The Green-Kehayopulu Relations
in le-$\Gamma$-Semigroups\textsuperscript{1}

Manoj Siripitukdet

Department of Mathematics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand
manojs@nu.ac.th

Aiyared Iampan

Department of Mathematics, School of Science and Technology
Naresuan University Phayao, Phayao 56000, Thailand
aiyaredi@nu.ac.th

Abstract

We can see that any semigroup can be considered as a $\Gamma$-semigroup. The concept of the Green-Kehayopulu relations in le-semigroups was introduced in 2002 by Petro and Pasku [3]. In this paper, we introduce the concept of the Green-Kehayopulu relations in le-$\Gamma$-semigroups mimics the definition of the Green-Kehayopulu relations in le-semigroups. We show that, an $H_\gamma$-class of an le-$\Gamma$-semigroup $M$ satisfies Green’s condition if and only if it contains a $\gamma$-idempotent and an $H_\gamma$-class of an le-$\Gamma$-semigroup $M$ is a subgroup of $<M_\gamma, \circ>$ if and only if it consists of a single idempotent.

Mathematics Subject Classification: 20N99, 06B99

Keywords: $\Gamma$-semigroup, le-$\Gamma$-semigroup, $\gamma$-idempotent, Green’s condition, Green-Kehayopulu relation

1 Introduction and Preliminaries

In 2002, Petraq Petro and Elton Pasku [3] introduced the concept of the Green-Kehayopulu relations in le-semigroups and showed that a nonsingleton $H$-class

\textsuperscript{1} This research is supported by a grant of the Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.
cannot be a subgroup and an \( H \)-class satisfying “Green’s condition” need not constitute a subsemigroup.

In this paper, we introduce the concept of the Green-Kehayopulu relations in \( le \)-\( \Gamma \)-semigroups and give necessary and sufficient conditions in order that an \( H \gamma \)-class of \( le \)-\( \Gamma \)-semigroup \( M \) is a subgroup or a subsemigroup of \( < M \gamma, \circ > \).

To present the main results we first recall the definition of a \( \Gamma \)-semigroup which is important here.

Let \( M \) and \( \Gamma \) be any two nonempty sets. \( M \) is called a \( \Gamma \)-semigroup [5] if there exists a mapping \( M \times \Gamma \times M \to M \), written as \((a, \gamma, b) \mapsto a\gamma b\), satisfying the following identity \((a\alpha b)\beta c = a\alpha (b\beta c)\) for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \). For any \( \gamma \in \Gamma \), an element \( x \) of a \( \Gamma \)-semigroup \( M \) is said to be a \( \gamma \)-idempotent \([6]\). For a \( \Gamma \)-semigroup \( M \) and any \( \gamma \in \Gamma \), if we define \( a \circ b = a\gamma b \) for all \( a, b \in M \), then \( M \) becomes a semigroup. We denote this semigroup by \( M \gamma \) [6].

Examples of \( \Gamma \)-semigroups can be seen in [1, 4] and [5], respectively.

The following definitions in this paper are introduced analogously to some definitions in [3].

A \( \Gamma \)-semigroup \( M \) is called an \( le \)-\( \Gamma \)-semigroup if \( < M; \lor, \land > \) is a lattice with a greatest element (the element is always denoted by \( e \) below) \([2]\) and for any \( a, b, c \in M \) and \( \gamma \in \Gamma \),

\[
   c\gamma(a \lor b) = c\gamma a \lor c\gamma b \quad \text{and} \quad (a \lor b)\gamma c = a\gamma c \lor b\gamma c.
\]

Throughout this paper \( M \) will stand for an \( le \)-\( \Gamma \)-semigroup. We shall consider the usual order relation \( \leq \) on \( M \) defined by for any \( a, b \in M \), \( a \leq b \) if and only if \( a \lor b = b \). Then we can show that for any \( a, b, c \in M \) and \( \gamma \in \Gamma \), \( a \leq b \) implies \( a\gamma c \leq b\gamma c \) and \( c\gamma a \leq c\gamma b \). Hence we have known that ordered \( \Gamma \)-semigroups (some author called \( po \)-\( \Gamma \)-semigroup) are a generalization of \( le \)-\( \Gamma \)-semigroups. For any \( \gamma \in \Gamma \), let the mappings \( l_\gamma, r_\gamma : M \to M \) be defined by for any \( x \in M \),

\[
   l_\gamma(x) = e\gamma x \lor x \quad \text{and} \quad r_\gamma(x) = x\gamma e \lor x.
\]

Then we define equivalence relations on \( M \) as follows:

\[
   \mathcal{L}_\gamma := \{(x, y) \in M \times M : l_\gamma(x) = l_\gamma(y)\},
\]

\[
   \mathcal{R}_\gamma := \{(x, y) \in M \times M : r_\gamma(x) = r_\gamma(y)\},
\]

\[
   \mathcal{H}_\gamma := \mathcal{L}_\gamma \cap \mathcal{R}_\gamma.
\]

We shall call the equivalences \( \mathcal{L}_\gamma, \mathcal{R}_\gamma \) and \( \mathcal{H}_\gamma \) the Green-Kehayopulu relations of \( M \). An element \( x \) of \( M \) is said to be a \( \gamma \)-left ideal (\( \gamma \)-right ideal) element if
$l_{\gamma}(x) = x$ $(r_{\gamma}(x) = x)$ and a $\gamma$-ideal element if it is both a $\gamma$-left ideal element and a $\gamma$-right ideal element; it is called a $\gamma$-quasi-ideal element if $e_{\gamma}x \wedge x_{\gamma}e \leq x$.

An element $x$ of $M$ is said to be a $\gamma$-regular element if $x \leq x_{\gamma}x_{\gamma}e$ and a $\gamma$-intra-regular element if $x \leq e_{\gamma}x_{\gamma}x_{\gamma}e$. An $\mathcal{H}_\gamma$-class $H$ of $M$ satisfying Green’s condition if there exist elements $a$ and $b$ of $M$ such that $a_{\gamma}b \in H$.

Before the characterizations of the $\mathcal{H}_\gamma$-class of $M$ for the main results, we give auxiliary results which are necessary in what follows.

**Lemma 1.1** For each $x \in M$ and $\gamma \in \Gamma$,

\[
l_{\gamma}(l_{\gamma}(x)) = l_{\gamma}(x) \text{ and } r_{\gamma}(r_{\gamma}(x)) = r_{\gamma}(x).
\]

**Proof.** From the definition of the mapping $l_\gamma$ it follows that $l_\gamma(l_\gamma(x)) = l_\gamma(e_{\gamma}x \lor x) = e_{\gamma}(e_{\gamma}x \lor x) \lor e_{\gamma}x = e_{\gamma}e_{\gamma}x \lor e_{\gamma}x \lor x = e_{\gamma}e_{\gamma}x \lor e_{\gamma}x \lor x$. Since $e$ is the greatest element in $M$, we also have $e_{\gamma}e_{\gamma} \leq e$. Thus $e_{\gamma}e_{\gamma}x \leq e_{\gamma}x$, so $e_{\gamma}e_{\gamma}x \lor e_{\gamma}x = e_{\gamma}x$. Hence $l_\gamma(l_\gamma(x)) = e_{\gamma}x \lor x = l_\gamma(x)$. By symmetry, $r_\gamma(r_\gamma(x)) = r_\gamma(x)$. □

**Lemma 1.2** If an element $a$ of $M$ is a $\gamma$-left ideal element and an element $b$ of $M$ is a $\gamma$-right ideal element, then $a \wedge b$ is a $\gamma$-quasi-ideal element.

**Proof.** Assume that $a$ is a $\gamma$-left ideal element and $b$ is a $\gamma$-right ideal element of $M$. Then $e_{\gamma}a \lor a = l_\gamma(a) = a$ and $b_{\gamma}e \lor b = r_\gamma(b) = b$, so $e_{\gamma}a \leq a$ and $b_{\gamma}e \leq b$. Hence $e_{\gamma}(a \wedge b) \land (a \wedge b)_{\gamma}e \leq e_{\gamma}a \land b_{\gamma}e \leq a \land b$. Therefore $a \land b$ is a $\gamma$-quasi-ideal element. □

**Lemma 1.3** For each $x \in M$ and $\gamma, \beta \in \Gamma$,

\[
l_{\beta}(l_{\beta}(x) \land r_{\gamma}(x)) = l_{\beta}(x) \text{ and } r_{\gamma}(l_{\beta}(x) \land r_{\gamma}(x)) = r_{\gamma}(x).
\]

**Proof.** Since $x = x \land x \leq l_{\beta}(x) \land r_{\gamma}(x) \leq l_{\beta}(x)$, it follows from Lemma 1.1 that $l_{\beta}(x) \leq l_{\beta}(l_{\beta}(x) \land r_{\gamma}(x)) \leq l_{\beta}(l_{\beta}(x)) = l_{\beta}(x)$. Hence $l_{\beta}(l_{\beta}(x) \land r_{\gamma}(x)) = l_{\beta}(x)$. By symmetry, $r_{\gamma}(l_{\beta}(x) \land r_{\gamma}(x)) = r_{\gamma}(x)$. □

**Lemma 1.4** Each $\mathcal{H}_\gamma$-class $H$ of $M$ has a greatest element which is equal to $l_\gamma(a) \land r_\gamma(a)$ where $a$ is an arbitrary element in $H$.

**Proof.** Let $a$ be an element of the $\mathcal{H}_\gamma$-class $H$ of $M$. By Lemma 1.3, we have $(l_{\gamma}(a) \land r_{\gamma}(a), a) \in \mathcal{L}_{\gamma}$ and $(l_{\gamma}(a) \land r_{\gamma}(a), a) \in \mathcal{R}_{\gamma}$. Thus $(l_{\gamma}(a) \land r_{\gamma}(a), a) \in \mathcal{H}_{\gamma}$, so $l_{\gamma}(a) \land r_{\gamma}(a) \in H$. Now let any $x \in H$. Then $(x, a) \in \mathcal{H}_{\gamma} = \mathcal{L}_{\gamma} \cap \mathcal{R}_{\gamma}$, this implies that $x \leq l_{\gamma}(x) = l_{\gamma}(a)$ and $x \leq r_{\gamma}(x) = r_{\gamma}(a)$. Hence $x \leq l_{\gamma}(a) \land r_{\gamma}(a)$, so $l_{\gamma}(a) \land r_{\gamma}(a)$ is a greatest element of $H$. □
Lemmas 1.1 and 1.2 imply that for each element \( a \) of \( M \), the meet \( l_\gamma(a) \land r_\gamma(a) \) is a \( \gamma \)-quasi-ideal element. Lemma 1.4 implies that for each element \( a \) of the \( H_\gamma \)-class \( H \), \( l_\gamma(a) \land r_\gamma(a) \) is a greatest element of \( H \). We call the element \( l_\gamma(a) \land r_\gamma(a) \) the representative \( \gamma \)-quasi-ideal element of the \( H_\gamma \)-class of \( a \); the representative \( \gamma \)-quasi-ideal element of an \( H_\gamma \)-class \( H \) will be denoted by \( q_H \).

From Lemma 1.4, the following properties of \( q_H \) hold.

1. \( q_H \in H \).
2. For each \( x \in H \), \( l_\gamma(x) \land r_\gamma(x) = q_H \); in particular, \( l_\gamma(q_H) \land r_\gamma(q_H) = q_H \).
3. For each \( x \in H \), \( x \leq q_H \).

Lemma 1.5 If elements \( x \) and \( y \) of \( M \) are \( R_\gamma \)-related (resp. \( L_\gamma \)-related), then \( x\gamma e = y\gamma e \) (resp. \( e\gamma x = e\gamma y \)).

Proof. Assume that \( (x, y) \in R_\gamma \). Then \( r_\gamma(x) = r_\gamma(y) \), so \( x\gamma e \lor y\gamma e = x\gamma e \lor y\gamma e \). This implies that \( x\gamma e \lor y\gamma e = (x\gamma e \lor x)\gamma e = (y\gamma e \lor y)\gamma e = y\gamma e \lor y\gamma e \). Since \( e\gamma e \leq e \), \( x\gamma e \leq x\gamma e \) and \( y\gamma e \leq y\gamma e \). Hence \( x\gamma e = x\gamma e \lor y\gamma e = x\gamma e \lor y\gamma e = y\gamma e \). Similarly, \( (x, y) \in L_\gamma \) implies \( e\gamma x = e\gamma y \). \( \square \)

Lemma 1.6 If \( H \) is an \( H_\gamma \)-class of \( M \) and \( x \in H \), then \( e\gamma x \land x\gamma e = e\gamma q_H \land q_H \gamma e \).

Proof. Assume that \( H \) is an \( H_\gamma \)-class of \( M \) and \( x \in H \). Then \( (x, q_H) \in H_\gamma \). It follows from Lemma 1.5 that \( e\gamma x = e\gamma q_H \) and \( x\gamma e = q_H \gamma e \). Hence \( e\gamma x \land x\gamma e = e\gamma q_H \land q_H \gamma e \). \( \square \)

2 Main Results

In this section, we characterize the relationship between the \( H_\gamma \)-classes of \( M \) satisfying Green’s condition and the semigroup \(< M_\gamma, \circ >\) and give some conditions which ensure that an \( H_\gamma \)-class of \( M \) forms a subgroup or a subsemigroup of the semigroup \(< M_\gamma, \circ >\).

The following theorems collect several properties that hold in every \( H_\gamma \)-class of \( M \) satisfying Green’s condition.

Theorem 2.1 Let \( H \) be an \( H_\gamma \)-class of \( M \) satisfying Green’s condition and \( q = q_H \). Then we have the following statements:

(a) \( q\gamma q \in H \) and \( q = e\gamma q \land q\gamma e \).

(b) The element \( q \) is the only \( \gamma \)-quasi-ideal element in \( H \).
(c) If \( x, y \in H \), then \( y \leq e \gamma x \) and \( y \leq x \gamma e \).

(d) For each integer \( n \geq 2 \), let \( \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in \{ \gamma \} \). Then \( q \gamma q = q \gamma e \gamma q = q \gamma q \gamma q \gamma q \gamma q \cdots \gamma q \gamma_{n-1} q \); in particular, \( q \gamma q \) is a \( \gamma \)-idempotent.

(e) Every element of \( H \) is a \( \gamma \)-intra-regular element.

(f) The element \( q \) is a \( \gamma \)-idempotent if and only if \( q \) is a \( \gamma \)-regular element in which case every element of \( H \) is a \( \gamma \)-regular element.

**Proof.** (a) Since \( H \) satisfies Green's condition, there exist \( b, c \in H \) such that \( b \gamma c \in H \). Since \( b, c \in H \), we have \( b \leq q \) and \( c \leq q \). Thus \( b \gamma c \leq q \gamma q \leq q \gamma e \), this implies that \( r_\gamma(b \gamma c) \leq r_\gamma(q \gamma q) \leq r_\gamma(q \gamma e) \). Since \( (b \gamma c, q) \in \mathcal{H}_\gamma \), \( (b \gamma c, q) \in \mathcal{R}_\gamma \). Thus \( r_\gamma(b \gamma c) = r_\gamma(q) \). On the other hand, since \( e \gamma e \leq e \), we have \( r_\gamma(q \gamma e) = q \gamma e \gamma e \vee q \gamma e = q \gamma e \leq q \gamma q \). Hence \( r_\gamma(q) = r_\gamma(b \gamma c) \leq r_\gamma(q \gamma q) \leq r_\gamma(q \gamma e) = q \gamma e \leq r_\gamma(q \gamma q) \), so \( r_\gamma(q) = r_\gamma(q \gamma q) = q \gamma e \). By symmetry, \( l_\gamma(q) = l_\gamma(q \gamma q) = e \gamma q \). Therefore \( (q, q \gamma q) \in \mathcal{H}_\gamma \), so \( q \gamma q \in H \). It follows that \( q = l_\gamma(q) \wedge r_\gamma(q) = e \gamma q \wedge q \gamma e \).

(b) By (a), \( q \) is a \( \gamma \)-quasi-ideal element in \( H \). Now let \( t \) be any \( \gamma \)-quasi-ideal element in \( H \). By (a) and Lemma 1.6, we have \( t \leq q = e \gamma q \wedge q \gamma e = e \gamma t \wedge t \gamma e \leq t \). Hence \( t = q \), so we conclude that \( q \) is the only \( \gamma \)-quasi-ideal element in \( H \).

(c) Let any \( x, y \in H \). By (a) and Lemma 1.6, we have \( y \leq q = e \gamma q \wedge q \gamma e = e \gamma x \wedge x \gamma e \). Hence \( y \leq e \gamma x \) and \( y \leq x \gamma e \).

(d) By (a), \( q = e \gamma q \wedge q \gamma e \leq q \gamma e \). Thus \( q \gamma q \leq q \gamma e \gamma q \). Since \( e \gamma q \leq e \), \( q \gamma e \gamma q \leq q \gamma e \). Thus \( q \gamma e \gamma q \leq q \gamma e \wedge q \gamma e = q \). Hence \( q \gamma q \gamma e \gamma q \leq q \gamma q \). By (a), we get \( (q \gamma q, q) \in \mathcal{R}_\gamma \). By Lemma 1.5, \( q \gamma e = q \gamma q \gamma e \) and it follows that \( q \gamma q = q \gamma q \gamma e \gamma q \). Hence \( q \gamma q \leq q \gamma e \gamma q \) and \( q \gamma e \gamma q \leq q \gamma q \). Therefore \( q \gamma q = q \gamma e \gamma q \). Now let any integer \( k \geq 2 \) and \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \in \{ \gamma \} \) be such that \( \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_{k-1} q = q \gamma q \). Then \( q \gamma_1 q \gamma_2 q \cdots q \gamma_{k-1} q = q \gamma q \gamma q = q \gamma (q \gamma q) \gamma q = (q \gamma q) \gamma q = q \gamma q \). In particular, \( (q \gamma q) \gamma (q \gamma q) = q \gamma q \). Hence \( q \gamma q \) is a \( \gamma \)-idempotent.

(e) Let any \( x \in H \). Then \( x \leq q \). By (a), we get \( q \leq e \gamma q \) and \( q \leq q \gamma q \). Thus \( x \leq e \gamma q \leq e \gamma q \gamma e \). By (a), we get \( (q \gamma q, q) \in \mathcal{R}_\gamma \). By Lemma 1.5, \( q \gamma e = q \gamma q \gamma e \). This implies that \( x \leq e \gamma q \gamma e = e \gamma (q \gamma q \gamma e) = (e \gamma q) \gamma (q \gamma e) \). Since \( (x, q) \in \mathcal{H}_\gamma \), it follows from Lemma 1.5 that \( e \gamma q = e \gamma x \) and \( q \gamma e = x \gamma e \). Hence \( x \leq e \gamma x x \gamma e \), so we conclude that \( x \) is a \( \gamma \)-intra-regular element.

(f) Assume that \( q = q \gamma q \). By (d), \( q \gamma q = q \gamma e \gamma q \). Thus \( q = q \gamma e \gamma q \), so \( q \) is a \( \gamma \)-regular element. If \( x \in H \), then \( x \leq q \). Since \( (x, q) \in \mathcal{H}_\gamma \), it follows from Lemma 1.5 that \( e \gamma q = e \gamma x \) and \( q \gamma e = x \gamma e \). Hence \( x \leq q = (q \gamma e) \gamma q = (x \gamma e) \gamma q = x \gamma (e \gamma q) = x \gamma e \gamma x \). Therefore \( x \) is a \( \gamma \)-regular element.

Conversely, assume that \( q \leq q \gamma e \gamma q \). By (d), \( q \gamma q = q \gamma e \gamma q \). Thus \( q \leq q \gamma q \). By (a), \( q \gamma q \in H \). Thus \( q \gamma q \leq q \). Hence \( q = q \gamma q \), so we conclude that \( q \) is a \( \gamma \)-idempotent.
Therefore we complete the proof of the theorem. \(\square\)

Using the Theorems 2.1 (a) and 2.1 (d), we have Corollary 2.2.

**Corollary 2.2** An \(H_\gamma\)-class \(H\) of \(M\) satisfies Green’s condition if and only if it contains a \(\gamma\)-idempotent.

**Theorem 2.3** An \(H_\gamma\)-class \(H\) of \(M\) is a subgroup of \(<M_\gamma, \circ>\) if and only if it consists of a single idempotent.

**Proof.** Assume that \(H\) is a subgroup of \(M_\gamma\) and let \(q = q_H\). Then \(q \gamma q = q \circ q \in H\), so \(q \gamma q \leq q\). Denote by \(i\) the identity element of \(H\). Then \(i \leq q\), so \(q \circ i = q \gamma i \leq q \gamma q = q \circ q\). Hence \(q \circ q = q\), so we conclude that \(q = i\). Now let \(t\) be an arbitrary element of \(H\). We denote by \(t^{-1}\) the inverse element of \(t\) in \(H\). Then \(t^{-1} \leq q\), so \(q = i = t \circ t^{-1} = t \gamma t^{-1} \leq t \gamma q = t \circ q = t \circ i = t\). On the other hand, \(t \leq q\). Therefore \(t = q\), so we conclude that \(H\) consists of a single idempotent.

The converse is obvious. \(\square\)

**Theorem 2.4** Let \(H\) be an \(H_\gamma\)-class of \(M\) and \(q = q_H\). Then the following statements are equivalent:

(a) An \(H_\gamma\)-class \(H\) is a subsemigroup of \(<M_\gamma, \circ>\).

(b) If \(x \in H\), then \(x \gamma x \in H\).

(c) An \(H_\gamma\)-class \(H\) satisfies Green’s condition and \(x \gamma q = q \gamma q = q \gamma x\) for every \(x \in H\).

**Proof.** Since \(H\) is a subsemigroup of \(M_\gamma\), we immediately have \(x \gamma x = x \circ x \in H\) for all \(x \in H\). Therefore (a) implies (b). Let any \(x \in H\). Then \(x \gamma x \in H\), so \(H\) satisfies Green’s condition and \((x, x \gamma x) \in \mathcal{H}_\gamma\). By Lemma 1.5, \(e \gamma x = e \gamma x \gamma x\) and \(x \gamma e = x \gamma x \gamma e\). Similarly, since \((x, q) \in \mathcal{H}_\gamma\), we get \(e \gamma x = e \gamma q\) and \(x \gamma e = q \gamma e\). By Theorem 2.1 (d), \(q \gamma q = q \gamma e \gamma q\). Hence \(x \gamma q \gamma q = x \gamma (q \gamma e \gamma q) = x \gamma (q \gamma e) \gamma q = x \gamma (x \gamma e) \gamma q = (x \gamma x \gamma e) \gamma q = (x \gamma e) \gamma q = (q \gamma e) \gamma q = q \gamma q\). Similarly, \(q \gamma q \gamma x = q \gamma q\). Since \(x, q \gamma q \in H\), we have \(x \leq q\) and \(q \gamma q \leq q\). Hence \(q \gamma q = x \gamma q \gamma q \leq x \gamma q \leq q \gamma q\), so we conclude that \(x \gamma q = q \gamma q\). Similarly, \(q \gamma x = q \gamma q\). Thus (b) implies (c). Let any \(x, y \in H\). Then \((y, q) \in \mathcal{H}_\gamma\), so \((y, q) \in \mathcal{R}_\gamma\). Thus \(r_\gamma(y) = r_\gamma(q)\), so \(y \gamma e \vee y = q \gamma e \vee q\). Hence \(r_\gamma(x \gamma y) = x \gamma y \gamma e \vee x \gamma y = x \gamma (y \gamma e \vee y) = x \gamma (q \gamma e \vee q) = x \gamma q \gamma e \vee x \gamma q = r_\gamma(x \gamma q)\). Since \(x \in H\), \(x \gamma q = q \gamma q\). This implies that \(r_\gamma(x \gamma y) = r_\gamma(q \gamma q)\). By Theorem 2.1 (a), \(q \gamma q \in H\). It follows that \(r_\gamma(q \gamma q) = r_\gamma(q)\). Hence \(r_\gamma(x \gamma y) = r_\gamma(q)\), so \((x \gamma y, q) \in \mathcal{R}_\gamma\). Similarly, since \((y, q) \in \mathcal{L}_\gamma\), we have \((x \gamma y, q) \in \mathcal{L}_\gamma\). We conclude that \((x \gamma y, q) \in \mathcal{H}_\gamma\), so
The Green-Kehayopulu relations in $\mathcal{H}$-semigroups

$x \circ y = x\gamma y \in H$. Therefore $H$ is a subsemigroup of $M_\gamma$, so we have that (c) implies (a).

Hence the theorem is now completed. \qed

As a consequence of Theorem 2.4, we immediately have Corollary 2.5.

**Corollary 2.5** If $H$ is an $\mathcal{H}_\gamma$-class of $M$ and $q_H \gamma x = q_H = x\gamma q_H$ for all $x \in H$, then $H$ is a subsemigroup of $<M_\gamma, \circ>$.

**Lemma 2.6** If $H$ is an $\mathcal{H}_\gamma$-class of $M$ satisfying Green’s condition and $q = q_H$ is a $\gamma$-ideal element, then $q\gamma x = q = x\gamma q$ for all $x \in H$.

**Proof.** Assume that $H$ is an $\mathcal{H}_\gamma$-class of $M$ satisfying Green’s condition and $q = q_H$ is a $\gamma$-ideal element. Then $l_\gamma(q) = q$ and $r_\gamma(q) = q$, so $e\gamma q \leq q$ and $q\gamma e \leq q$. By Theorem 2.1 (c), we have $q \leq e\gamma q$ and $q \leq q\gamma e$. This implies that $e\gamma q = q = q\gamma e$. By Theorem 2.1 (a), $q\gamma q \in H$. Thus $(q, q\gamma q) \in L_\gamma$. It follows from Lemma 1.5 that $e\gamma q = e\gamma q\gamma q$. Therefore $q\gamma e\gamma q = (e\gamma q)\gamma q = e\gamma q = q$. Now let $x$ be an arbitrary element of $H$. By Lemma 1.5, we have $e\gamma x = e\gamma q$ and $x\gamma e = q\gamma e$. Hence $x\gamma q = x\gamma(e\gamma q) = (x\gamma e)\gamma q = (q\gamma e)\gamma q = q\gamma x = (q\gamma e)\gamma x = q\gamma(e\gamma x) = q\gamma(e\gamma q) = q$. Therefore $q\gamma x = q = x\gamma q$ for all $x \in H$.

Hence the proof of the lemma is completed. \qed

Immediately from Corollary 2.5 and Lemma 2.6, we have Corollary 2.7.

**Corollary 2.7** If $H$ is an $\mathcal{H}_\gamma$-class of $M$ satisfying Green’s condition and $q = q_H$ is a $\gamma$-ideal element, then $H$ is a subsemigroup of $<M_\gamma, \circ>$.

**Corollary 2.8** An $\mathcal{H}_\gamma$-class $H$ of the greatest element $e$ of $M$ is a subsemigroup of $<M_\gamma, \circ>$ if and only if $e$ is a $\gamma$-idempotent.

**Proof.** Assume that an $\mathcal{H}_\gamma$-class $H$ of the greatest element $e$ of $M$ is a subsemigroup of $M_\gamma$. Then $e\gamma e = e \circ e \in H$, so $H$ satisfies Green’s condition. Since $e \in H$, $e \leq q_H$. Thus $q_H = e$. Since $e \leq e\gamma e \vee e = l_\gamma(e) = r_\gamma(e) \leq e$, we have $l_\gamma(e) = e = r_\gamma(e)$. Hence $e$ is a $\gamma$-ideal element. By Lemma 2.6, $e\gamma x = e = x\gamma e$ for all $x \in H$. Hence $e = e\gamma e$, so $e$ is a $\gamma$-idempotent.

Conversely, assume that $e$ is a $\gamma$-idempotent in an $\mathcal{H}_\gamma$-class $H$. Then $e\gamma e = e \in H$, so $H$ satisfies Green’s condition. By the above proof, $q_H = e$ and $e$ is a $\gamma$-ideal element. It follows from Corollary 2.7 that $H$ is a subsemigroup of $M_\gamma$.

Hence the proof is completed. \qed
Theorem 2.9 Let $H$ be an $H\gamma$-class of $M$ such that its representative $\gamma$-quasi-ideal element $q = q_H$ is minimal in the set of all $\gamma$-quasi-ideal elements of $M$. Then $H = \{x \in M : x \leq q\}$ is a subsemigroup of $<M_\gamma, \circ>$.

Proof. If $x \in H$, then $x \leq q$. Now assume that $x$ is an element of $M$ such that $x \leq q$. Then $l_\gamma(x) \land r_\gamma(x) \leq l_\gamma(q) \land r_\gamma(q) = q$. By Lemmas 1.1 and 1.2, $l_\gamma(x) \land r_\gamma(x)$ is a $\gamma$-quasi-ideal element. Since $q$ is a minimal $\gamma$-quasi-ideal element, $l_\gamma(x) \land r_\gamma(x) = q$. Thus $q \leq l_\gamma(x)$ and $q \leq r_\gamma(x)$.

By Lemma 1.1, we have $l_\gamma(q) \leq l_\gamma(l_\gamma(x)) = l_\gamma(x)$ and $r_\gamma(q) \leq r_\gamma(r_\gamma(x)) = r_\gamma(x)$. Since $x \leq q$, we have $l_\gamma(x) \leq l_\gamma(q)$ and $r_\gamma(x) \leq r_\gamma(q)$. Hence $l_\gamma(x) = l_\gamma(q)$ and $r_\gamma(x) = r_\gamma(q)$, so $(x, q) \in L_\gamma \cap R_\gamma = H_\gamma$. Therefore $x \in H$, so we conclude that $H = \{x \in M : x \leq q\}$. Now let $x$ be an arbitrary element of $H$. Then $x \leq q$. Since $x \leq e$, we have $x\gamma x \leq e\gamma q \land q\gamma e \leq l_\gamma(q) \land r_\gamma(q) = q$. This implies that $x\gamma x \in H$. It follows from Theorem 2.4 that $H$ is a subsemigroup of $M_\gamma$.

Therefore the proof of the theorem is completed. \qed

ACKNOWLEDGEMENTS. The authors would like to thank the referee for the useful comments and suggestions given in an earlier version of this paper.

References


Received: September 10, 2007