A New Characterization of the $p$-Adic Spectrum

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Abstract

The main purpose of this paper is to give a new characterization of the $p$-adic spectrum of the polynomial ring $\mathbb{Q}_p[X_1, \ldots, X_m]$ in $m$ indeterminates over the $p$-adic number field $\mathbb{Q}_p$ comparable to the real case. Using this new characterization, we prove a relationship between the dimension of a $p$-adic semi-algebraic set and that of the constructible set of the $p$-adic spectrum of $\mathbb{Q}_p[X_1, \ldots, X_m]$ associated to this set by the $p$-adic tilde identification.

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0 Introduction

The real spectrum of a ring is a powerful and useful tool in both real algebraic and analytic geometry. It can be used to understand and prove many important results. It was introduced by Cost-Roy in [6]. For a more detailed study of this notion, we refer the interested reader to [4] and [1]. Let us recall here that a point of the real spectrum of the polynomial ring $\mathbb{R}[X_1, \ldots, X_m]$ over $\mathbb{R}$ can be regarded as an ordered pair $(I, \leq)$, where $I$ is a real prime ideal of $\mathbb{R}[X_1, \ldots, X_m]$ and $\leq$ a linear order on the residue field $k(I)$ of $I$. But to give a linear order $\leq$ on $k(I)$ is equivalent to give its positive cone

$$\{x \in k(I) | 0 \leq x\} = \{x \in k(I) | x = y^2 \text{ and } y \in R\} = k(I) \cap R^{(2)},$$

where $R$ is a real closed field. Let us remark that

$$k(I) \cap R^{(2n)} = k(I) \cap R^{(2)} \quad \text{and} \quad k(I) \cap R^{(2n+1)} = k(I) \quad \text{for all } n \geq 1.$$ 

Thus the subset $k(I) \cap R^{(2)}$ is sufficient to determine a linear order on $k(I)$. Therefore a point of the real spectrum of $\mathbb{R}[X_1, \ldots, X_m]$ may be viewed as an
ordered pair \((I, k(I) \cap R^{(2)})\), where \(I\) is a real prime ideal of \(\mathbb{R}[X_1, \ldots, X_m]\) and \(R\) is a real closed extension field of the residue field \(k(I)\) of \(I\). For further details on the theory of real closed fields, we refer the reader to [4] and [9].

Let \(p\) be a fixed prime number. The strong and well-known analogy between the theory of real closed fields and that of \(p\)-adically closed fields has motivated many mathematicians to translate results and proofs from the real case to the \(p\)-adic one. For instance, Macintyre’s theorem was shown in [8] to give a \(p\)-adic analogue of Tarski’s theorem. In the same way, we are interested here by the \(p\)-adic spectrum of the polynomial ring \(\mathbb{Q}_p[X_1, \ldots, X_m]\) over \(\mathbb{Q}_p\). More precisely, we will give a new characterization of the points of the \(p\)-adic spectrum of the polynomial ring \(\mathbb{Q}_p[X_1, \ldots, X_m]\) over \(\mathbb{Q}_p\) comparable to that of the points of real spectrum of the polynomial ring \(\mathbb{R}[X_1, \ldots, X_m]\) as remarked above.

The paper is organized as follows. In section 1 we give briefly some basic facts about \(p\)-adic algebra used later. Section 2 is devoted to the \(p\)-adic spectrum; first we recall briefly the construction of the \(p\)-adic spectrum and then we give some elementary results which are mostly well-known. In section 3 we give the main result of the paper (see Theorem 3.2). Finally, in section 4 as an illustration of the new characterization, we establish a relationship between the dimension of a \(p\)-adic semi-algebraic set and that of the constructible set of the \(p\)-adic spectrum of \(\mathbb{Q}_p[X_1, \ldots, X_m]\) associated to this set by the \(p\)-adic tilde identification.

1 Preliminary notes

Let us first introduce some notations and terminologies. Throughout this paper, any ring is assumed to be a commutative ring with unit. We will denote by \(\mathbf{x}\) the \(m\)-tuple \((x_1, \ldots, x_m)\), with \(m\) from \(\mathbb{N}^*\).

Let \(K\) be a \(p\)-adically closed field, i.e. a \(p\)-adically closed field of \(p\)-rank 1 in the sense of [10]. For example, the \(p\)-adic number field \(\mathbb{Q}_p\) is a \(p\)-adically closed field. We will denote by \(K^{*\,(n)}\) (resp. \(K^{(n)}\)) the subset of \(K\) defined as follows

\[
K^{*\,(n)} = \{ x \in K^* \mid \exists y \in K^* \quad x = y^n \}.
\]

(resp. \(K^{(n)} = \{ x \in K \mid x = 0 \text{ or } x \in K^{*\,(n)} \}\).

We denote by \(K[\mathbf{X}] = K[X_1, \ldots, X_m]\) the polynomial ring over \(K\), and by \(K(\mathbf{X})\) the quotient field of \(K[\mathbf{X}]\). The \(p\)-adic Kochen operator is defined by:

\[
\gamma(\mathbf{X}) = \frac{1}{p} \cdot \frac{X^p - X}{(X^p - X)^2 - 1}.
\]

The \(p\)-adic Kochen ring is the subring of \(K(\mathbf{X})\) defined by:

\[
\Lambda = \left\{ \frac{t}{1 + ps} \mid t, s \in \mathbb{Z}[\gamma(K(\mathbf{X}))] \right\}.
\]
We denote by $\Lambda \cdot K[X]$ the subring of $K(X)$ defined by:

$$\Lambda \cdot K[X] = \left\{ \frac{t}{1 + ps} \mid t \in K[X, \gamma(K(X))] \text{ and } s \in \mathbb{Z}[\gamma(K(X))] \right\}.$$ 

For an ideal $I$ of $K[X]$, $Z(I)$ will denote the algebraic set of $K^m$ defined by

$$Z(I) = \{ x \in K^m \mid f(x) = 0 \text{ for all } f \in I \},$$

and $J(Z(I))$ will denote the ideal of $K[X]$ associated to $I$, that is:

$$J(Z(I)) = \{ f \in K[X] \mid f(x) = 0 \text{ for all } x \in Z(I) \}.$$ 

For a prime ideal $I$ of $K[X]$, $k(I)$ denotes the residue field of $I$, that is the quotient field of the integral domain $K[X]/I$.

La notion of $p$-adic ideal was introduced in [13] as a $p$-adic analogue of that of real ideal. It was used with the model-completenes of the theory of $p$-adically closed fields to give an alternative proof of the $p$-adic Nullsellsatz. In this paper a $p$-adic ideal is in fact a $p$-adic ideal of $p$-rank 1 in the sense of [13]. Let us recall the definition of this notion:

**Definition 1.1.** Let $I$ be an ideal of the polynomial ring $K[X]$ generated by the polynomials $f_1, \ldots, f_r$. We say that $I$ is a $p$-adic ideal if for every polynomial $g \in K[X]$, for every $q \in \mathbb{N}^*$ and for every $\lambda_1, \ldots, \lambda_r \in \Lambda \cdot K[X]$ such that $g^q = \lambda_1 f_1 + \cdots + \lambda_r f_r$ then we have $g \in I$.

For $p$-adic prime ideals of $K[X]$, we have the following characterization:

**Proposition 1.2 ([13]).** Let $I$ be a prime ideal of $K[X]$. Then $I$ is a $p$-adic ideal of $K[X]$ if and only if the residue field $k(I)$ of $I$ is a formally $p$-adic field.

In particular, it is clear to see that the prime ideal $(X_1, \ldots, X_i)$ of $K[X]$ generated by $X_1, \ldots, X_i$ is a $p$-adic ideal for every integer $i$ such that $1 \leq i \leq m$. We need also the following result from [13]:

**Proposition 1.3.** An ideal $I$ of $K[X]$ is $p$-adic if and only if $J(Z(I)) = I$.

The first important model-theoretic result about $p$-adically closed fields is the model-completenes, which says that the theory of $p$-adically closed fields, denoted by $\text{Th}(\mathbb{Q}_p)$, is model-complete in the language of rings. For $n \in \mathbb{N}^*$, let $P_n$ (resp. $P_n^*$) denote the unary predicate interpreted as follows

$$\forall x \left( \text{Th}(\mathbb{Q}_p) \models P_n(x) \iff \exists y, x = y^n \right)$$

(resp. $\forall x \left( \text{Th}(\mathbb{Q}_p) \models P_n^*(x) \iff x \neq 0 \land P_n(x) \right)$).
The other important model-theoretic result is elimination of quantifiers. In this context, Macintyre’s theorem [8] states that \( \text{Th}(\mathbb{Q}_p) \) admits elimination of quantifiers in the language of rings extended by the predicates \( P_n^* \) for \( n \geq 1 \). This result was generalized by in [10] to the theory of \( p \)-adically closed fields of \( p \)-rank \( d \).

2 The \( p \)-adic spectrum

The \( p \)-adic spectrum of a ring was introduced by E. Robinson [11] in order to give a \( p \)-adic analogue of the real spectrum of a ring. An alternative definition of the \( p \)-adic spectrum was given in [5]. Here we follow that of [2]. Let \( A \) be a ring, and let us consider the relation \( \sim_p \) on the homomorphisms \( A \to L \), where \( L \models \text{Th}(\mathbb{Q}_p) \). Let \( f : A \to L \) and \( g : A \to M \) be two homomorphisms, where \( L, M \models \text{Th}(\mathbb{Q}_p) \). We say that \( f \) and \( g \) are equivalent, denoted \( f \sim_p g \), if there exists \( \varphi : L \to M \) such that \( g = \varphi \circ f \). The model-completeness of the theory \( \text{Th}(\mathbb{Q}_p) \) insures that the relation \( \sim_p \) is an equivalence relation. We can now state:

**Definition 2.1.** Let \( A \) be a ring. The \( p \)-adic spectrum of \( A \), denoted by \( \text{Spec}_p(A) \), is the topological space defined by

\[
\text{Spec}_p(A) = \left\{ A \to L \mid \text{where } L \text{ is a } p\text{-adically closed field} \right\} / \sim_p,
\]

and whose topology is given by the open basis

\[
\left\{ D_n(a) \bigg| n_i \in \mathbb{N} \text{ and } a_i \in A \text{ for } 1 \leq i \leq r \right\},
\]

where

\[
D_n(a) = \left\{ (A \xrightarrow{\varphi} L) / \sim_p \mid L \models \bigwedge_{i=1}^r P_n^*(\varphi(a_i)) \right\}.
\]

Using Macintyre’s theorem, it is not difficult to see that \( D_n(a) \) are well-defined.

**Definition 2.2.** Let \( A \) be a ring. We call a constructible subset of \( \text{Spec}_p(A) \) any subset which can be obtained by taking a finite number of intersections, unions and complements from open basis.

**Remark 2.3.** Let \( A \) be a ring. Then the \( p \)-adic spectrum \( \text{Spec}_p(A) \) of \( A \) is a quasi-compact space. Every constructible subset is also a quasi-compact space.

The coset of a homomorphism \( \alpha : A \to k(\alpha) \) modulo \( \sim_p \), where \( k(\alpha) \) is a \( p \)-adically closed field, will also denoted by \( \alpha \). For every \( a \) in \( A \), we will denote
$a(\alpha) = \alpha(a)$. Let $\alpha$ and $\beta$ be in $\text{Spec}_p(A)$. We say that $\beta$ is a specialization of $\alpha$ (or $\alpha$ is a generalization of $\beta$) if $\beta \in \overline{\{\alpha\}}$, that is
\[ \forall a \in A, \quad (a(\beta) \in k(\beta)^{(n)} \implies a(\alpha) \in k(\alpha)^{(n)}). \]

The specialization yields a partial order on $\text{Spec}_p(A)$: $\alpha \leq \beta$ if and only if $\beta$ is a specialization of $\alpha$. A specialization chain of length $r$ in $\text{Spec}_p(A)$ is a sequence $(\alpha_i)_{0 \leq i \leq r}$ of elements of $\text{Spec}_p(A)$ such that $\alpha_0 < \alpha_1 < \ldots < \alpha_r$. The dimension of a constructible subset $C$ of $\text{Spec}_p(A)$, denoted by $\dim(C)$, is the maximum length of specialization chains of $\text{Spec}_p(A)$ contained in $C$, if there exist. Otherwise, the dimension is infinite.

**Proposition 2.4.** Let $A$ be a ring. Let $C$ be a constructible set of the $p$-adic spectrum $\text{Spec}_p(A)$ of $A$. Then
\[ \overline{C} = \{ \beta \in \text{Spec}_p(A) \mid \exists \alpha \in C \text{ such that } \beta \text{ is a specialization of } \alpha \}. \]

**Proof.** Let $\beta \in \text{Spec}_p(A)$ such that there exists $\alpha \in C$ with $\beta$ is a specialization of $\alpha$. Since $\beta \in \overline{\{\alpha\}}$, we have $\{\alpha\} \cap V \neq \emptyset$ for every neighborhood $V$ of $\beta$. But $\{\alpha\} \subseteq C$ implies that $C \cap V \neq \emptyset$ for every neighborhood $V$ of $\beta$. Thus, $\beta \in \overline{C}$.

Conversely, let $\beta \in \overline{C}$. For every open basis subset $D_{\overline{a}}(\underline{a})$ which contains $\beta$, the set $D_{\overline{a}}(\underline{a}) \cap C$ is not empty. Since the $D_{\overline{a}}(\underline{a})'$s and $C$ are quasi-compact, there is a point $\alpha \in C$ which belongs to the $D_{\overline{a}}(\underline{a})'$s. Therefore $\beta$ is a specialization of $\alpha$.

**Corollary 2.5.** Let $A$ be a ring. Let $C$ and $D$ be two constructible sets of $\text{Spec}_p(A)$. Then $C$ is closed (resp. open) in $D$ if and only if it is closed with respect to specialization (resp. generalization).

### 3 Main results

Let us consider the canonical injective map $K^m \longrightarrow \text{Spec}_p(K[X])$, $x \longmapsto \varphi_x$ defined by the equality $\varphi_x(f) = f(x)$ for all $f \in K[X]$. We shall identify $K^m$ with its image in $\text{Spec}_p(K[X])$ by mean of this map. The proof of our main result is based on the following lemma which gives a characterization of the points the $p$-adic spectrum $\text{Spec}_p(K[X])$ in terms of $p$-adic prime ideals and homomorphisms:

**Lemma 3.1.** The following statements are equivalent:

1) There exists a point $\overline{\alpha}$ of $\text{Spec}_p(K[X])$.

2) There exists an ordered pair $(I_\alpha, \Psi_\alpha)$, where $I_\alpha$ is a $p$-adic prime ideal of $K[X]$ and $\Psi_\alpha$ a $K$-homomorphism from the residue field $k(I_\alpha)$ of the ideal $I_\alpha$ to some $p$-adically closed field.
**Proof.** Let $\alpha : K[X] \rightarrow L$ be a $K$-homomorphism, where $L$ is a $p$-adically closed field. Put $I_\alpha = \ker(\alpha)$. It is clear that $I_\alpha$ is a prime ideal of $K[X]$. According to proposition 1.3 to show that $I_\alpha$ is $p$-adic, it suffices to check that $\mathcal{J}(\mathcal{Z}(I_\alpha)) = I_\alpha$. Let $f \in \mathcal{J}(\mathcal{Z}(I_\alpha))$. Let us put $x = (x_1, \ldots, x_m)$, where for each integer $i$ such that $1 \leq i \leq m$, one has $x_i = \alpha(X_i)$. Since $g(x) = \alpha(g(X_1, \ldots, X_m)) = \alpha(g) = 0$ for all $g \in I_\alpha$, we deduce that $x \in \mathcal{Z}(I_\alpha)$. It follows that $f(x) = 0$. But $f(x) = \alpha(f)$. Therefore $\alpha(f) = 0$. Thus $f \in I_\alpha$ and $\mathcal{J}(\mathcal{Z}(I_\alpha)) = I_\alpha$. According to proposition 1.2, the residue field $k(I)$ of $I$ is a formally $p$-adic field. Let $k(\alpha)$ denote the $p$-adic closure of $k(I)$. Then $k(\alpha)$ is a $p$-adically closed field. The canonical map $\Psi_\alpha : k(I_\alpha) \rightarrow k(\alpha)$ is a $K$-homomorphism. Thus we have an ordered pair $(I_\alpha, \Psi_\alpha)$. If $\beta : K[X] \rightarrow M$ is an $K$-homomorphism, where $M$ is a $p$-adically closes field. Then there exists a $K$-homomorphism $i : L \rightarrow M$ such that $\beta = i \circ \alpha$. Since

$$\forall f \in K[X], \quad \alpha(f) = 0 \iff \ i(\alpha(f)) = 0 \iff \beta(f) = 0,$$

we have $I_\beta = I_\alpha$. Therefore $k(I_\beta) = k(I_\alpha)$ and $\Psi_\beta = \Psi_\alpha$. Consequently, $(I_\alpha, \Psi_\alpha) = (I_\beta, \Psi_\beta)$.

Conversely, let $(I_\alpha, \Psi_\alpha)$ be an ordered pair, where $I_\alpha$ is a $p$-adic prime ideal of $K[X]$ and $\Psi_\alpha$ a $K$-homomorphism from the residue field $k(I_\alpha)$ of $I_\alpha$ to some $p$-adically closed field $k(\alpha)$. Let $s : K[X] \rightarrow K[X]/I_\alpha$ be the canonical map and let $i_\alpha : K[X]/I_\alpha \rightarrow k(I_\alpha)$ be the injection map. Then

$$K[X] \xrightarrow{s} K[X]/I_\alpha \xrightarrow{i_\alpha} k(I_\alpha) \xrightarrow{\Psi_\alpha} k(\alpha).$$

The map $\varphi = \Psi_\alpha \circ i_\alpha \circ s$ is a $K$-homomorphism from $K[X]$ to $k(\alpha)$. The coset of $\varphi$ modulo $\sim_p$ provides a point of the $p$-adic spectrum $\text{Spec}_p(K[X])$.

Now we can state the main result of the paper which gives a new characterization of the points of the $p$-adic spectrum:

**Theorem 3.2 (Main result).** The following statements are equivalent:

1. There exists a point $\overline{\sigma}$ of $\text{Spec}_p(K[X])$.

2. There exists $(I_\alpha, k(I_\alpha) \cap L^{(n)})_{n \geq 1}$, where $I_\alpha$ is a $p$-adic prime ideal of $K[X]$ and $L$ a $p$-adically closed extension field of the residue field $k(I_\alpha)$ of $I_\alpha$.

**Proof.** According to Lemma 3.1, to give a point $\overline{\sigma}$ of $\text{Spec}_p(K[X])$ is equivalent to give an ordered pair $(I_\alpha, \Psi_\varphi)$, where $I_\alpha$ is a $p$-adic prime ideal of $K[X]$ and $\Psi_\alpha$ is a $K$-homomorphism from the residue field $k(I_\alpha)$ to a $p$-adically closed field $L$. Let us consider $k(I_\alpha) \cap L^{(n)}$, for every $n \geq 1$ and where $L$ is a $p$-adically closed extension field of $k(I_\alpha)$. Thus, to give a point
\( \sigma \) of \( \text{Spec}_p(K[X]) \) is equivalent to give an ordered pair \((I_\alpha, (k(I_\alpha) \cap L^{(n)})_{n \geq 1})\), where \( I_\alpha \) is a \( p \)-adic prime ideal of \( K[X] \) and \( L \) is a \( p \)-adically closed extension field of \( k(I_\alpha) \).

The following result is the \( p \)-adic analogue of proposition 7.5.3 of [4]:

**Corollary 3.3.** There is a specialization chain \( \alpha_0 < \alpha_1 < \ldots < \alpha_m \) in the \( p \)-adic spectrum \( \text{Spec}_p(K[X]) \) of length \( m \).

**Proof.** According to Proposition 1.2, the prime ideal \( I_0 = (X_1, \ldots, X_m) \) is a \( p \)-adic ideal of \( K[X] \). On the other hand, we know that \( K[X]/(X_1, \ldots, X_m) \cong K \). Then \( k(I_0) \cong K \). From Theorem 3.2, \( \alpha_0 = (I_0, (K^{(n)})_{n \geq 1}) \) is a point of \( \text{Spec}_p(K[X]) \). Similarly, the prime ideal \( I_1 = (X_2, \ldots, X_m) \) of \( K[X] \) is a \( p \)-adic ideal of \( K[X] \). We have also \( K[X]/(X_2, \ldots, X_m) \cong K[X_1] \). Thus, \( k(I_1) \cong K(X_1) \). According to example 2.2 of [9], \( K(X_1) \) is a formally \( p \)-adic field. Let \( k(\alpha_1) \) denote the \( p \)-adic closure of this field. Then \( \alpha_1 = (I_1, (K(X_1) \cap k(\alpha_1)^{(n)})_{n \geq 1}) \) is a point of \( \text{Spec}_p(K[X]) \). Moreover, one has \( \alpha_0 < \alpha_1 \). Indeed, let \( f \in K[X] \) such that \( f(\alpha_1) \in k(\alpha_1)^{(n)} \). i.e. \( f(X_1, 0, \ldots, 0) \in k(\alpha_1)^{(n)} \). But \( f(\alpha_0) = f(0, 0, \ldots, 0) \). Therefore \( f(\alpha_0) \in k(\alpha_0)^{(n)} \). Since \( \alpha_0 \not\in \{\alpha_1\} \), one has \( \alpha_0 < \alpha_1 \).

Let \( \alpha_i = (I_i, (k(I_i) \cap k(\alpha_i)^{(n)})_{n \geq 1}) \), with \( I_i = (X_{i+1}, \ldots, X_m) \), \( k(I_i) \) the residue field of \( I_i \) and \( k(\alpha_i) \) the \( p \)-adic closure of \( k(I_i) \) for \( 1 \leq i < m \). We have to construct the point \( \alpha_{i+1} \). As above, the prime ideal \( I_{i+1} = (X_{i+2}, \ldots, X_m) \) is a \( p \)-adic ideal of \( K[X] \). But \( K[X]/I_{i+1} \cong K[X_1, \ldots, X_{i+1}] \). Therefore \( k(I_{i+1}) \cong k(I_i)(X_{i+1}) \). Thus \( k(I_{i+1}) \) is a formally \( p \)-adic field over \( k(I_i) \). Let \( k(\alpha_{i+1}) \) denote the \( p \)-adic closure of \( k(I_{i+1}) \). Put \( \alpha_{i+1} = (I_{i+1}, (k(I_{i+1}) \cap k(\alpha_{i+1})^{(n)})_{n \geq 1}) \). Then \( \alpha_{i+1} \) is a point of \( \text{Spec}_p(K[X]) \) and \( \alpha_i < \alpha_{i+1} \). We put \( \alpha_m = ((0), (K(X) \cap k(\alpha_m)^{(n)})_{n \geq 1}) \).

4 **Applications to semi-algebraic sets**

Let us recall that a \( p \)-adic semi-algebraic set of \( K^m \) is a subset which can be obtained by taking a finite number of intersections, unions and complements from subsets of the form \( \{x \in K^m \mid f(x) \in K^{(n)}\} \), with \( f \in K[X] \). It is easy to see that any semi-algebraic set \( S \) of \( K^m \) can be written of the form

\[
S = \bigcup_{j=1}^{r} \bigcap_{i=1}^{q} \left\{ x \in K^m \mid g_{ij}(x) = 0 \text{ and } f_{ij}(x) \in K^{(n_{ij})} \right\},
\]

where \( g_{ij}, f_{ij} \in K[X] \) and \( n_{ij} \in \mathbb{N}^* \).

Let us remark that the \( p \)-adic semi-algebraic sets are the \( p \)-adic analogue of the real semi-algebraic sets, that is the subset of \( \mathbb{R}^m \) which can be obtained by taking a finite number of intersections, unions and complements from subsets
of the form \( \{ x \in \mathbb{R}^m \mid f(x) \geq 0 \} \), with \( f \in \mathbb{R}[X] \) (see [4]). Macintyre’s theorem enables us to deduce that the projection of a semi-algebraic subsets of \( K^{m+1} \) is a semi-algebraic subsets of \( K^m \). Thus, the closure and the interior of a \( p \)-adic semi-algebraic set are also \( p \)-adic semi-algebraic sets. Using Macintyre’s theorem, we can also conclude that the \( p \)-adic semi-algebraic sets of \( K^m \) are exactly the definable subsets of \( K^m \). We refer the reader to [3] for more details about definable subsets.

The dimension of a \( p \)-adic semi-algebraic set \( S \) of \( K^m \), denoted \( \dim(S) \), is the Krull dimension of the ring \( K[X] / \mathcal{J}(S) \), where \( \mathcal{J}(S) \) is the ideal of \( K[X] \) defined by

\[
\mathcal{J}(S) = \{ f \in K[X] \mid f(x) = 0 \text{ for all } x \in S \}.
\]

A \( p \)-adic semi-algebraic function is a function \( f : S \rightarrow K \) whose graph is a \( p \)-adic semi-algebraic subset of \( K^{m+1} \). For more details about \( p \)-adic semi-algebraic subsets and functions, we refer the reader to [7] and [12]. Here we will need the following result (see [12]):

**Proposition 4.1 (Corollary 3.1, [12]).** Let \( S \) be a \( p \)-adic semi-algebraic subset of \( K^m \). Then there is a finite partition of \( S \) into \( p \)-adic semi-algebraic subsets \( S_1, \ldots, S_q \), where each \( S_i \) is homeomorphic (by means of a semi-algebraic function) to a \( p \)-adic semi-algebraic open subset of \( K^{r_i} \) with \( r_i \leq m \).

We will need the following result about the dimension of \( p \)-adic semi-algebraic sets:

**Proposition 4.2 (Theorem 3.2, [12]).** Let \( S = S_1 \cup \cdots \cup S_q \) be a \( p \)-adic semi-algebraic subset of \( K^m \), where each \( S_i \) is homeomorphic (by means of a semi-algebraic function) to a \( p \)-adic semi-algebraic open subset of \( K^{r_i} \) with \( r_i \leq m \). Then we have \( \dim(S) = \max(\dim(S_1), \ldots, \dim(S_r)) \).

As in the real case, we can define the \( p \)-adic tilde operation as follows:

**Definition 4.3.** The \( p \)-adic tilde operation is the correspondance \( S \mapsto \tilde{S} \), where

\[
S = \bigcup_{j=1}^r \bigcap_{i=1}^q \left\{ x \in K^m \mid g_j(x) = 0 \text{ and } f_{ij}(x) \in K^{*(n_{ij})} \right\}
\]

is a \( p \)-adic semi-algebraic subset of \( K^m \) and

\[
\tilde{S} = \bigcup_{j=1}^r \bigcap_{i=1}^q \left\{ \alpha \in \text{Spec}_p(K[X]) \mid g_j(\alpha) = 0 \text{ and } f_{ij}(\alpha) \in k(\alpha)^{*(n_{ij})} \right\}
\]

is a constructible subset of \( \text{Spec}_p(K[X]) \), where \( f_{ij}, g_j \in K[X] \).

The following result summarizes the main properties of the \( p \)-adic tilde operation:
Theorem 4.4. i) The map $S \mapsto \tilde{S}$ is an isomorphism between the $p$-adic semi-algebraic subsets of $K^m$ and the constructible subsets of the $p$-adic spectrum $\text{Spec}_p(K[X])$.

(ii) A semi-algebraic subset $S$ is open in $K^m$ if and only if $\tilde{S}$ is open in $\text{Spec}_p(K[X])$.

Thus, the isomorphism $S \mapsto \tilde{S}$ yields a bijection between the $p$-adic semi-algebraic open subsets of $K^m$ and the quasi-compact constructible subsets of $\text{Spec}_p(K[X])$.

(iii) The $p$-adic tilde operation commutes with the closure and the interior, that is, if $S$ is a $p$-adic semi-algebraic subset of $K^m$, then: $\tilde{\tilde{S}} = \tilde{S}$ and $\tilde{S} = \tilde{\tilde{S}}$.

Proof. i) Let $S$ and $T$ be two $p$-adic semi-algebraic subsets of $K^m$ such that $S = T$. The model-completeness of the theory $\text{Th}(\mathbb{Q}_p)$ implies that $S = T$ on $L$, where $L$ is any $p$-adically closed fields such that $K \subseteq L$. In particular, we have $S = T$ on $k(\alpha)$ for any $\alpha \in \text{Spec}_p(K[X])$. Thus, $\tilde{S} = \tilde{T}$. It follows that the map $S \mapsto \tilde{S}$ is well-defined. Conversely, if $\tilde{S} = \tilde{T}$. Then $\tilde{S} \cap K^m = \tilde{T} \cap K^m$. Therefore $S = T$. Thus, the map $S \mapsto \tilde{S}$ is injective. On the other hand, it is clear that this map is an homomorphism surjective from the Boole algebra of $p$-adic semi-algebraic subsets of $K^m$ to that of the constructible of $\text{Spec}_p(K[X])$.

ii) Let $S$ be an open semi-algebraic subsets of $K^m$. According to the $p$-adic finiteness theorem, $S$ is a finite union of open of the type

$$S_{f_1, \ldots, f_r} = \left\{ x \in K^m \mid f_1(x) \in K^{*^{(n_1)}}, \ldots, f_r(x) \in K^{*^{(n_r)}} \right\},$$

where $f_1, \ldots, f_r \in K[X]$. Thus, $\tilde{S}$ is a finite union of open of the type $\tilde{S}_{\tilde{f}_1, \ldots, \tilde{f}_r}$.

iii) Since quasi-compact open form an open basis of the space $\text{Spec}_p(K[X])$, we have

$$\tilde{S} = \bigcup_{\alpha \in S} \tilde{O} = \bigcup_{\alpha \in \tilde{S}} O,$$

where $O$ are open semi-algebraic contained in $S$.

But $\tilde{S}$ is a $p$-adic semi-algebraic. Therefore $\tilde{\tilde{S}} = \tilde{S}$. Similarly for the closure. $\square$

The following result is the $p$-adic analogue of proposition 7.5.6 of [4]:

**Proposition 4.5.** Let $S$ be a semi-algebraic of $K^m$. Then $\dim(S) = \dim(\tilde{S})$.

Proof. Let $\alpha_r < \ldots < \alpha_0$ be a specialization chain of length $r$ of $\text{Spec}_p(K[X])$ contained in $\tilde{S}$. Let us put $I_i = \ker(\alpha_i)$ for each integer $i$ such that $0 \leq i \leq r$. Then, we have a chain

$$I_0 \subset I_1 \subset \ldots \subset I_r$$

of prime ideals of $K[X]$ of length $r$. On the other hand, if $\alpha \in \tilde{S}$ then we have $f(\alpha) = 0$ for all $f \in \mathcal{J}(S)$. It follows that $\mathcal{J}(S) \subset I_i$ for each integer $i$ such that $0 \leq i \leq r$. Therefore we we have a chain

$$I_0/\mathcal{J}(S) \subset I_1/\mathcal{J}(S) \subset \ldots \subset I_r/\mathcal{J}(S)$$
of prime ideals of $K[\overline{X}]/\mathcal{J}(S)$ of length $r$. Hence $\dim(\overline{S}) \leq \dim(S)$.

Conversely, we have to show that $\dim(S) \leq \dim(\overline{S})$. Let $d = \dim(S)$. According to Proposition 4.2, we know that $S$ contains a $p$-adic semi-algebraic subset $T$ such that $T$ is homeomorphic (by means of a semi-algebraic function) to a $p$-adic semi-algebraic open subset $U$ of $K^d$ which contains the origin. The $p$-adic semi-algebraic homeomorphism between $T$ and $U$ extends to a homeomorphism between $\overline{T}$ and $\overline{U}$. From Corollary 3.3, there exists in $\text{Spec}_p(K[X_1, \ldots, X_d])$ a specialization chain of length $d$ which ends by the origin. According to corollary 2.4, $\overline{U}$ is closed with respect to generalization, and this specialization chain is contained in $\overline{U}$. Thus, we obtain a specialization chain of length $d$ contained in $\overline{S}$, and also in $\overline{T}$, and also in $\overline{S}$. Hence $\dim(S) \leq \dim(\overline{S})$. \(\square\)

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References


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