On Green's Relations for Γ-semigroups and Reductive Γ-semigroups

R. Chinram and P. Siammai

Prince of Songkla University, Department of Mathematics, Faculty of Science, Hat Yai, Songklha 90112, Thailand
ronnason.c@psu.ac.th

Abstract

The notion of a Γ-semigroup has been introduced by M. K. Sen in the year 1981. In this paper, we consider Green's relations for Γ-semigroups and reductive Γ-semigroups.

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1 Introduction

The notion of a Γ-semigroup has been introduced by M. K. Sen in [6] the year 1981. Many classical notions of semigroup have been extended to Γ-semigroup (see [6], [7], [1], [2]). Green's relations for semigroups, first studied by J. A. Green [4], have played a fundamental role in the development to semigroup theory (see [5]). In [8], G. Thierrin has introduced a reductive semigroup. A. Fattahi and H. R. E. Vishki have given a characterization for regular reductive semigroups in [3]. In this paper, we consider Green's relations for Γ-semigroups and reductive Γ-semigroups. Moreover, we give a characterization for regular reductive Γ-semigroups.

2 Preliminaries

Let $S$ and $\Gamma$ be nonempty sets. If there exists a mapping $S \times \Gamma \times S \to S$, written $(a, \gamma, b)$ by $a\gamma b$, $S$ is called a Γ-semigroup if $S$ satisfies the identities

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$(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.

Let $S$ be an arbitrary semigroup and $\Gamma$ be any nonempty set. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that $S$ is a $\Gamma$-semigroup. Thus a semigroup can be considered to be a $\Gamma$-semigroup.

Let $S$ be a $\Gamma$-semigroup and $\alpha$ be a fixed element in $\Gamma$. We define $a \cdot b = a\alpha b$ for all $a, b \in S$. We can show that $(S, \cdot)$ is a semigroup and we denote this semigroup by $S_\alpha$.

An element $a$ of a $\Gamma$-semigroup $S$ is said to be regular if there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A regular $\Gamma$-semigroup is a $\Gamma$-semigroup each element of which is regular. Let $a$ be an element of a $\Gamma$-semigroup $S$ and $\alpha, \beta \in \Gamma$. An element $b$ of $S$ is called an $(\alpha, \beta)$-inverse of $a$ if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$.

Let $S$ be a $\Gamma$-semigroup and $\gamma \in \Gamma$. An element $e$ in $S$ is said to be a $\gamma$-idempotent if $e\gamma e = e$. The set of all $\gamma$-idempotents is denoted by $E_\gamma$. We denote $\bigcup_{\gamma \in \Gamma} E_\gamma$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of $S$. A $\Gamma$-semigroup $S$ is called an idempotent $\Gamma$-semigroup if $S = E(S)$.

3 Main results

The Green’s equivalence relations $L, R, H$ and $D$ on a $\Gamma$-semigroup $S$ are defined by the following rules:

(i) $aLb$ if and only if $S^1\Gamma a = S^1\Gamma b$ where $S^1\Gamma a = S\Gamma a \cup \{a\}$.
(ii) $aRb$ if and only if $a\Gamma S^1 = b\Gamma S^1$ where $a\Gamma S^1 = a\Gamma S \cup \{a\}$.
(iii) $H = L \cap R$.
(iv) $D = L \circ R$.

Remark We have

(i) $aLb$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$.
(ii) $aRb$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$.
(iii) $aHb$ if and only if $aLb$ and $aRb$.
(iv) $aDb$ if and only if there exists $c \in S$ such that $aLc$ and $cRb$.

Theorem 3.1 $L \circ R = R \circ L$.

Proof. Let $(a, b) \in L \circ R$. Then there exists $c \in S$ such that $aLc$ and $cRb$.

Case 1: $a = c$. Then $aRb$. Since $aLa$ and $aRb$, $(a, b) \in L \circ R$.

Case 2: $b = c$. Then $aLb$. Since $aLb$ and $bRb$, $(a, b) \in L \circ R$.
Case 3: \(a \neq c\) and \(b \neq c\). Since \(a \mathcal{L} c\) and \(c \mathcal{R} b\), there exist \(x, y, u, v \in S\) and \(\gamma, \mu, \eta, \theta \in \Gamma\) such that
\[
x \gamma a = c, \quad y \mu c = a, \quad c \eta u = b, \quad b \theta v = c.
\]
Let \(d = y \mu c \eta u\). Then
\[
a \eta u = y \mu c \eta u = d
\]
and
\[
d \theta v = y \mu c \eta u \theta v = y \mu b \theta v = y \mu c = a
\]
from which it follows \(a \mathcal{R} d\). Also,
\[
y \mu b = y \mu c \eta u = d
\]
and
\[
x \gamma d = x \gamma y \mu c \eta u = x \gamma a \eta u = c \eta u = b,
\]
so \(d \mathcal{L} b\). We deduce that \((a, b) \in \mathcal{R} \circ \mathcal{L}\). Therefore \(\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}\).

Similarly, we can prove that \(\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}\).

The \(\mathcal{L}\)-class (resp. \(\mathcal{R}\)-class, \(\mathcal{H}\)-class, \(\mathcal{D}\)-class) containing the element \(a\) will be written \(L_a\) (resp. \(R_a, H_a, D_a\)).

**Theorem 3.2** Let \(S\) be a \(\Gamma\)-semigroup, \(\alpha \in \Gamma\) and \(e\) be an \(\alpha\)-idempotent. Then
\begin{enumerate}[(i)]  
  
  \item \(a \alpha e = a\) for all \(a \in L_e\).
  
  \item \(e \alpha a = a\) for all \(a \in R_e\).
  
  \item \(a \alpha e = a = e \alpha a\) for all \(a \in H_e\).
  
  \item For all \(a \in S\), \(|H_a \cap E_a| \leq 1\).
\end{enumerate}

**Proof.** (i) Let \(a \in L_e\). Then \(a \mathcal{L} e\). It follows that \(S^1 \Gamma a = S^1 \Gamma e\). Then \(a = e\) or there exist \(x \in S\) and \(\gamma \in \Gamma\) such that \(a = x \gamma e\). If \(a = e\), then \(a \alpha e = e \alpha e = e = a\). If \(a = x \gamma e\), then \(a \alpha e = (x \gamma e) \alpha e = x \gamma (e \alpha e) = x \gamma e = a\).

(ii) It is similar to (i).

(iii) It follows from (i) and (ii).

(iv) Let \(e, f \in H_a \cap E_a\). Then \(e \mathcal{H} f\). So \(e \mathcal{L} f\) and \(e \mathcal{R} f\). Then \(f \in L_e\) and \(e \in R_f\). By (i) and (ii), respectively, we have \(f \alpha e = f\) and \(f \alpha e = e\). Therefore \(e = f\). It follows that \(|H_a \cap E_a| \leq 1\).

**Theorem 3.3** If \(a\) is a regular element of a \(\Gamma\)-semigroup \(S\), then every element of \(D_a\) is regular.
Proof. Since $a$ is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Let $b \in D_a$. So $aDb$. Then $aLc$ and $cRb$ for some $c \in S$. Since $aLc$, $a = c$ or there exist $u, v \in S$ and $\gamma, \mu \in \Gamma$ such that

$$u\gamma a = c$$

and $v\mu c = a$

Since $cRb$, $b = c$ or there exist $z, t \in S$ and $\eta, \theta \in \Gamma$ such that

$$c\eta z = b$$

and $b\theta t = c$

Case 1: $a = c$ and $c = b$. Then $a = b$, so $b$ is regular.

Case 2: $a = c$ and $c\eta z = b$ and $b\theta t = c$. Then

$$b\theta(t\alpha x)\beta b = c\alpha x\beta c\eta z = a\alpha x\beta a\eta z = a\eta z = c\eta z = b$$

Case 3: $u\gamma a = c$ and $v\mu c = a$, and $b = c$. Then

$$b\alpha(x\beta v)\mu b = c\alpha x\beta v\mu b = u\gamma a\alpha x\beta a = u\gamma a = c = b$$

Case 4: $u\gamma a = c$ and $v\mu c = a$, and $c\eta z = b$ and $b\theta t = c$. Then

$$b\theta(t\alpha x\beta v)\mu b = c\alpha x\beta v\mu c\eta z = u\gamma a\alpha x\beta a\eta z = u\gamma a\eta z = c\eta z = b.$$

Therefore $b$ is a regular element.

Let $D$ be a $D$-class. Then either every element of $D$ is regular or no element of $D$ is regular. We call the $D$-class regular if all its elements are regular.

Theorem 3.4 In a regular $D$-class, each $L$-class and each $R$-class contains at least one idempotent.

Proof. Let $a$ be an element of a regular $D$-class $D$ in a $\Gamma$-semigroup $S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Then $x\beta a = x\beta(a\alpha x\beta a) = (x\beta a)\alpha(x\beta a)$. Thus $x\beta a$ is an $\alpha$-idempotent. Since $a = a\alpha(x\beta a)$, $aLx\beta a$. Similarly, $a\alpha x$ is a $\beta$-idempotent and $aRa\alpha x$.

Theorem 3.5 Let $a$ be an element of a regular $D$-class $D$ in a $\Gamma$-semigroup $S$. Then

(i) If $a'$ is an $(\alpha, \beta)$-inverse of $a$, then $a' \in D$ and the two $H$-classes $R_a \cap L_{a'}$ and $L_a \cap R_{a'}$, contain a $\beta$-idempotent $a\alpha a'$ and an $\alpha$-idempotent $a'\beta a$, respectively.

(ii) If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain a $\beta$-idempotent $e$ and an $\alpha$-idempotent $f$, respectively, then $H_b$ contains an $(\alpha, \beta)$-inverse $a^*$ of $a$ such that $a\alpha a^* = e$ and $a^*\beta a = f$.

(iii) No $H$-class contains more than one $(\alpha, \beta)$-inverse of $a$ for all ordered pair $(\alpha, \beta) \in \Gamma \times \Gamma$. 

Proof. (i) Let $a'$ be an $(\alpha, \beta)$-inverse of $a$. Then $a = a\alpha a' \beta a$ and $a' = a' \beta a a a'$. Thus
\[ a \mathcal{L} a' \beta a, \quad a a a' \mathcal{R} a, \quad a' \mathcal{L} a a a', \quad a' \beta a \mathcal{R} a'. \]
Thus $a' D a, \ a a a' \in R_a \cap L_{a'}$ and $a' \beta a \in L_a \cap R_a$. Therefore $a' \in D$. Since $a = a a a' \beta a, a' \beta a = a' \beta a a \alpha a'$ and $a a a' = a a a' a a a'$. Therefore $a' \beta a$ is an $a$-idempotent and $a a a'$ is a $b$-idempotent.

(ii) Since $a \mathcal{R} e$, by Theorem 3.2(ii), $e \beta a = a$. Similarly, from $a \mathcal{L} f$ we deduce that $a \alpha f = a$ by Theorem 3.2(i). Again from $a \mathcal{R} e$ it follows that $a = e$ or there exist $x \in S$ and $\gamma \in \Gamma$ such that $a \gamma x = e$.

Case 1: $a = e$. Let $a^* = f \beta e$. Then
\[ a a a^* \beta a = a \alpha (f \beta e) \beta a = (a \alpha f) \beta (e \beta a) = a \beta a = e \beta a = a \]
and
\[ a^* \beta a a a^* = (f \beta e) \beta a a (f \beta e) = f \beta (e \beta a) \alpha f \beta e = f \beta (a \alpha f) \beta e = f \beta (a \beta e) = f \beta e = a^*. \]
Then $a^*$ is an $(\alpha, \beta)$-inverse of $a$. Moreover
\[ a \alpha a^* = a \alpha f \beta e = a \beta e = e \beta e = e. \]
Further, since $a \mathcal{L} f, \ a = f$ or $f = y \theta a$ for some $y \in S$ and $\theta \in \Gamma$. If $a = f$, then $a^* \beta a = f \beta e \beta a = e \beta e \beta e = e = f$. If $f = y \theta a$, then $a^* \beta a = f \beta e \beta a = y \theta a \beta e \beta e = y \theta a = f$. It now follows easily that $a^* \in L_e \cap R_f = L_b \cap R_b = H_b$.

Case 2: $a \gamma x = e$. Let $a^* = f \gamma x \beta e$. Then
\[ a a a^* \beta a = a \alpha (f \gamma x \beta e) \beta a = (a \alpha f) \gamma x \beta (e \beta a) = a \gamma x \beta a = e \beta a = a \]
and
\[ a^* \beta a a a^* = (f \gamma x \beta e) \beta a a (f \gamma x \beta e) = f \gamma x \beta (e \beta a) \alpha f \gamma x \beta e = f \gamma x \beta (a \alpha f) \gamma x \beta e = f \gamma x \beta (a \gamma x) \beta e = f \gamma x \beta e = a^*. \]
Then $a^*$ is an $(\alpha, \beta)$-inverse of $a$. Moreover
\[ a \alpha a^* = a \alpha f \gamma x \beta e = a \gamma x \beta e = e \beta e = e. \]
Since $a \mathcal{L} f, \ a = f$ or there exist $y \in S$ and $\theta \in \Gamma$ such that $f = y \theta a$. If $a = f$, then $a^* \beta a = f \gamma x \beta e \beta a = a \gamma x \beta e \beta a = e \beta e \beta a = e \beta a = a = f$. If $f = y \theta a$, then $a^* \beta a = f \gamma x \beta e \beta a = y \theta (a \gamma x) \beta e \beta a = y \theta (e \beta e) \beta a = y \theta (e \beta a) = y \theta a = f$. It now follows easily that $a^* \in L_e \cap R_f = L_b \cap R_b = H_b$.

(iii) Suppose that $a'$ and $a^*$ are both $(\alpha, \beta)$-inverses of $a$ inside the single $\mathcal{H}$-class $H_b$. Since $a a a'$ and $a \alpha a^*$ are $\beta$-idempotents in the $\mathcal{H}$-class $R_a \cap L_b$, $a \alpha a^* = a a a'$ by Theorem 3.2(iv). Similarly, $a' \beta a = a^* \beta a$ because both are $\alpha$-idempotents in the $\mathcal{H}$-class $L_a \cap R_b$. Then $a' = a' \beta a a a' = a^* \beta a a a^* = a^*$. $\blacksquare$
Let $S$ be a $\Gamma$-semigroup. An equivalence relation $\rho$ on $S$ is called a right [resp. left] congruence on $S$ if for each $a, b \in S$, $(a, b) \in \rho$ implies $(a \gamma t, b \gamma t) \in \rho$ [resp. $(t \gamma a, t \gamma b) \in \rho$] for all $t \in S$ and $\gamma \in \Gamma$. An equivalence relation $\rho$ on $S$ is called a congruence if $\rho$ is both a right and left congruence. It is easy to prove that $\mathcal{L}$ is a right congruence on $S$ and $\mathcal{R}$ is a left congruence on $S$.

Let $S$ be a $\Gamma$-semigroup and $\rho$ be a congruence on $S$. For $a \rho, b \rho \in S/\rho$ and $\gamma \in \Gamma$, let $(a \rho) \gamma (b \rho) = (a \gamma b) \rho$. This is well-defined, since for all $a, a', b, b' \in S$ and $\gamma \in \Gamma$,

$$a \rho = a' \rho \text{ and } b \rho = b' \rho \Rightarrow (a, a'), (b, b') \in \rho$$

$$\Rightarrow (a \gamma b, a' \gamma b), (a' \gamma b, a' \gamma b') \in \rho$$

$$\Rightarrow (a \gamma b, a' \gamma b') \in \rho$$

$$\Rightarrow (a \gamma b) \rho = (a' \gamma b') \rho.$$

Let $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. We have

$$(a \rho \gamma b) \mu c \rho = ((a \gamma b) \rho) \mu c \rho = ((a \gamma b) \mu c) \rho = (a \gamma (b \mu c)) \rho = a \rho \gamma (b \mu c) \rho = a \rho \gamma (b \mu c) \rho.$$

Then the quotient set $S/\rho$ is a $\Gamma$-semigroup.

**Theorem 3.6** Let $S$ be a $\Gamma$-semigroup and $\rho$ be a congruence on $S$. Then

(i) If $\rho \subseteq \mathcal{L}$ then for all $a, b \in S$, $a \mathcal{L} b$ if and only if $a \rho \mathcal{L} b \rho$ in $S/\rho$.

(ii) If $\rho \subseteq \mathcal{R}$ then for all $a, b \in S$, $a \mathcal{R} b$ if and only if $a \rho \mathcal{R} b \rho$ in $S/\rho$.

(iii) If $\rho \subseteq \mathcal{H}$ then for all $a, b \in S$, $a \mathcal{H} b$ if and only if $a \rho \mathcal{H} b \rho$ in $S/\rho$.

**Proof.** (i) Let $a, b \in S$ such that $a \mathcal{L} b$. Then $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = xab$ and $b = y\beta a$.

Case 1: $a = b$. Then $a \rho = b \rho$.

Case 2: $a = xab$ and $b = y\beta a$. Then $a \rho = (xab) \rho = (xp) \alpha (bp)$ and $b \rho = (y\beta a) \rho = (yp) \beta (ap)$. Therefore $a \rho \mathcal{L} b \rho$.

Conversely, let $a, b \in S$. Assume $a \rho \mathcal{L} b \rho$. Then $a \rho = b \rho$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (xp) \alpha (bp)$ and $b = (yp) \beta (ap)$.

Case 1: $a \rho = b \rho$. Then $(a, b) \in \rho$. Since $\rho \subseteq \mathcal{L}$, $(a, b) \in \mathcal{L}$. So $a \mathcal{L} b$.

Case 2: $a \rho = (xp) \alpha (bp)$ and $b \rho = (yp) \beta (ap)$. Then $a \mathcal{L} b \rho$ and $b \mathcal{L} (y\beta a) \rho$. Hence $a \mathcal{L} b$.

(ii) It is similar to (i).

(iii) It follows from (i) and (ii). \[\blacksquare\]

A congruence $\rho$ on $S$ is called right [resp. left] reductive if for each $a, b \in S$, $(a \gamma t, b \gamma t) \in \rho$ [resp. $(t \gamma a, t \gamma b) \in \rho$] for all $t \in S$ and $\gamma \in \Gamma$ implies $(a, b) \in \rho$.

A $\Gamma$-semigroup $S$ is called right [resp. left] reductive if equality on $S$ is a right
[resp. left] reductive congruence. In other words, $S$ is called right [resp. left] reductive if for each $a, b \in S$, $a \gamma t = b \gamma t$ [resp. $t \gamma a = t \gamma b$] for all $t \in S$ and $\gamma \in \Gamma$ implies $a = b$. A $\Gamma$-semigroup is called reductive if it is both right and left reductive.

**Theorem 3.7** Let $S$ be a $\Gamma$-semigroup and $\rho$ be a congruence on $S$. The following statements are true.

(i) $\rho$ is a right reductive congruence if and only if $S/\rho$ is a right reductive $\Gamma$-semigroup.

(ii) $\rho$ is a left reductive congruence if and only if $S/\rho$ is a left reductive $\Gamma$-semigroup.

(iii) $\rho$ is a reductive congruence if and only if $S/\rho$ is a reductive $\Gamma$-semigroup.

**Proof.** (i) Let $\rho$ be a right reductive congruence. Let $a, b \in S$ such that $(a \rho \gamma t = b \rho \gamma t)$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a \gamma t, b \gamma t) \in \rho$ for all $t \in S$ and $\gamma \in \Gamma$. Since $\rho$ is right reductive, $(a, b) \in \rho$. Hence $a \gamma t = b \gamma t$.

Conversely, suppose $S/\rho$ is a right reductive $\Gamma$-semigroup. Let $a, b \in S$ such that $(a \gamma t, b \gamma t) \in \rho$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a \gamma t \rho = b \gamma t \rho)$ for all $t \in S$ and $\gamma \in \Gamma$. Thus $(a \rho \gamma t = b \rho \gamma t)$ for all $t \in S$ and $\gamma \in \Gamma$. Since $S/\rho$ is a right reductive $\Gamma$-semigroup, $a \rho = b \rho$. Therefore $(a, b) \in \rho$.

(ii) It is similar to (i).

(iii) It follows by (i) and (ii).

Define two congruence $\rho_r$ and $\rho_l$ on a $\Gamma$-semigroup $S$ as follows:

$$\rho_r = \{(a, b) \in S \times S \mid a \gamma t = b \gamma t \text{ for all } t \in S \text{ and for all } \gamma \in \Gamma\}$$
$$\rho_l = \{(a, b) \in S \times S \mid t \gamma a = t \gamma b \text{ for all } t \in S \text{ and for all } \gamma \in \Gamma\}.$$  

The three following theorems hold.

**Theorem 3.8** Let $S$ be a $\Gamma$-semigroup. Then

(i) $S$ is a right reductive $\Gamma$-semigroup if and only if $\rho_r = 1_S$.

(ii) $S$ is a left reductive $\Gamma$-semigroup if and only if $\rho_l = 1_S$.

**Proof.** (i) Assume $S$ is a right reductive $\Gamma$-semigroup. Let $a, b \in S$ such that $a \rho \gamma t = b \gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Since $S$ is right reductive, $a = b$.

Conversely, suppose $\rho_r = 1_S$. Let $a, b \in S$ such that $a \gamma t = b \gamma t$ for all $t \in S$ and $\gamma \in \Gamma$. Then $(a, b) \in \rho_r$. Since $\rho_r = 1_S$, $a = b$. Hence $S$ is a right reductive $\Gamma$-semigroup.

(ii) It is similar to (i).

**Theorem 3.9** Let $S$ be a regular $\Gamma$-semigroup. Then

(i) $\rho_r \subseteq R$.

(ii) $\rho_l \subseteq L$. 
Proof. (i) Let \((a, b) \in \rho_r\). Then \(a \gamma t = b \gamma t\) for all \(t \in S\) and for all \(\gamma \in \Gamma\). So \(a \Gamma S = b \Gamma S\). Since \(a \in a \Gamma S\) and \(b \in b \Gamma S\) because \(S\) is regular, \(a \Gamma S^1 = b \Gamma S^1\). Therefore \((a, b) \in \mathcal{R}\). Thus \(\rho_r \subseteq \mathcal{R}\).

(ii) It is similar to (i).

\[\blacksquare\]

**Theorem 3.10** Let \(S\) be a regular \(\Gamma\)-semigroup. Then

(i) \(\rho_r\) is the minimum right reductive congruence on \(S\).

(ii) \(\rho_l\) is the minimum left reductive congruence on \(S\).

Proof. (i) Let \(a, b \in S\). Assume that \((a \gamma t, b \gamma t) \in \rho_r\) for all \(t \in S\) and \(\gamma \in \Gamma\). Then \(a \gamma t \beta t' = b \gamma t \beta t'\) for all \(t, t' \in S\) and \(\gamma, \beta \in \Gamma\). Thus \(a \alpha t'' = b \alpha t''\) for all \(t'' \in S\) and \(\alpha \in \Gamma\) because \(S\) is regular. So \((a, b) \in \rho_r\). Therefore \(\rho_r\) is a right reductive congruence on \(S\).

Next, let \(\rho\) be any right reductive congruence on \(S\). Let \((a, b) \in \rho_r\). Then \(a \gamma t = b \gamma t\) for all \(t \in S\) and \(\gamma \in \Gamma\). Since \(\rho\) is reflexive, \((a \gamma t, b \gamma t) \in \rho\). Therefore \((a, b) \in \rho\) because \(\rho\) is right reductive.

(ii) It is similar to (i).

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