Some Equalities between Elliptic Dilogarithm of 2-Isogenous Elliptic Curves

Nouressadat Touafek

Laboratoire de Physique Théorique
Equipe de Théorie des Nombres
Université de Jijel, Algeria
nstouafek@yahoo.fr

Abstract

In this paper we establish some equalities between elliptic dilogarithm of the 2-isogenous curves 14A and 14B. This allows us to give a new exotic relation for the curve 14B.

Mathematics Subject Classification: 11G05, 14G05, 14H52

Keywords: Elliptic curves, Elliptic dilogarithm, Exotic relations

1 Introduction

For some elliptic curves, the elliptic dilogarithm satisfies linear relations other than distribution ones and called exotic by Bloch and Grayson [4].

Bloch and Grayson conjectured the following fact.

Conjecture 1.1 Suppose that $E(\mathbb{Q})_{\text{tors}}$ is cyclic and $d = \# E(\mathbb{Q})_{\text{tors}} > 2$. Write $\Sigma$ for the number of fibres of type $I_\nu$ with $\nu \geq 3$ in the Néron model, and suppose $\left\lfloor \frac{d-1}{2} \right\rfloor - \Sigma > 1$. Then there should be at least $\left\lfloor \frac{d-1}{2} \right\rfloor - \Sigma - 1$ exotic relations

$$\sum_{r=1}^{\left\lfloor \frac{d-1}{2} \right\rfloor} a_r D^E(rP) = 0$$

where $P$ is a $d$-torsion point and $a_r \in \mathbb{Z}$.

In particular Bloch and Grayson conjectured for the curve $E_1 = 14A$ the exotic relation

$$2D^E(P_1) + 5D^E(2P_1) = 0$$

where $P_1 = (1,0)$ is the 6-torsion point of the curve 14A and the notation $\overset{\circ}{A} = B$, means ” A is conjectured to be equal to B ”, that is A and B are numerically equal to at least 25 decimal places. Recently (2004) this relation was proved by Bertin [2].
By the help of some equalities between elliptic dilogarithm of the 2-isogenous curves 14A and 14B of Cremona’s tables [5], we give a new exotic relation for the curve 14B.

In [4] only elliptic curves with negative discriminant are considered, so our new exotic relation do not appear in the list of Bloch and Grayson since the curve 14B have positive discriminant.

2 The elliptic dilogarithm

Let $E$ be an elliptic curve defined over $\mathbb{Q}$.

Throughout this paper, the notation $E = [a_1, a_2, a_3, a_4, a_6]$ means that the elliptic curve $E$ is in the Weierstrass form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$ 

We have two representations for $E(\mathbb{C})$

$$E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \xrightarrow{\sim} \mathbb{C}^*/q^{\mathbb{Z}}$$

$$\varphi(u), \varphi'(u) \rightarrow u(\text{mod } \Lambda) \rightarrow z = e^{2\pi i u}$$

where $\varphi$ is the Weierstrass function, $\Lambda = \{1, \tau\}$ the lattice associated to the elliptic curve and $q = e^{2\pi i \tau}$.

Definition 2.1 The elliptic dilogarithm $D^E$ [3] is defined by

$$D^E(P) = \sum_{n=-\infty}^{n=+\infty} D(q^n z)$$

where $P \in E(\mathbb{C})$ is the image of $z \in \mathbb{C}^*$, $q = e^{2\pi i \tau}$ and $D$ is the Bloch-Wigner dilogarithm,

$$D : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{R}$$

$$z \mapsto \Im(L_i^c(z) + \log |z| \log^{|c|}(1 - z))$$

where $c$ is a loop from $\frac{1}{2}$ to $z$ in $\mathbb{C} \setminus \{0, 1\}$.

In section 3 we use the two following properties of the Bloch-Wigner dilogarithm

$$D(z) = -D(z^{-1})$$

$$D(z^2) = 2(D(z) + D(-z)).$$

Remark 2.2 There is a second representation of the elliptic dilogarithm given by Bloch [3], [9] in terms of Eisenstein-Kronecker series

$$D^E(P) = \frac{(3\tau)^2}{\pi} \Re\left( \sum_{m,n \in \mathbb{Z}} \exp\left( \frac{2\pi i (n\xi - m\eta)}{(m\tau + n)^2 (m\tau + n)} \right) \right)$$

where $z = e^{2\pi i u}$ and $u = \xi \tau + \eta$. 


3 Equalities between elliptic dilogarithm: a new exotic relation

Let $E_1$ be the elliptic curve 14A with equation

$$Y_1^2 + X_1Y_1 + Y_1 = X_1^3 - X_1$$

and its 6-torsion point $P_1 = (1, 0)$. Let $E_2$ be the elliptic curve 14B with equation

$$Y_2^2 + X_2Y_2 + Y_2 = X_2^3 - 11X_2 + 12$$

and its 6-torsion point $P_2 = (0, 3)$.

**Lemma 3.1** Using definition 2.1, we find that

\[ D_{E_2}(P_2) = \sum_{n=\infty}^{+\infty} D(je^{2\pi i(2n+1)\tau_1}), \]

\[ D_{E_2}(2P_2) = \sum_{n=\infty}^{+\infty} D(je^{2\pi i(2n\tau_1)}), \]

\[ D_{E_1}(P_1) = \sum_{n=\infty}^{+\infty} D(-je^{2\pi in\tau_1}), \]

and

\[ D_{E_1}(2P_1) = \sum_{n=\infty}^{+\infty} D(je^{2\pi in\tau_1}) \]

where $j = e^{2\pi i}$.

**Proof.** By definition 2.1,

\[ D_{E_2}(P_2) = \sum_{n=\infty}^{+\infty} D(q^n z) = \sum_{n=\infty}^{+\infty} D(e^{2\pi in\tau_2} e^{2\pi iu}), \]

using the fact that $\tau_2 = 2\tau_1 + 1$ and $u = \frac{1}{2}\tau_2 + \frac{1}{6}$, we get

\[ D_{E_2}(P_2) = \sum_{n=\infty}^{+\infty} D(je^{2\pi i(2n+1)\tau_1}), \]

To prove the remaining equalities it suffices to use the fact that $u = \frac{1}{3}$ for $2P_2$, $u = \frac{5}{6}$ for $P_1$ and $u = \frac{2}{3}$ for $2P_1$. □

Now, we can prove the following theorem.
Theorem 3.2 We have the following equalities

1) $D_{E_2}(P_2) = -2D_{E_1}(P_1) + 3D_{E_1}(2P_1)$
2) $D_{E_2}(2P_2) = -2D_{E_1}(P_1) + 2D_{E_1}(2P_1)$.

Proof. 1) We get from (4)

$$D_{E_1}(2P_1) = \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi in\tau_1})$$

$$= \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(2n+1)\tau_1})$$

$$= \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(2n+1)\tau_1}) + \sum_{n=-\infty}^{+\infty} D((je^{2\pi in\tau_1})^2),$$

using (1), (3) and the distribution property of the dilogarithm

$$\sum_{n=-\infty}^{+\infty} D((je^{2\pi in\tau_1})^2) = 2 \sum_{n=-\infty}^{+\infty} D(-je^{2\pi i(n\tau_1)}) + 2 \sum_{n=-\infty}^{+\infty} D(je^{2\pi i(n\tau_1)})$$

we get,

$$D_{E_1}(2P_1) = D_{E_2}(P_2) + 2D_{E_1}(P_1) + \sum_{n=-\infty}^{+\infty} D(je^{2\pi i(n\tau_1)});$$

hence,

$$D_{E_1}(2P_1) = D_{E_2}(P_2) + 2D_{E_1}(P_1) = -2 \sum_{n=-\infty}^{+\infty} D((je^{2\pi i(n\tau_1)})^{-1})$$

$$= D_{E_2}(P_2) + 2D_{E_1}(P_1) - 2 \sum_{n=-\infty}^{+\infty} D(j^2 e^{-2\pi i(n\tau_1)})$$

$$= D_{E_2}(P_2) + 2D_{E_1}(P_1) - 2 \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(n\tau_1)}).$$

So, by (4) we get

$$D_{E_1}(2P_1) = D_{E_2}(P_2) + 2D_{E_1}(P_1) - 2D_{E_1}(2P_1).$$
2) We get from (2)

\[
\begin{align*}
D^E(2P_2) &= \sum_{n=-\infty}^{+\infty} D(je^{2\pi i(2n)\tau_1}) = -\sum_{n=-\infty}^{+\infty} D((je^{2\pi i(2n)\tau_1})^{-1}) \\
&= -\sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(2n)\tau_1}) = -\sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(2n)\tau_1}) \\
&= -2 \sum_{n=-\infty}^{+\infty} D(-je^{2\pi i\tau_1}) - 2 \sum_{n=-\infty}^{+\infty} D(je^{2\pi i\tau_1})
\end{align*}
\]

which becomes by (3)

\[
D^E(2P_2) = -2D^E_1(P_1) - 2 \sum_{n=-\infty}^{+\infty} D(je^{2\pi i\tau_1});
\]

hence,

\[
\begin{align*}
D^E_2(2P_2) &= -2D^E_1(P_1) + 2 \sum_{n=-\infty}^{+\infty} D((je^{2\pi i\tau_1})^{-1}) \\
&= -2D^E_1(P_1) + 2 \sum_{n=-\infty}^{+\infty} D(j^2 e^{-2\pi i\tau_1}) \\
&= -2D^E_1(P_1) + 2 \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i\tau_1}).
\end{align*}
\]

So, by (4) we get

\[
D^E_2(2P_2) = -2D^E_1(P_1) + 2D^E_1(2P_1).
\]

For the elliptic curve 14B, we have \( E(\mathbb{Q})_{\text{tors}} \) cyclic, \( d = \# E(\mathbb{Q})_{\text{tors}} = 6 \) and \( \Sigma = 0 \). So the conditions of the conjecture of Bloch and Grayson are satisfied and the curve 14B must satisfy an exotic relation. Using the above theorem, we can give a new exotic relation for the curve 14B.

**Corollary 3.3** The elliptic curve 14B satisfies the following exotic relation

\[
7D^E_2(P_2) - 8D^E_2(2P_2) = 0.
\]

**Proof.** By theorem 3.2 we get the equivalence

\[
7D^E_2(P_2) - 8D^E_2(2P_2) = 0 \iff 2D^E_1(P_1) + 5D^E_1(2P_1) = 0.
\]
The exotic relation

\[ 2D^{E_1}(P_1) + 5D^{E_1}(2P_1) = 0 \]

was proved by Bertin [2], so we get

\[ 7D^{E_2}(P_2) - 8D^{E_2}(2P_2) = 0. \]

\[ \blacksquare \]

**Remark 3.4** We note that in some cases we can see exotic relations as relations between elliptic regulators, see Bertin [1, 2] and Touafek [8]. For more details about the elliptic regulator, see Bloch [3] and Rodriguez-Villegas [6, 7].

**References**


Received: July 26, 2007