like normality, permutability and $S$-permutability, is not a transitive relation in an arbitrary group. For example, $S_4$, the symmetric group of degree 4, is a counterexample. In this paper, we study the structure of groups in which semipermutability or $S$-semipermutability is transitive, that is, groups $G$ such that $H$ semipermutable (resp. $S$-semipermutable) in $K$ and $K$ semipermutable (resp. $S$-semipermutable) in $G$ imply that $H$ is semipermutable (resp. $S$-semipermutable) in $G$. Such groups are called $BT$-groups (resp. $SBT$-groups). A byproduct of our results is that solvable $BT$-group and solvable $SBT$-group coincide. We also give the structure of the minimal non-$BT$-groups (resp. minimal non-$SBT$-groups), i.e., the group which is not a $BT$-group (resp. $SBT$-group), but every proper subgroup of which is a $BT$-group (resp. $SBT$-group).

**Remark**: It is easy to see that every nilpotent group is a $BT$-group ($SBT$-group).

## 2 Preliminaries

In this section, we give some results which will be useful in the sequel.

Deducing from the definitions directly, we get:

**Lemma 2.1** Suppose that $H$ is $(S$-)semipermutable in a $G$, $K \leq G$. Then:
1. If $H \leq K$, then $H$ is $(S$-)semipermutable in $K$;
2. If $N$ is a normal $p$-subgroup of $G$, then $HN/N$ is $(S$-)semipermutable in $G/N$.

**Lemma 2.2** Suppose that $N$ is a normal $p$-subgroup of $G$, where $p \in \pi(G)$. If all $p'$-elements of $G$ induce power automorphisms in $N$ by conjugate, then $G/C_G(N)$ is nilpotent.

**Proof.** If $N$ is non-abelian, all $p'$-power automorphisms are trivial by [6]. Hence, $G/C_G(N)$ is a $p$-group. If $N$ is abelian, the power automorphisms are in the center of $\text{Aut}(N)$. Hence $G/C_G(N)$ is nilpotent.

**Lemma 2.3** Let $G$ be a group. Then the following statements are equivalent:
1. Every subgroup of $G$ is semipermutable in $G$;
(ii) For any $p, q \in \pi(G)$, and $p \neq q$, let $\langle x \rangle$ be a $p$–subgroup and $\langle y \rangle$ a $q$–subgroup. Then $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$.

Proof. (i) $\implies$ (ii). It is clear.

(ii) $\implies$ (i). Take any subgroup $A$ of $G$. Let $B$ be a subgroup of $G$ such that $|A|, |B| = 1$. Clearly, any group can be generated by some elements of prime power order. Hence, we can assume that $A = \langle a_1, a_2, \ldots , a_m \rangle$ and $B = \langle b_1, b_2, \ldots , b_n \rangle$, where $a_i, b_j$ are elements of $G$ of prime power order. Since $|A|, |B| = 1$, it follows that $(o(a_i), o(b_j)) = 1$. By hypotheses, we have $\langle a_i \rangle \langle b_j \rangle = \langle b_j \rangle \langle a_i \rangle$. So

$$AB = \langle a_1, a_2, \ldots , a_m \rangle \langle b_1, b_2, \ldots , b_n \rangle = \langle b_1, b_2, \ldots , b_n \rangle \langle a_1, a_2, \ldots , a_m \rangle = BA.$$ 

Therefore, $A$ is semipermutable in $G$. Thus, Lemma 2.3 is proved.

**Lemma 2.4** Let $N$ be a solvable minimal normal subgroup of a $BT$–group. Then $N$ is a cyclic subgroup of prime order.

Proof. By hypotheses, we can suppose that $N$ is an elementary abelian $p$–group for some $p \in \pi(G)$. Since $N$ is a $p$–subgroup, every subgroup of $N$ is semipermutable in $N$. It is clear that $N$ is semipermutable in $G$. Since $G$ is a $BT$–group, it follows that every subgroup of $N$ is semipermutable in $G$.

Let $G_p$ be a Sylow $p$–subgroup of $G$. Then $N \leq G_p$ and $N \cap Z(G_p) \neq 1$. Take a subgroup $A$ of $N \cap Z(G_p)$ of order $p$. Then $A$ is semipermutable in $G$. Let $Q$ be any $p'$–subgroup of $G$. Then $AQ = QA$. Since $[Q, A] \leq N \cap AQ = A$, we have that $Q \leq N_G(A)$. Hence $O^p(G) \leq N_G(A)$. Since $A \leq Z(G_p)$, it follows that $A \leq G$. By the minimality of $N$, we have that $N = A$ is a cyclic subgroup of order $p$.

**Lemma 2.5** If $G$ is a solvable $BT$–group, then $G$ is supersolvable.

Proof. Let $N$ be a minimal normal subgroup of $G$. By Lemma 2.1, we have that $G/N$ is a $BT$–group. Hence $G/N$ is supersolvable by the induction on the order of $G$. From Lemma 2.4, we know that $N$ is a cyclic subgroup of prime order. Hence $G$ is supersolvable.

**Lemma 2.6** Suppose that $G$ is a group. Then $G$ is supersolvable if every cyclic subgroup of $G$ of prime order or order 4 is $S$-semipermutable in $G$.

Proof. See [14, Lemma 5].

**Lemma 2.7** Suppose that $G$ is a group and $P$ is a normal $p$–subgroup of $G$ for some $p \in \pi(G)$. If $G/C_G(P)$ is a $p$–group, then $P \leq Z(G)$.

Proof: See [10, Theorem VI. 3]].
3 The structure of $BT$–group or $BST$-group

**Theorem 3.1** Let $G$ be a group. The following statements are equivalent:

1. $G$ is a solvable $BT$-group;
2. $G$ is a solvable $SBT$-group;
3. every subgroup of $G$ of prime power order is semipermutable in $G$;
4. every subgroup of $G$ is semipermutable in $G$;
5. every subgroup of $G$ is $S$-semipermutable in $G$;
6. every subgroup of $G$ of prime power order is $S$-semipermutable in $G$;
7. there exists an abelian normal Hall subgroup $L$ of $G$ of odd order such that $G/L$ is nilpotent and the elements of $G$ induce power automorphisms in $L$. Moreover, for any two distinct primes $p, q \notin \pi(L)$, $[G_p, G_q] = 1$.

**Proof:** We get the theorem by proving $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (4) \Rightarrow (2) \Rightarrow (3)$ and $(4) \Rightarrow (1)$.

$(1) \Rightarrow (3)$ Suppose that $\pi(G) = \{p_1, p_2, \ldots, p_n\}$, where $p_1 < p_2 < \cdots < p_n$. Applying Lemma 2.5, we have that the Sylow $p_n$-subgroup $G_{p_n}$ of $G$ is normal in $G$. Therefore, $G_{p_n}$ is semipermutable in $G$. Clearly, every subgroup of $G_{p_n}$ is semipermutable in $G_{p_n}$. Hence, every subgroup of $G_{p_n}$ is semipermutable in $G$ as $G$ is a $BT$-group.

Again by Lemma 2.5, $G_{p_n} G_{p_{n-1}} \leq G$, where $G_{p_{n-1}}$ is a Sylow $p_{n-1}$-subgroup of $G$. Since every subgroup of $P_n$ is semipermutable in $G$, we have that every subgroup of $P_{n-1}$ is semipermutable in $P_{n-1} P_n$. Hence every subgroup of $P_{n-1}$ is semipermutable in $G$ as $G$ is a $BT$-group.

Continuing this procedure, we can obtain that every subgroup of $G_{p_i}$ is semipermutable in $G$, where $G_{p_i} \in \text{Syl}_{p_i}(G)$, $i = 1, 2, \ldots, n$. Therefore every subgroup of $G$ of prime power order is semipermutable in $G$.

$(3) \Rightarrow (4)$. Let $H \leq G$. Since $G$ is solvable, we can choose a Sylow base $\{H_{p_1}, H_{p_2}, \ldots, H_{p_s}\}$ of $H$. Then $H = H_{p_1} H_{p_2} \cdots H_{p_s}$. By the hypotheses, every $H_{p_i}$ is semipermutable in $G$. Therefore, $H$ is semipermutable in $G$.

$(4) \Rightarrow (5)$ and $(5) \Rightarrow (6)$ are obvious.

$(6) \Rightarrow (7)$ Suppose that every subgroup of $G$ of prime order is semipermutable in $G$. We prove $(7)$ in following steps:

(i) $G$ is supersolvable.

By hypotheses, we know that every cyclic subgroup of $G$ of prime order or order 4 is $S$-semipermutable. By Lemma 2.6, we have that $G$ is supersolvable.

(ii) If $N$ is a normal $p$–subgroup of $G$, then $p'$–elements of $G$ induce power automorphisms in $N$.

Take any $a \in N$. Let $x$ be a $p'$–element of $G$. Then $x \in G_{p'}$ for some
\(p'-\text{Hall subgroup of } G\). Then \(\langle a \rangle G_{p'}\) is a group. Hence
\[
a^{(x)} = a^{(x)} \cap \langle a \rangle G_{p'} = \langle a \rangle (a^{(x)} \cap G_{p'}) = \langle a \rangle (N \cap G_{p'}) = \langle a \rangle.
\]
Therefore \(x\) induce a power automorphism in \(\langle a \rangle\).

(iii) Let \(L = G^N\) be the nilpotent residual of \(G\). Then \(L\) is a abelian Hall subgroup of \(G\).

Let \(p\) be the largest prime dividing \(|G|\), \(P \in \text{Syl}_p(G)\). Since \(G\) is supersolvable by step (i), we know that \(P \leq G\). Now, we consider the quotient group \(G/P\). By Lemma 2.1, all subgroups of \(G/P\) is semipermutable in \(G/P\). By induction, \((G/P)^N = G^N P/P = LP/P\) is an abelian Hall subgroup of \(G/P\).

(a) If \(G/C_G(P)\) is a \(p\)-group, then \(G = PC_G(P)\). By Lemma 2.7, \(P \leq Z_\infty(G)\). Let \(L_p \in \text{Syl}_p(L)\). Suppose that \(L_p \neq 1\), by \(L_p \leq P\) and \(G = PC_G(P)\), we have \(L_p = [L_p, G] = [L_p, PC_G(P)] = [L_p, P] < L_p\), a contradiction. Hence \(L_p = 1\) and \(L\) is a \(p'\)-group. Therefore, \(L \cong LP/P\) is a normal Hall subgroup of \(G\).

(b) If \(G/C_G(P)\) is not a \(p\)-subgroup, then there exists a \(p'\)-element \(x\) such that \(P = [P, x]\) since \(G/C_G(P) \leq \text{Aut}(P)\). Hence \(P = [P, G] \leq L\). By (ii) and Lemma 2.2, we know that \(G/C_G(P)\) is nilpotent. So \(L \leq C_G(P)\) and \(P \leq Z(L)\). Since \(L/P\) is a normal abelian Hall subgroup of \(G/P\), it follows that \(L = P \times L_{p'}\) is a normal Hall subgroup of \(G\), where \(L_{p'}\) is the \(p'\)-Hall subgroup of \(L\).

(iv) \(L\) has odd order.

Let \(L_2\) be a Sylow 2-subgroup of \(L\). If \(L_2 \neq 1\), then all 2'-elements of \(G\) induce power automorphism in \(L_2\) by (ii). Since 2 is the smallest prime dividing \(|G|\), then \([L_2, G_2] = 1\), where \(G_2\) is a Hall 2'-Hall subgroup of \(G\). Hence, \(L_2 = [L_2, G] = [L_2, G_2] < L_2\), a contradiction, where \(G_2 \in \text{Syl}_2(G)\). So, \(L_2 = 1\) and \(L\) has odd order.

(v) The elements of \(G\) induce power automorphisms in \(L\).

This follows from (ii) and (iii).

(vi) For any two distinct primes \(p, q \not\in \pi(L)\), \([G_p, G_q] = 1\).

By the hypotheses, \(G_p G_q\) is a group. Since \(G_p G_q \cong G_p G_q L/L \leq G/L\), \(G_p G_q\) is nilpotent. Hence \([G_p, G_q] = 1\).

(vii) \(\Rightarrow\) (iv) we prove that every subgroup of \(G\) is semipermutable in \(G\).

By Lemma 2.3, we only need to prove \(\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle\), for any \(p\)-element \(x\) and \(q\)-element \(y\) in \(G\).

If \(p, q \in \pi(L)\), since \(L\) is a normal abelian Hall subgroup of \(G\), we have that \(x, y \in L\) and \(\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle\).

If \(p, q \not\in \pi(L)\), by hypotheses, \([x, y] = 1\). Hence \(\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle\).
Suppose that \( p \in \pi(L) \) or \( q \in \pi(L) \). Without lose generality, let \( p \in \pi(L) \). Then \( x \in L \) and \( \langle x \rangle \trianglelefteq G \). Hence \( \langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle \).

(4) \Rightarrow (2) and (4) \Rightarrow (1) Obviously.

(2) \Rightarrow (6) Repeating the argument in (1) \Rightarrow (3).

Hence we get the equivalence of (1)-(7).

**Corollary 3.2** Every solvable \( BT \)-group is a subgroup closed and quotient closed \( BT \)-group.

From Theorem 3.1 and the structure of \( PST \)-group \([1]\), we can obtain the following result:

**Corollary 3.3** If \( G \) is a finite solvable \( BT \)-group, then \( G \) is a \( PST \)-group.

The following example indicates that \( BT \)-group is a proper subclass of \( PST \)-group.

**Example.** Let \( G = \langle a \rangle \rtimes (\langle x \rangle \times \langle y \rangle) \), where \( a^7 = x^2 = y^3 = 1, a^x = a^{-1}, a^y = a^4 \). Since the nilpotent residual \( L \) of \( G \) is \( \langle a \rangle \). By Agrawal’s theorem \([1]\), we know that \( G \) is a \( PST \)-group. However, since \( \langle x^a \rangle \langle y \rangle \neq \langle y \rangle \langle x^a \rangle \), by Theorem 3.1, we know that \( G \) is not a \( BT \)-group.

The following result tells us under certain conditions \( BT \)-groups and \( PST \)-groups may coincide.

**Corollary 3.4** Let \( L \) be the nilpotent residual of a solvable group \( G \). If \( \pi(G)/\pi(L) = \{p\} \), then \( G \) is a \( BT \)-group if and only if \( G \) is a \( PST \)-group.

4 **Minimal non-\( BT \)-group**

In this section, we use the Theorem 3.1 to determine the structure of minimal non-\( BT \)-group. First, we give the following result:

**Theorem 4.1** Let \( G \) be a minimal non-\( BT \)-group. Then

(i) \( G \) is solvable;

(ii) \( G \) has a normal Sylow subgroup;

(iii) \( |\pi(G)| = 2 \) or 3.

Proof. Assume that \( G \) is insolvable. Then \( G \) contains a minimal insolvable subgroup \( G_1 \). Since \( G \) is a minimal non-\( BT \)-group, it follows that each proper subgroup of \( G_1 \) is a solvable \( BT \)-group. By Lemma 2.5, each proper subgroup of \( G_1 \) is supersolvable. Hence \( G_1 \) is a solvable, a contradiction. Therefore, \( G \) is solvable. (i) is proved.
If $G$ is supersolvable, then $G$ has a normal Sylow subgroup. If $G$ is non-supersolvable, then $G$ is a minimal nonsupersolvable group by Theorem 3.1. Applying a result of Doerk [3], $G$ has a normal Sylow subgroup. Hence (ii) is proved.

Denote the normal $p$–Sylow subgroup of $G$ by $P$.

Since $G$ is a minimal non-$BT$–group, from Theorem 3.1 and Lemma 2.3 there exist elements $x, y$ of $G$ of prime power order such that $(o(x), o(y)) = 1$ and $⟨x⟩⟨y⟩ = ⟨y⟩⟨x⟩$. By the minimality of $G$ and without losing generality, we can suppose that $G = ⟨x, y⟩$.

If one of $x, y$ is a $p$–element, $x$ say, then $x ∈ P$. This implies that $G = ⟨x, y⟩ = P⟨y⟩$. Hence $|\pi(G)| = 2$.

Therefore assume that both $x$ and $y$ are not $p$–elements. Let $H$ be a $p'$–Hall subgroup of $G$. Then $G/P ≅ H$ is an $BT$–group. By Theorem 3.1, each subgroup of $G/P$ is semipermutable. This implies that $(⟨x⟩P/P) · (⟨y⟩P/P) = (⟨y⟩P/P) · (⟨x⟩P/P)$. Since $G = ⟨x, y⟩$, it follows that $G = ⟨x⟩⟨y⟩P$. Hence $|\pi(G)| = 3$. (iii) is proved.

Next, we study the structure of the minimal non-$BT$–group $G$.

**Theorem 4.2** Suppose $G$ is a minimal non-$BT$–group.

1. If $|\pi(G)| = 2$, then $G$ is a minimal non-$PST$–group.

2. If $|\pi(G)| = 3$, then $G = P × (Q × R)$, where $P$ is a normal abelian $p$–Sylow subgroup ($p$ is odd), $Q = ⟨a⟩$, $R = ⟨b⟩$ are cyclic $q$–Sylow subgroup and $r$–Sylow subgroup, respectively. The elements of $G$ induce power automorphisms in $P$, and for any $g ∈ G$, $[a^g, b^g] = 1$, $[a^g, b^r] = 1$.

**Proof.** (1) Suppose that $|\pi(G)| = 2$. By Corollary 3.3, we have that each proper subgroup of $G$ is a $PST$–group. Clearly, the nilpotent residual $G^N$ of $G$ is nontrivial. So, if $G$ is a $PST$–group, then by Corollary 3.4, $G$ is an $BT$–group, which is a contradiction. Hence, $G$ is a minimal non-$PST$–group.

By the proof of Theorem 4.1, we have that $G = ⟨x, y⟩ = P⟨x⟩⟨y⟩$, where $P$ is the normal $p$–Sylow subgroup, $⟨x⟩$, $⟨y⟩$ are cyclic $q$–Sylow subgroup and $r$–Sylow subgroup, respectively, for some prime $p, q, r ∈ \pi(G)$ and $⟨x⟩$ and $⟨y⟩$ don’t permute.

Since $G$ is a minimal non-$BT$–group, it follows from Corollary 3.3 that each proper subgroup of $G$ is $PST$–group. By [9, Theorem 1], we have that $G$ is a $PST$–group. Let $L = G^N$ be the nilpotent residual of $G$. By [1], $L$ is an abelian normal Hall subgroup of odd order and the elements of $G$ induce power automorphisms in $L$.

Suppose that $q || L$. Since $L$ is a Hall subgroup of $G$, we have that $⟨x⟩ ≤ L$. So $⟨x⟩ ≤ G$ and $G = ⟨x, y⟩ = ⟨x⟩⟨y⟩$, a contradiction. Hence $q ∤ |L|$. Similarly, $r ∤ |L|$.

If $L = 1$, then $G$ is nilpotent and $G$ is an $BT$–group, a contradiction.
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Hence $L \neq 1$. Since $q \nmid |L|$, $r \nmid |L|$ and $L$ is a Hall subgroup of $G$, it follows that $L = P$ is abelian and the elements of $G$ induce power automorphisms in $P$.

Let $H$ be a $p'$-Hall subgroup of $G$ such that $\langle x\rangle \leq H$. Then there exists $g_0 \in G$ such that $\langle y \rangle^{g_0} \leq H$. Clearly, $H = \langle x \rangle \langle y \rangle^{g_0}$. Since $G/P \cong H$ is nilpotent, $H = \langle x \rangle \times \langle y \rangle^{g_0}$. Let $x = a$, $y^{g_0} = b$, $Q = \langle a \rangle$, $R = \langle b \rangle$, then $G = P \rtimes (Q \times R)$.

Let $M_1 = P\langle a^q \rangle \langle b \rangle$ and $M_2 = P\langle a \rangle \langle b^r \rangle$, then $M_1, M_2$ are proper normal subgroups of $G$. So $M_1, M_2$ are $BT$-groups. By $L = P$, we have that the nilpotent residual $M_i^N \leq P$, $i = 1, 2$. Since $M_i$ is an $BT$-group, $M_i^N$ is Hall subgroup of $M_i$. Hence $M_i^N = 1$ or $P$. If $M_1^N = 1$, then $M_1$ is nilpotent and $\langle b \rangle$ is the normal $q$-subgroup of $M_1$ and $\langle b \rangle \trianglelefteq G$, then this will imply that $\langle x \rangle$ and $\langle y \rangle$ permute, a contradiction. Hence $M_1^N \neq 1$. Similarly, $M_2^N \neq 1$. So $M_1^N = M_2^N = P$. By $M_1 \leq G$, $M_2 \leq G$, we have that $\langle b^g \rangle \leq M_1$, $\langle a^q \rangle \leq M_2$ for any $g \in G$. Applying Theorem 3.1, $[a^q, b^g] = 1$ and $[a^q, b^r] = 1$. The proof is complete.

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