Strong Zero-Divisor Graphs of Non-Commutative Rings\textsuperscript{1}

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Abstract
An element $a$ in a ring $R$ is called a strong zero-divisor if, either $\langle a \rangle \langle b \rangle = 0$ or $\langle b \rangle \langle a \rangle = 0$, for some $0 \neq b \in R$ ($\langle x \rangle$ is the ideal generated by $x \in R$). Let $S(R)$ denote the set of all strong zero-divisors of $R$. This notion of strong zero-divisor has been extensively studied by these authors in [8]. In this paper, for any ring $R$, we associate an undirected graph $\tilde{\Gamma}(R)$ with vertices $S(R)\ast S(R)\setminus \{0\}$, where distinct vertices $a$ and $b$ are adjacent if and only if either $\langle a \rangle \langle b \rangle = 0$ or $\langle b \rangle \langle a \rangle = 0$. We investigate the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of $\tilde{\Gamma}(R)$. It is shown that for every ring $R$, every two vertices in $\tilde{\Gamma}(R)$ are connected by a path of length at most 3, and if $\tilde{\Gamma}(R)$ contains a cycle, then the length of the shortest cycle in $\tilde{\Gamma}(R)$, is at most 4. Also we characterize all rings $R$ whose $\tilde{\Gamma}(R)$ is a complete graph or a star graph. Also, the interplay of between the ring-theoretic properties of a ring $R$ and the graph-theoretic properties of $\tilde{\Gamma}(M_n(R))$, are fully investigated.

Mathematics Subject Classification: 16U99; 05C50; 16P10; 16S50

Keywords: Non-commutative ring; Strong zero-divisor; Zero-divisor; Graph

\textsuperscript{1}The research of the first author was in part supported by a grant from IPM (No. 85130016).
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1. Introduction

Throughout the paper, $R$ denotes a ring (not necessarily commutative or with identity element). If $X$ is either an element or a subset of $R$, then the ideal generated by $X$ is denoted by $\langle X \rangle$. Also, the left (resp. right) ideal generated by $X$ is denoted by $\langle X \rangle$ (resp. $[X]$). The left annihilator of $X$ is the left ideal $\text{Ann}_L(X) = \{a \in R : aX = 0\}$ and the right annihilator of $X$, denoted by $\text{Ann}_R(X)$, is similarly defined. We denote the finite field of order $q$ by $F_q$. Also, for any subset $Y$ of $R$, we define $Y^* = Y \setminus \{0\}$.

For a non-commutative ring $R$, let $Z(R)$ be the set of all zero-divisors of $R$. The undirected zero-divisor graph of $R$ (denoted by $\tilde{\Gamma}(R)$) is a simple graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$, where distinct vertices $a$ and $b$ are adjacent if and only if either $ab = 0$ or $ba = 0$ (see for example [1,14] for more detail). This is a generalization of the zero-divisor graph of commutative rings. The concept of a zero-divisor graph of a commutative ring $R$ (denoted by $\Gamma(R)$) was introduced by I. Beck [7], and it was mainly concerned with colorings of rings. In Anderson and Livingston [5], the vertex set of $\Gamma(R)$ was chosen to be $Z(R)^*$, and the authors studied the interplay between the ring-theoretic properties of a commutative ring $R$ and the graph-theoretic properties of $\Gamma(R)$. The zero-divisor graph of a commutative ring has also been studied by several other authors (see for example, [2,3,6,12,13]). The zero-divisor graph has also been introduced and studied for semigroups by DeMeyer and Schneider in [10], for near-rings by Cannon et al. in [9].

In this paper, we use a new generalization of the notion of zero-divisor element of a commutative ring to non-commutative rings. An element $a$ in a ring $R$ is called a left (resp. right) strong zero-divisor if, there exists a nonzero $b \in R$ such that $\langle a \rangle \langle b \rangle = 0$ (resp. $\langle b \rangle \langle a \rangle = 0$). A strong zero-divisor in $R$ is an element of $R$ which is either a left or a right strong zero-divisor. Recently, this notion of strong zero-divisor has been extensively studied by these authors in [8]. For a ring $R$ we denote $S_L(R)$, $S_R(R)$ and $S(R)$, for the set of all left strong zero-divisors, right strong zero-divisors and strong zero-divisors of $R$, respectively. It is clear that $S(R) \subseteq Z(R)$ and for a commutative ring $R$, the set of zero-divisors and the set of strong zero-divisors of $R$ coincide. Clearly, $S(R) = \{0\}$ if and only if $R$ is a prime ring.

For a ring $R$, we associate an undirected graph $\tilde{\Gamma}(R)$ with vertices $S(R)^*$, where distinct vertices $a$ and $b$ are adjacent if and only if either $\langle a \rangle \langle b \rangle = 0$ or $\langle b \rangle \langle a \rangle = 0$. $\tilde{\Gamma}(R)$ is called strong zero-divisor graph of $R$, and we note that for a commutative ring $R$, the zero-divisor graph $\Gamma(R)$ considered by Anderson and Livingston [5], coincides with the $\tilde{\Gamma}(R)$. But, for a non-commutative ring
$R$, $\tilde{\Gamma}(R)$ is a subgraph (not necessarily an induced subgraph) of $\Gamma(R)$ (see Examples 2.4 and 2.5). Also, $\Gamma(R)$ is the empty graph if and only if $R$ is a domain, but $\tilde{\Gamma}(R)$ is the empty graph if and only if $R$ is a prime ring. We note that a generalization of commutative domains is the notion of non-commutative prime ring, which inherits many properties of commutative domains. Thus $\Gamma(R)$ determines, in a sense how far the ring is from being a domain, but $\tilde{\Gamma}(R)$ determines how far the ring is from being prime, and hence, the significance of our study of the strong zero-divisor graphs of a non-commutative ring becomes apparent. In Section 2, we investigate the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of $\tilde{\Gamma}(R)$. A graph $\Gamma \neq \emptyset$ is a complete graph if there is an edge between every pair of the vertices, and it is a star graph if $\Gamma$ contains one vertex to which all other vertices are joined and $\Gamma$ has no other edges. It is shown that for any ring $R$, every two vertices in $\tilde{\Gamma}(R)$ are connected by a path of length at most 3, and if $\tilde{\Gamma}(R)$ contains a cycle, then the length of the shortest cycle in $\tilde{\Gamma}(R)$, is at most 4. Finally, we characterize all rings $R$ whose $\tilde{\Gamma}(R)$ is a complete graph or a star graph. In Section 3, the interplay of between the ring-theoretic properties of a ring $R$ and the graph-theoretic properties of $\tilde{\Gamma}(M_n(R))$, are fully investigated.

2. Some properties of the strong zero-divisor graphs

Let $R$ be a ring. We note that for an element $a \in R$, $\langle a \rangle = Ra + Za$, $[a] = aR + Za$ and $\langle a \rangle = \{\sum_{i=1}^{n} r_i s_i + ra + as + ma \mid n \in \mathbb{N}, m \in \mathbb{Z}, r, s, r_i, s_i \in R\}$. Clearly if $R$ is a null ring, then $\langle a \rangle = Za$, and if $R$ is a ring with identity element, then $\langle a \rangle = Ra$, $[a] = aR$ and $\langle a \rangle = \{\sum_{i=1}^{n} r_i s_i \mid n \in \mathbb{N}, r, s, r_i, s_i \in R\}$. Also, for every $a, b \in R$ the following statements are equivalent:

1. $\langle a \rangle \langle b \rangle = 0$;
2. $a\langle b \rangle = 0$;
3. $\langle a \rangle b = 0$;
4. $a\langle b \rangle = 0$;
5. $[a]b = 0$;
6. $aRb = 0$ and $ab = 0$.

**Definition 2.1** Let $R$ be a ring. We define an undirected graph $\tilde{\Gamma}(R)$ with vertices $S^*(R)$, where every two distinct vertices $x$ and $y$ are adjacent (i.e, $x$ --- $y$ is an edge in $\tilde{\Gamma}(R)$) if and only if either $\langle x \rangle \langle y \rangle = 0$ or $\langle y \rangle \langle x \rangle = 0$. 
Lemma 2.2. [Theorem 1.1] Let \( R \) be a ring. If \( S_\ell(R) \) is finite, then \( R \) is a finite ring or a prime ring. Moreover, if \( 1 < |S_\ell(R)| < \infty \), then \( R \) is a finite ring.

Note that in lemma above we can replace \( S_\ell(R) \) with \( S_r(R) \) or \( S(R) \).

Corollary 2.3. Let \( R \) be a ring. If \( R \) is not a prime ring, then \( \tilde{\Gamma}(R) \) is a finite graph if and only if \( R \) is a finite ring.

For any ring \( R \), \( \tilde{\Gamma}(R) \) is called the strong zero-divisor graph of \( R \), and we note that for a commutative ring \( R \), the zero-divisor graph \( \Gamma(R) \) considered by Anderson and Livingston [5], coincides with the \( \tilde{\Gamma}(R) \). Also, for a non-commutative ring \( R \), \( \tilde{\Gamma}(R) \) is a subgraph of \( \bar{\Gamma}(R) \), but the following examples show that in general, \( \tilde{\Gamma}(R) \) is not an induced subgraph of \( \bar{\Gamma}(R) \) (even if \( S(R) = Z(R) \)).

Example 2.4. Let \( R = M_2(\mathbb{Z}_6) \), and let
\[
a = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \in R
\]
Clearly \( aRc = b Rc = 0 \) and so \( a, b, c \in S(R)^* \). On the other hand, \( ab = 0 \) i.e., \( a \rightarrow b \) is an edge in \( \Gamma(R) \), but \( aRb \neq 0 \) and \( bRa \neq 0 \), and hence, the vertices \( a \) and \( b \) are not adjacent in \( \tilde{\Gamma}(R) \).

Example 2.5. Let \( N \) be a null ring (i.e., \( N^2 = \{0\} \)), and let \( R = N \times T \), where \( T \) is any ring. Clearly \( S(R) = Z(R) = R \). Now if \( S(T) \neq Z(T) \), then \( \Gamma(R) \neq \tilde{\Gamma}(R) \). In particular, if \( T = M_n(D) \), where \( D \) is a division ring and \( n \geq 2 \), then \( S(N \times M_n(D)) = Z(N \times M_n(D)) \), but \( \tilde{\Gamma}(N \times M_n(D)) \) is not an induced subgraph of \( \bar{\Gamma}(N \times M_n(D)) \).

Recall that a graph \( \Gamma \) is connected if there is a path between any two distinct vertices. For distinct vertices \( x \) and \( y \) of \( \Gamma \), let \( d(x, y) \) be the length of the shortest path from \( x \) to \( y \) and if there is no such path we define \( d(x, y) = \infty \). The diameter of \( \Gamma \) is \( \text{diam}(\Gamma) = \sup \{d(x, y) : x \text{ and } y \text{ are distinct vertices of } \Gamma\} \). The girth of \( \Gamma \), denoted by \( g(\Gamma) \), is defined as the length of the shortest cycle in \( \Gamma \) (\( g(\Gamma) = \infty \) if \( \Gamma \) contains no cycles).

By Anderson and Livingston [5, Theorem 2.3], for every commutative ring \( R \), \( \Gamma(R) \) is a connected graph and \( \text{diam}(\Gamma(R)) \leq 3 \). Moreover, if \( \Gamma(R) \) contains a cycle, then \( g(\Gamma(R)) \leq 4 \) (see [13]). These facts later were developed by Redmond [14], for the undirected zero-divisor graph \( \bar{\Gamma}(R) \) of a non-commutative
ring $R$. Although by examples above, $\bar{\Gamma}(R) \subseteq \bar{\Gamma}(R)$ is not necessarily an induced subgraph, but we will show that $\bar{\Gamma}(R)$ is also a connected graph and $\text{diam}\bar{\Gamma}(R) \leq 3$ and, if $\bar{\Gamma}(R)$ contains a cycle, then $g(\bar{\Gamma}(R)) \leq 4$ (see Theorem 2.7).

First we need the following useful lemma.

Lemma 2.6. Let $R$ be a ring and $x, y \in S^*(R)$. If $x \longrightarrow y$ is an edge in $\bar{\Gamma}(R)$, then for each $0 \neq r \in R$, either $ry = 0$ (resp. $yr = 0$) or $x \longrightarrow ry$ (resp. $x \longrightarrow yr$) is also an edge in $\bar{\Gamma}(R)$.

Proof. Without loss of generality we can assume that $\langle x \rangle \langle y \rangle = 0$. $ry \neq 0$, where $r \in R$. It is clear that $\langle ry \rangle \subseteq \langle y \rangle$ and so $\langle x \rangle \langle ry \rangle \subseteq \langle x \rangle \langle y \rangle = 0$. Thus $x \longrightarrow ry$ is also an edge in $\bar{\Gamma}(R)$. □

Theorem 2.7. For any ring $R$, $\bar{\Gamma}(R)$ is a connected graph and $\text{diam}\bar{\Gamma}(R) \leq 3$. Moreover, if $\bar{\Gamma}(R)$ contains a cycle, then $g(\bar{\Gamma}(R)) \leq 4$.

Proof. Let $x, y \in S^*(R)$ be distinct. If either $\langle x \rangle \langle y \rangle = 0$ or $\langle y \rangle \langle x \rangle = 0$, then $d(x, y) = 1$. So suppose that $\langle x \rangle \langle y \rangle$ and $\langle y \rangle \langle x \rangle$ are nonzero. Since $\langle x \rangle \langle y \rangle + \langle y \rangle \langle x \rangle \subseteq \langle x \rangle \langle y \rangle$, $\langle x \rangle \langle y \rangle \neq 0$.

Case 1: $\langle x \rangle^2 = \langle y \rangle^2 = 0$. Then for each $0 \neq z \in \langle x \rangle \langle y \rangle$, $\langle z \rangle \subseteq \langle x \rangle \langle y \rangle$ and so $x \longrightarrow z \longrightarrow y$ is a path of length 2; thus $d(x, y) = 2$.

Case 2: $\langle x \rangle^2 = 0$ and $\langle y \rangle^2 \neq 0$. Then there is a $b \in \bar{Z}(R) \setminus \{x, y\}$ such that either $\langle b \rangle \langle y \rangle = 0$ or $\langle y \rangle \langle b \rangle = 0$. If either $\langle b \rangle \langle x \rangle = 0$ or $\langle x \rangle \langle b \rangle = 0$, then $x \longrightarrow b \longrightarrow y$ is a path of length 2. Let $\langle x \rangle \langle b \rangle + \langle b \rangle \langle x \rangle \neq 0$. Since $\langle x \rangle \langle b \rangle + \langle b \rangle \langle x \rangle \subseteq \langle b \rangle \langle x \rangle$, for each $0 \neq c \in \langle b \rangle \langle x \rangle$, $\langle c \rangle \subseteq \langle b \rangle \langle x \rangle$ and so $x \longrightarrow c \longrightarrow y$ is a path of length 2. A similar argument holds if $\langle y \rangle^2 = 0$ and $\langle x \rangle^2 \neq 0$.

Case 3: $\langle x \rangle \langle y \rangle$, $\langle y \rangle \langle x \rangle$, $\langle x \rangle^2$ and $\langle y \rangle^2$ are all nonzero. Thus there exist elements $a$ and $b$ in $S^*(R) \setminus \{x, y\}$ such that either $\langle a \rangle \langle x \rangle = 0$ or $\langle x \rangle \langle a \rangle = 0$ and either $\langle b \rangle \langle y \rangle = 0$ or $\langle y \rangle \langle b \rangle = 0$. If $\langle a \rangle = \langle b \rangle$, then $x \longrightarrow a \longrightarrow y$ is a path of length 2. Thus we may assume that $\langle a \rangle \neq \langle b \rangle$. If either $\langle a \rangle \langle b \rangle = 0$ or $\langle b \rangle \langle a \rangle = 0$, then $x \longrightarrow a \longrightarrow b \longrightarrow y$ is a path of length 3, and hence $d(x, y) \leq 3$. If $\langle a \rangle \langle b \rangle + \langle b \rangle \langle a \rangle \neq 0$, then $\langle a \rangle \langle b \rangle \neq 0$ and for every $0 \neq d \in \langle a \rangle \langle b \rangle$, $x \longrightarrow d \longrightarrow y$ is a path of length 2; thus $d(x, y) = 2$.

Thus $d(x, y) \leq 3$, for all distinct vertexes $x, y \in S(R)^*$. Hence $\bar{\Gamma}(R)$ is connected and also $\text{diam}(\bar{\Gamma}(R)) \leq 3$.

For the 'moreover' statement let $(x_1, \ldots, x_n)$ be a cycle with length $n$. Note
that $n \geq 3$. Define $x_0 := x_n$ and $x_{n+1} := x_1 = x$. If there is an $i \in \{1, 2, n\}$ such that $\langle x_i \rangle \cap \{x_{i-1}, x_{i+1}\} \neq \emptyset$, then let

$$l(i) = \begin{cases} (x, x_2, x_n) & \text{if } i = 1, \\ (x, x_2, x_3) & \text{if } i = 2, \\ (x, x_{n-1}, x_n) & \text{if } i = n. \end{cases}$$

By Lemma 2.6, it is easy to see that $l(i)$ is a cycle, for $i = 1, 2, n$. Hence $g(\Gamma(R)) \leq 3$. Henceforth assume $\langle x_i \rangle \cap \{x_{i-1}, x_{i+1}\} = \emptyset$ for all $i \in \{1, 2, n\}$. Suppose $\langle x_i \rangle \subseteq \{x_{i-1}, x_i, x_{i+1}, 0\}$ for all $i \in \{1, 2, n\}$. Then we must have $\langle x_i \rangle = \{x_i, 0\}$ for all $i \in \{1, 2, n\}$. Consequently $\langle x_2 \rangle \cap \langle x_n \rangle = 0$. In this case $\langle x_2 \rangle \subseteq \langle x_2 \rangle \cap \langle x_n \rangle = 0$ and so $(x, x_2, x_n)$ is a cycle with length 3 (hence $g(\Gamma(R)) \leq 3$). Finally suppose there is an $i \in \{1, 2, n\}$ such that $\langle x_i \rangle \nsubseteq \{x_{i-1}, x_i, x_{i+1}, 0\}$. Pick $y \in \langle x_i \rangle \setminus \{x_{i-1}, x_i, x_{i+1}, 0\}$. Define

$$l(i, y) = \begin{cases} (x, x_2, y, x_n) & \text{if } i = 1, \\ (x, x_2, x_3, y) & \text{if } i = 2, \\ (x, y, x_{n-1}, x_n) & \text{if } i = n. \end{cases}$$

By Lemma 2.6, it is straightforward to verify that $l(i, y)$ is a cycle, for all $i = 1, 2, n$ and hence $g(\Gamma(R)) \leq 4$. □

**Remark 2.8.** Let $R$ be ring (not necessarily with identity) with a non-zero idempotent $e$. Then $R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)R(1 - e)$, where $eRe = \{ere \mid r \in R\}$, $(1 - e)Re = \{re - ere \mid r \in R\}$, $eR(1 - e) = \{er - ere \mid r \in R\}$ and $(1 - e)R(1 - e) = \{r - re - er + ere \mid r \in R\}$.

Next, we determine when $\Gamma(R)$ has a vertex adjacent to every other vertex; i.e., when $\Gamma(R)$ has a spanning tree which is a star graph. Special cases of this are when either $\Gamma(R)$ is a complete graph or a star graph. In fact, the following theorem is the key concept in characterizing these graphs (see also [5, Theorem 2.5]).

**Theorem 2.9.** Let $R$ be a ring. Then $\Gamma(R)$ has a vertex adjacent to every other vertex if and only if either $R \cong \mathbb{Z}_2 \times R_1$, where $R_1$ is a prime ring, or $S(R) = \text{Ann}_e(\langle a \rangle) \cup \text{Ann}_r(\langle a \rangle)$ for some $0 \neq a \in R$.

**Proof.** For one direction, the proof is straightforward. For the other direction assume that there is a vertex $e \in S(R)^*$ which is adjacent to every other vertex of $\Gamma(R)$. Clearly every nonzero element of $\langle e \rangle$ is also adjacent to every other
Then the vertices $\langle e \rangle = \{0,e\}$ and $e^2 = 0$. Then $\langle e \rangle^2 = 0$, i.e., $S(R) = \text{Ann}_e(\langle e \rangle) \cup \text{Ann}_r(\langle e \rangle)$.

**Case 2:** $\langle e \rangle = \{0,e\}$ and $e^2 \neq 0$. This means that $e$ is an idempotent element and by Remark 2.8, we can write $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. We claim that $eR(1-e) \cup (1-e)Re = \{0\}$, for if not, then either $er(1-e) = e$ or $(1-e)re = e$, for some $r \in R$ (since $eR(1-e) \cup (1-e)Re = \{0\} \subseteq \langle e \rangle$). It follows that either $e = e^2 = er(1-e)e = 0$ or $e = e^2 = e(1-e)re = 0$, a contradiction. Thus $R = eRe \oplus (1-e)R(1-e)$. Clearly $eRe$ and $(1-e)R(1-e)$ are two ideals of $R$ and so $R \cong \mathbb{Z}_2 \oplus A$, where $A = (1-e)R(1-e)$. Now we show that $A$ is a prime ring. Let $aAb = 0$ and $ab = 0$, for some nonzero $a, b \in A$. Then $(e,a)R(0,b) = 0$ and $(e,a)(0,b) = 0$. It follows that $(e,a) \in S(R)^*$. Thus either $(e,0)R(e,a) = 0$ and $(e,0)(e,a) = 0$ or $(e,a)R(e,0) = 0$ and $(e,a)(e,0) = 0$. Therefore $e^2 = 0$, a contradiction. Thus $R \cong \mathbb{Z}_2 \oplus A$, where $A$ is a prime ring.

**Case 3:** $\langle e \rangle \neq \{0,e\}$ i.e., there exists a nonzero element $e' \in \langle e \rangle$ different from $e$. Since $\langle e' \rangle \subseteq \langle e \rangle$, $e'$ also adjacent to every other vertex $\Gamma(R)$. Moreover, either $\langle e' \rangle \langle e \rangle = 0$ or $\langle e \rangle \langle e' \rangle = 0$ and so $\langle e' \rangle^2 = 0$. It follows that $S(R) = \text{Ann}_e(\langle e' \rangle) \cup \text{Ann}_r(\langle e' \rangle)$. □

We can repeat a portion of the proof of Theorem 2.9 for the following important corollary.

**Corollary 2.10.** Let $R$ be a semiprime ring. Then $\Gamma(R)$ has a vertex adjacent to every other vertex if and only if $R \cong \mathbb{Z}_2 \times R_1$, where $R_1$ is a prime ring.

Let $R$ be a ring. By [1, Theorem 5], $\Gamma(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R)$ is an ideal of $R$ and $Z(R)^2 = \{0\}$. Now here we give the same result for strong zero-divisor graphs of rings.

**Theorem 2.11.** Let $R$ be a ring. Then $\Gamma(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $S(R)^2 = \{0\}$. Moreover in the latter case $S(R)$ is an ideal of $R$.

**Proof.** For one direction, the proof is straightforward. For the other direction assume that $\Gamma(R)$ is a complete graph. First suppose that there exists a strong zero-divisor $e \in R$ such that $e^2 \neq 0$. We show that $e^2 = e$. If not, then the vertices $e$ and $e^2$ are adjacent, and hence, either $\langle e^3 \rangle \subseteq \langle e \rangle \langle e^2 \rangle = 0$ or $\langle e^3 \rangle \subseteq \langle e^2 \rangle \langle e \rangle = 0$. Thus either $\langle e^2 \rangle \langle e - e^2 \rangle = 0$ or $\langle e - e^2 \rangle \langle e \rangle = 0$ and
so $e - e^2$ is a non-zero strong zero-divisor different from $e$. Since $\tilde{\Gamma}(R)$ is a complete graph, we conclude that either $\langle e \rangle \langle e - e^2 \rangle = 0$ or $\langle e \rangle \langle e - e^2 \rangle \langle e \rangle = 0$. Since $\langle e^3 \rangle = 0$, $\langle e^2 \rangle \subseteq \langle e - e^2 \rangle \langle e \rangle = 0$ (or $\langle e^2 \rangle \subseteq \langle e \rangle \langle e - e^2 \rangle = 0$). This follows that $e^2 = 0$, a contradiction. So $e$ is an idempotent element of $R$. Thus by the proof of Theorem 2.9, $R \cong \mathbb{Z}_2 \times R_1$, where $R_1$ is a prime ring.

Clearly $(0, a) \in S(R)$, for each $0 \neq a \in R_1$. Let $0 \neq a \in R_1$. Since $\tilde{\Gamma}(R)$ is a complete graph, for each $b \in R_1$ different from $a$, either $\langle (0, a) \rangle \langle (0, b) \rangle = 0$ or $\langle (0, b) \rangle \langle (0, a) \rangle = 0$ i.e., either $aR_1b = 0$ and $ab = 0$ or $bR_1a = 0$ and $ba = 0$. Since $R_1$ is a prime ring, $b = 0$ and so $R_1 = \{0, a\}$. If $a^2 = 0$, then $R_1^2 = 0$ i.e., $R_1 = 0$, a contradiction. Thus $a^2 = a$ and this implies that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Next suppose that $z^2 = 0$, for all $z \in S(R)$. We claim that $\langle z \rangle^2 = 0$, for all $z \in S(R)$. Let $0 \neq z \in S(R)$ and $T := \langle z \rangle$. Then $T$ is a subring of $R$ and $T \subseteq S(R)$. Since $\tilde{\Gamma}(R)$ is a complete graph, $\tilde{\Gamma}(T)$ is also a complete graph with $S(T) = T$. Thus $Z(T) = T$ and $\tilde{\Gamma}(T)$ (undirected zero-divisor graph of $T$) is also a complete graph. By [1, Theorem 5], either $T \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(T)$ is an ideal of $R$ and $Z(T)^2 = \{0\}$. Since $t^2 = 0$, for all $t \in T$, $T \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and hence, $T^2 = Z(T)^2 = 0$. Thus $\langle z \rangle^2 = 0$, for any $z \in S(R)$. Clearly, for every $z_1, z_2 \in S(R)$, $\langle z_1 + z_2 \rangle^2 = 0$ and so $z_1 + z_2 \in S(R)$. Since $S(R)$ is closed under the operation of multiplication of $R$ (see also, [8, Proposition 1.2]) $S(R))$ is an ideal of $R$ and the proof is complete. □

**Corollary 2.12.** Let $R$ be a ring. If $\tilde{\Gamma}(R)$ is a complete graph, then $\tilde{\Gamma}(R)$ is also a complete graph (i.e., $\tilde{\Gamma}(R)$ is an induced subgraph $\tilde{\Gamma}(R))$.

**Proof.** Let $\tilde{\Gamma}(R)$ be a complete graph. By [1, Theorem 5], either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $Z(R)^2 = \{0\}$. Since $S(R) \subseteq Z(R)$, $Z(R)^2 = \{0\}$ implies that $S(R)^2 = \{0\}$. Now apply Theorem 2.11. □

The following corollary shows that if $\tilde{\Gamma}(R)$ is a complete graph, then $\tilde{\Gamma}(R)$ is not necessarily complete.

**Example 2.13.** Let $R = M_2(\mathbb{Z}_4)$. One can easily check that $S(R) = M_2(U)$, where $U = \{0, 2\}$. Since $S(R)^2 = 0$, by Theorem 2.11, $\tilde{\Gamma}(R)$ is a complete graph. On the other hand, $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Z(R)$ is not an ideal of $R$. Thus by [1, Theorem 5], $\tilde{\Gamma}(R)$ is not complete.

Let $R \cong R_1 \times R_2$, where $R_1$ and $R_2$ are rings. Clearly $R_1 \times \{0\} \cup \{0\} \times R_2 \subseteq S(R)$. This yields that $S(R)$ is an ideal of $R$ if and only if $S(R) = R$. Thus
we have the following corollary.

**Corollary 2.14.** Let \( R \cong R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are rings. Then \( \tilde{\Gamma}(R) \) is a complete graph if and only if either \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( R \) is a null ring.

Also, in view Theorem 2.11, Corollary 2.10 and corollary above we have the following corollary.

**Corollary 2.15.** Let \( R \) be a semiprime ring. Then \( \tilde{\Gamma}(R) \) is a complete graph if and only if \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

The following theorem has been established in [6], for every commutative ring with identity. Also in [1, Theorem 6], the authors have generalized this result to undirected zero-divisor graphs \( \Gamma(R), \Gamma(R[x]) \) and \( \Gamma(R[[x]]) \), for any arbitrary ring \( R \). Now using Theorem 2.11, and the same method as [1], we extend this fact to strong zero-divisor graphs of rings.

**Theorem 2.16.** Let \( R \) be a ring which is not isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). If one of the graphs \( \tilde{\Gamma}(R), \tilde{\Gamma}(R[x]), \) and \( \tilde{\Gamma}(R[[x]]) \) is a complete graph, then the two other graphs are also complete graphs.

**Proof.** Since \( \tilde{\Gamma}(R) \) is an induced subgraph of \( \tilde{\Gamma}(R[x]) \) and \( \tilde{\Gamma}(R[x]) \) is an induced graph of \( \tilde{\Gamma}(R[[x]]) \), it is enough to prove that if \( \tilde{\Gamma}(R) \) is a complete graph, then \( \tilde{\Gamma}(R[[x]]) \) is also a complete graph. Suppose that \( \tilde{\Gamma}(R) \) is complete graph. Since \( R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \), Theorem 2.11 implies that \( S(R) \) is an ideal of \( R \) with \( S(R)^2 = \{0\} \). To complete the proof, one needs only verify that \( S(R[[x]]) \subseteq S(R)[[x]] \). Assume that \( f(x) = \sum_{n=0}^{\infty} f_n x^n \) is an element of \( S(R)[[x]] \setminus S(R)[[x]] \), thus there exists \( 0 \neq g(x) = \sum_{n=0}^{\infty} g_n x^n \) such that either \( \langle f(x)\rangle \langle g(x)\rangle = 0 \) or \( \langle g(x)\rangle \langle f(x)\rangle = 0 \). Without loss of generality we can assume that \( \langle f(x)\rangle \langle g(x)\rangle = 0 \). We note that \( S(R) \) is a prime ideal of \( R \), for if \( IJ \subseteq S(R) \) where \( I \) and \( J \) are ideal of \( R \), then \( IJJ = 0 \), so \( I(JIJ) = 0 \), now, if \( JIJ \neq 0 \), then \( I \subseteq S(R) \), so we assume that \( JIJ = 0 \). Now if \( IJ \neq 0 \), then \( J \subseteq S(R) \). Otherwise \( IJ = 0 \), and this implies \( I \subseteq S(R) \) and \( J \subseteq S(R) \). Therefore \( S(R) \) is a prime ideal of \( R \) i.e., \( \frac{R}{S(R)} \) is a prime ring, and it follows that \( \frac{R}{S(R)}[[x]] \cong \frac{R[[x]]}{S(R)[[x]]} \) is also prime i.e., \( S(R)[[x]] \) is a prime ideal \( R[[x]] \). Since \( \langle f(x)\rangle \langle g(x)\rangle = 0 \subseteq S(R)[[x]] \) and \( f(x) \notin S(R)[[X]], \langle g(x)\rangle \subseteq S(R)[[x]] \). Assume that \( r \) and \( s \) are the smallest indices such that \( f_r \notin S(R) \) and \( g_s \neq 0 \). Since \( S(R)^2 = 0 \) and \( f_i g(x) = 0 \), for all \( i < r \),
Let $\Gamma$ be a star graph and $a$ be a vertex in $\Gamma$. We say that $a$ is a **radix vertex** of $\Gamma$ if all other vertices are joined to $a$.

**Theorem 2.17.** Let $R$ be a ring. Then $\tilde{\Gamma}(R)$ is a star graph with radix vertex $a$ with $a^2 \neq 0$ if and only if $R \cong \mathbb{Z}_2 \times A$, where $A$ is a prime ring.

**Proof.** For one direction, the proof is clear. Since $a$ is adjacent to all other vertices, every nonzero element of $\langle a \rangle$ is also adjacent to all other vertices different from $a$. Thus $\langle a \rangle = \{0, a\}$. Since $a^2 \neq 0$, $\langle a \rangle^2 \neq 0$ i.e., $a^2 = a$, and by Remark 2.8, we can write $R = aRa \oplus aR(1 - a) \oplus (1 - a)Ra \oplus (1 - a)R(1 - a)$. Since $aR(1 - a) \subseteq \langle a \rangle \subseteq \{0, a\}$ and $a^2 = a$, $aR(1 - a) = \{0\}$. Similarly, $(1 - a)Ra = \{0\}$, and hence, $R = aRa \oplus (1 - a)R(1 - a)$. If $(1 - a)R(1 - a) = \{0\}$, then $R = aRa = \{0, a\}$ and so $S(R) = \{0\}$, a contradiction. Thus $(1 - a)R(1 - a) \neq \{0\}$. Since $a \notin (1 - a)R(1 - a)^*$ and all vertices of the set $(1 - a)R(1 - a)^*$ are adjacent to all vertices of $aRa^*$ and noting that $\tilde{\Gamma}(R)$ is a star graph, we conclude that $aRa = \{0, a\}$. Therefore $R = \langle a \rangle \oplus (1 - a)R(1 - a)$ and so $R_1 := (1 - a)R(1 - a)$ is an ideal of $R$. Furthermore, because of $\tilde{\Gamma}(R)$ is a star graph, we claim that $R_1$ is a prime ring, for if not, then by Theorem 2.7, $\tilde{\Gamma}(R_1)$ is a connected graph with no edge, i.e., the ring $R_1$ has exactly one non-zero strong zero-divisor, say $c$. Clearly, $cR_1c = 0$ and $c^2 = 0$ and so $\langle c \rangle^2 = 0$. It follows that $(a + c) = c$ is an edge of $\tilde{\Gamma}(R)$, a contradiction. Hence $R_1$ is a prime ring. Thus $R \cong \mathbb{Z}_2 \times R_1$, where $R_1$ is the prime ring. \[\Box\]

**Theorem 2.18.** Let $R$ be a ring. If $\tilde{\Gamma}(R)$ is a star graph with radix vertex $a$ with $a^2 = 0$, then $I = \{0, a\}$ is a null ideal of $R$ and either $R = \text{Ann}_e(\langle a \rangle)$ (or $R = \text{Ann}_r(\langle a \rangle)$) or $S(R) = P_1 \cup P_2$, where $P_1 = \text{Ann}_e(\langle a \rangle)$ and $P_2 = \text{Ann}_r(\langle a \rangle)$ and those are prime ideals of $R$.

**Proof.** Since $a$ is the radix vertex of $\tilde{\Gamma}(R)$, every nonzero element of $\langle a \rangle$ is also adjacent to all other vertices different from $a$. Thus $\langle a \rangle = \{0, a\}$. Since $a^2 = 0$, $\langle a \rangle$ is a null ring (i.e., $\langle a \rangle^2 = 0$) and so for each $b \in S(R)$, either $\langle a \rangle b = 0$ or $b\langle a \rangle = 0$. Thus $S(R) \subseteq \text{Ann}_e(\langle a \rangle) \cup \text{Ann}_r(\langle a \rangle)$. On the other hand $\text{Ann}_e(\langle a \rangle)$ and $\text{Ann}_r(\langle a \rangle)$ contain in $S(R)$. Since $\text{Ann}_e(\langle a \rangle) = \text{Ann}_e(a)$ and $\text{Ann}_r(\langle a \rangle) = \text{Ann}_r(a)$, we have

\[S(R) = \text{Ann}_e(\langle a \rangle) \cup \text{Ann}_r(\langle a \rangle) = \text{Ann}_e(a) \cup \text{Ann}_r(a) \quad (*) \]
Case 1: $S(R) = R$. Then equation (**) above implies that either $Ann_r(\langle a \rangle) \subseteq Ann_l(\langle a \rangle)$ or $Ann_l(\langle a \rangle) \subseteq Ann_r(\langle a \rangle)$. Thus either $R = Ann_l(\langle a \rangle)$ or $R = Ann_r(\langle a \rangle)$.

Case 2: $S(R) \neq R$. Then equation (**) above implies that $P_1 = Ann_r(\langle a \rangle)$ and $P_2 = Ann_l(\langle a \rangle)$ are proper ideals of $R$. We claim that $P_1$ and $P_2$ are prime ideals of $R$. Also, we have $S(R) = (\bigcup_{I \in B} I) \bigcup (\bigcup_{L \in C} L)$ where

$$B = \{I : I \text{ is an ideal of } R \text{ such that } IK = 0 \text{ for some nonzero ideal } K \text{ of } R\},$$

$$C = \{L : L \text{ is an ideal of } R \text{ such that } KL = 0 \text{ for some nonzero ideal } K \text{ of } R\}.$$

Clearly, $P_1 \in C$ and $P_2 \in B$. We show that $P_1$ is a maximal element of $C$ and $P_2$ is a maximal element of $B$. Let $P \in C$ and $P_1 \not\subseteq P$. Then $KP = 0$ for some nonzero ideal $K$ of $R$. Since $\Gamma(R)$ is a star graph with the radix vertex $a$, either $K = \{0, a\}$ or $P = \{0, a\}$. But, $\{0, a\} \subseteq P_1 \not\subseteq P$ and hence $P \neq \{0, a\}$. Thus $K = \{0, a\}$ and so $P \subseteq Ann_r(\langle a \rangle) = P_1$, a contradiction. Thus $P_1$ is a maximal ideal of $C$. Similarly, $P_2$ is a maximal element of $B$. We show that $P_1$ is a prime ideal. Suppose $IJ \subseteq P_1$, where $I$ and $J$ are ideals of $R$; i.e., $\langle a \rangle IJ = 0$. If $\langle a \rangle I = 0$, then $I \subseteq P_1$. If $\langle a \rangle I \neq 0$, then by the maximality of $P_1$, $Ann_r(\langle a \rangle) = Ann_r(\langle a \rangle I)$ and so $J \subseteq P_1$. Thus $P_1$ is prime. By similar argument $P_2$ is prime. This completes the proof. □

Now by Theorem 2.17, and Theorem 2.18 we have the following corollary.

**Corollary 2.19.** Let $R$ be a semiprime ring. Then $\Gamma(R)$ is a star graph if and only if $R \cong \mathbb{Z}_2 \times R_1$, where $R_1$ is a prime ring.

**Theorem 2.20.** Let $R$ be a left Artinian ring. If $\Gamma(R)$ is a star graph, then one of the following holds:

(i) $R \cong \mathbb{Z}_2 \times M_n(D)$, for some $n \geq 1$ and a division ring $D$.

(ii) $S(R) = R$ and there exists $0 \neq a \in R$ such that either $R = Ann_l(\langle a \rangle)$ or $R = Ann_r(\langle a \rangle)$.

(iii) $S(R) = P_1 \cup P_2$, where $P_1 = Ann_l(\langle a \rangle)$ and $P_2 = Ann_r(\langle a \rangle)$ for some $0 \neq a \in R$ and also $P_1$ and $P_2$ are maximal ideals of $R$.

**Proof.** Assume that $\Gamma(R)$ is a star graph and $a$ is the radix vertex of $\Gamma(R)$. If $a^2 \neq 0$, then by Theorem 2.17, $R \cong \mathbb{Z}_2 \oplus A$, where $A$ is a prime ring. Since $R$ is left Artinian, $A$ is a left Artinian prime ring, i.e., $A \cong M_n(D)$, for some $n \in \mathbb{N}$ and a division ring $D$. Thus $R \cong \mathbb{Z}_2 \oplus M_n(D)$. Now let $a^2 = 0$. Then by Theorem 2.18, either $R = Ann_l(\langle a \rangle)$ (or $R = Ann_r(\langle a \rangle)$) or $S(R) = P_1 \cup P_2$,
where \( P_1 = \text{Ann}_t(\langle a \rangle) \) and \( P_2 = \text{Ann}_r(\langle a \rangle) \) and those are prime ideals of \( R \). Since \( R \) is a Artinian ring, by, every prime ideal of \( R \) is a maximal ideal (see [8, the proof of Theorem 1.12]) and this completes the proof. □

The next two corollaries are immediate.

**Corollary 2.21.** Let \( R \) be a left Artinian ring with identity. If \( \tilde{\Gamma}(R) \) is a star graph, then either \( R \cong \mathbb{Z}_2 \times M_n(D) \), for some \( n \geq 1 \) and a division ring \( D \) or \( S(R) = P_1 \cup P_2 \), where \( P_1 \) and \( P_2 \) are maximal ideals of \( R \) such that \( P_1 = \text{Ann}_t(\langle a \rangle) \) and \( P_2 = \text{Ann}_r(\langle a \rangle) \) for some \( 0 \neq a \in R \).

**Corollary 2.22.** Let \( R \) be a left Artinian ring. If \( R \) is not a reduced ring, then \( \tilde{\Gamma}(R) \) is a star graph if and only if \( R \cong \mathbb{Z}_2 \times M_n(D) \), for some \( n \geq 1 \) and a division ring \( D \).

The following interesting result shows that if \( \tilde{\Gamma}(R) \) is a star graph, then \( \Gamma(R) \) and \( \tilde{\Gamma}(R) \) coincide. Thus if \( \Gamma(R) \) is a star graph, then \( \tilde{\Gamma}(R) \) is also a star graph but, the converse is not true (for example, one can easily see that \( \tilde{\Gamma}(\mathbb{Z}_2 \times M_2(\mathbb{Z}_2)) \) is a star graph but \( \tilde{\Gamma}(\mathbb{Z}_2 \times M_2(\mathbb{Z}_2)) \) is not a star graph).

**Proposition 2.23.** Let \( R \) be a ring such that \( \tilde{\Gamma}(R) \) is a star graph. Then \( \tilde{\Gamma}(R) = \tilde{\Gamma}(R) \).

**Proof.** Assume that \( a \) is the radix vertex of \( \tilde{\Gamma}(R) \).

Case 1: \( a^2 = 0 \). Then \( Ra \cup aR \subseteq Z(R) \).

Subcase 1: \( Z(R) = \{0, a\} \). Then \( \langle a \rangle = \{0, a\} \) and so \( S(R) = \{0, a\} \) i.e., \( \tilde{\Gamma}(R) = \Gamma(R) \).

Subcase 2: \( Z(R) \neq \{0, a\} \). Then there exists \( 0 \neq b \in Z(R) \setminus \{0, a\} \), such that either \( ab = 0 \) or \( ba = 0 \). If \( ab = 0 \), then \( Rab = 0 \) and since \( \Gamma(R) \) is a star graph, \( Ra \subseteq \{0, a\} \). Thus \( aRa \subseteq a\{0, a\} = \{0\} \), i.e., \( \langle a \rangle = \{0, a\} \) and \( \langle a \rangle^2 = 0 \). Clearly, \( ac = 0 \) (\( ca = 0 \)) if and only if \( \langle a \rangle c = 0 \) (\( c\langle a \rangle = 0 \)). This shows that \( S(R) = Z(R) \) and \( \tilde{\Gamma}(R) \) is a star graph with radix vertex \( a \). Therefore, \( \tilde{\Gamma}(R) = \tilde{\Gamma}(R) \).

Case 2: \( a^2 = a \). By Remark 2.8, we can write \( R = aRa \oplus aR(1-a) \oplus (1-a)Ra \oplus (1-a)R(1-a) \). If \( (1-a)R(1-a) = \{0\} \), then for any elements \( x \) and \( y \) of the set \( aR(1-a) \cup (1-a)Ra \) we have either \( xy = 0 \) or \( yx = 0 \). Since \( a \neq 1 \), this set cannot be empty and hence it has just one element. Without loss of generality, assume that \( aR(1-a) = \{0, b\} \) and \( (1-a)Ra = \{0\} \). We have \( b(a-b) = 0 \).
So \( a - b \) is a vertex of \( \tilde{\Gamma}(R) \) which is not adjacent to \( a \), a contradiction. Thus \( (1-a)R(1-a) \neq \{0\} \). Since all vertices of the set \( (1-a)R(1-a)^* \) are adjacent to all vertices of \( aRa^* \cup aR(1-a)^* \cup (1-a)Ra^* \) and noting that \( \tilde{\Gamma}(R) \) is a star graph, we conclude that \( aRa = \{0,a\} \) and \( aR(1-a) = (1-a)Ra = \{0\} \). Therefore \( aRa \) and \( (1-a)R(1-a) \) are two ideals of \( R \). Furthermore, because of \( \tilde{\Gamma}(R) \) is a star graph, \( (1-a)R(1-a)^* \) is an independent set. Suppose that \( (1-a)R(1-a) \) is not a domain. Since by [14, Theorem 3.2], \( \tilde{\Gamma}((1-a)R(1-a)) \) is a connected graph with no edge, the ring \( (1-a)R(1-a) \) has exactly one non-zero zero-divisor, say \( c \). Clearly \( c^2 = 0 \) and so \( (a+c)-c \) is an edge of \( \tilde{\Gamma}(R) \), a contradiction. Hence \( (1-a)R(1-a) \) is a domain. Therefore, \( R \cong \mathbb{Z}_2 \oplus R_1 \), where \( R_1 = (1-a)R(1-a) \) is a domain. Now by Theorem 2.17, \( \tilde{\Gamma}(R) \) is a star graph. Clearly, \( S(R) = Z(R) = \{(0,x) : x \in R_1\} \cup \{(1,0)\} \) and \( (1,0) \) is the radix vertex in \( \tilde{\Gamma}(R) \) and also in \( \tilde{\Gamma}(R) \). Thus \( \tilde{\Gamma}(R) = \tilde{\Gamma}(R) \) and the proof is complete. \( \square \)

We conclude this section with the following corollaries.

**Corollary 2.24.** Let \( R \) be a ring. If \( \tilde{\Gamma}(R) \) is a star graph, then one of the following holds:

(i) \( R \cong \mathbb{Z}_2 \times R_1 \), where \( R_1 \) is a domain.
(ii) \( S(R) = R \) and there exists \( 0 \neq a \in R \) such that either \( R = \text{Ann}_r(\langle a \rangle) \) or \( R = \text{Ann}_l(\langle a \rangle) \).
(iii) \( S(R) = P_1 \cup P_2 \), where \( P_1 = \text{Ann}_l(\langle a \rangle) \) and \( P_2 = \text{Ann}_r(\langle a \rangle) \) for some \( 0 \neq a \in R \) and also \( P_1 \) and \( P_2 \) are prime ideals of \( R \).

**Proof.** By Theorem 2.17, 2.18 and Proposition 2.23, is clear. \( \square \)

**Corollary 2.25.** Let \( R \) be a semiprime ring. Then \( \tilde{\Gamma}(R) \) is a star graph if and only if \( R \cong \mathbb{Z}_2 \times R_1 \), where \( R_1 \) is a domain.

**Proof.** By Corollary 2.19 and Proposition 2.23, is clear. \( \square \)

### 3. On strong zero-divisors of matrix rings

Let \( R \) be a ring with identity. We denote \( M_n(R) \) the ring of all \( n \times n \) matrices over \( R \). If \( A \in M_n(R) \), we denote by \( [A]_{i,j} \) the \((i,j)\)-entry of \( A \). For
a nonempty subset \( I \subseteq R \), we define 
\[ M_n(I) = \{ A \in M_n(R) \mid [A]_{ij} \in I \text{ for all } i, j = 1, \ldots, n \}. \]

It is clear that if \( I \) is an ideal of \( R \), then \( M_n(I) \) is an ideal of \( M_n(R) \). In fact, every ideal of \( M_n(R) \) is of this form (see for example [11]). Also, for each \( A \in M_n(R) \), we denote the ideal in \( R \) generated by all of entries of \( A \) by \( U_A \). Then one can easily see that \( \langle A \rangle = M_n(U_A) \). Thus, if \( A, B \in M_n(R) \), then \( \langle A \rangle \langle B \rangle = 0 \) if and only if \( U_A U_B = 0 \). Also, for each \( 0 \neq b \in R \) and \( A \in M_n(R) \), \( \langle b \rangle U_A = 0 \) if and only if \( \langle b \rangle M_n(U_A) = 0 \), if and only if \( M_n(\langle b \rangle) M_n(U_A) = 0 \). Thus we have the following evident result.

**Proposition 3.1.** Let \( R \) be a ring with identity and \( A \in M_n(R) \). Let \( U_A \) be the ideal in \( R \) generated by all of entries of \( A \). Then the following statements are equivalent:

1. \( A \) is a strongly zero divisor in \( M_n(R) \);
2. there exits \( 0 \neq B \in M_n(R) \) such that either \( U_B U_A = 0 \) or \( U_A U_B = 0 \);
3. there exits \( 0 \neq b \in R \) such that either \( \langle b \rangle A = 0 \) or \( A \langle b \rangle = 0 \);
4. there exits \( 0 \neq b \in R \) such that either \( \langle b \rangle M_n(U_A) = 0 \) or \( M_n(U_A) \langle b \rangle = 0 \);
5. there exits nonzero ideal \( U \) of \( R \) such that either \( A M_n(U) = 0 \) or \( M_n(U) A = 0 \);
6. \( M_n(U_A) \subseteq S(M_n(R)) \).

**Lemma 3.2.** Let \( R \) be a ring with identity. Then \( S(M_n(R)) \subseteq M_n(S(R)) \).

**Proof.** Suppose \( A \in S(M_n(R)) \). Then by Proposition 3.1, there exits \( 0 \neq B \in M_n(R) \) such that either \( U_A U_B = 0 \) or \( U_B U_A = 0 \). It follows that \( U_A \subseteq S(R) \) and so \( A \in M_n(U_A) \subseteq M_n(S(R)) \). \( \square \)

**Proposition 3.3.** Let \( R \) be a left Artinian ring with identity. Then the following statements are equivalent:

1. \( S(R) \) is an ideal of \( R \);
2. \( S(M_n(R)) \) is an ideal of \( M_n(R) \), for all integer \( n \geq 1 \);
3. \( S(M_n(R)) \) is an ideal of \( M_n(R) \), for some integer \( n \geq 1 \);
4. \( S(M_n(R)) = M_n(S(R)) \), for all integer \( n \geq 1 \);
5. \( S(M_n(R)) = M_n(S(R)) \), for some integer \( n \geq 1 \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( n \) be a positive integer. By Lemma 3.2, \( S(M_n(R)) \subseteq M_n(S(R)) \). Since \( R \) is a left Artinian ring, \( M_n(R) \) is also left Artinian and
so by Theorem [8, Theorem 1.12], $S(M_n(R))$ contains a maximal ideal $P$ of $M_n(R)$. Thus $P \subseteq S(M_n(R)) \subseteq M_n(S(R))$ i.e., $P = S(M_n(R)) = M_n(S(R))$. Thus $S(M_n(R))$ is an ideal of $M_n(R)$

$$(2) \Rightarrow (3)$$ is trivial.

$$(3) \Rightarrow (4)$$ It is sufficient to show that $S(R)$ is an ideal of $R$, i.e., $s_1 + s_2 \in S(R)$, for all $s_1, s_2 \in S(R)$. Let $s_1, s_2 \in S(R)$. Then $s_1E_{11} + s_2E_{12} \in M_n(S(R))$. Thus by the given hypothesis $s_1E_{11} + s_2E_{12} \in S(M_n(R))$. By Proposition 3.1, $(s_1E_{11} + s_2E_{12})M_n(U) = 0$ for some nonzero ideal $U$ of $R$. It follows that

$$(s_1 + s_2)E_{11}M_n(U) = (s_1E_{11} + s_2E_{12})(E_{11} + E_{21})M_n(U) = 0.$$

Thus $(s_1 + s_2)E_{11}(uE_{11}) = 0$, for all $u \in U$ i.e., $(s_1 + s_2)U = 0$. Since $U$ is a nonzero ideal of $R$, $s_1 + s_2 \in S(R)$.

$$(4) \Rightarrow (5)$$ is trivial.

$$(5) \Rightarrow (1)$$ Let $S(M_n(R)) = M_n(S(R))$, for some integer $n \geq 1$. Since $R$ is a left Artinian ring, $M_n(R)$ is also left Artinian and so by [8, Theorem 1.12], $S(M_n(R))$ contains a maximal ideal $P$ of $M_n(R)$. Thus $P \subseteq S(M_n(R)) \subseteq M_n(S(R))$ i.e., $P = S(M_n(R)) = M_n(S(R))$. Thus $S(R)$ is an ideal of $R$. □

**Lemma 3.4.** Let $R$ be a ring with identity. If $s \in S(R)$, then $M_n(\langle s \rangle) \subseteq S(M_n(R))$ for all integer $n \geq 1$.

**Proof.** Since $s \in S(R)$, there exits $0 \neq t \in R$ such that either $\langle s \rangle \langle t \rangle = 0$ or $\langle t \rangle \langle s \rangle = 0$. Then either $M_n(\langle s \rangle)M_n(\langle t \rangle) = 0$ or $M_n(\langle t \rangle)M_n(\langle s \rangle) = 0$. Thus $M_n(\langle s \rangle) \subseteq S(M_n(R))$. □

**Theorem 3.5.** Let $R$ be a ring with identity and $n \geq 2$. Then there is a vertex of $\Gamma(M_n(R))$ which is adjacent to every other vertex if and only if $S(M_n(R)) = Ann_e(\langle A \rangle) \cup Ann_r(\langle A \rangle)$ for some $0 \neq A \in M_n(R)$.

**Proof.** ($\Rightarrow$). Let $n \geq 2$ be an integer and there is a vertex of $\Gamma(M_n(R))$ which is adjacent to every other vertex. By Theorem 2.9, either $M_n(R) \simeq \mathbb{Z}_2 \times R_1$, where $R_1$ is a prime ring or $S(M_n(R)) = Ann_e(\langle A \rangle) \cup Ann_r(\langle A \rangle)$ for some $0 \neq A \in M_n(R)$. Clearly, for each prime ring $R_1$, $(1, 0) \in \mathbb{Z}_2 \times R_1$ is the only vertex of $\Gamma(\mathbb{Z}_2 \times R_1)$ which is adjacent to every other vertex, and hence, by Lemma 3.4, $M_n(\langle A \rangle) \not\subseteq \mathbb{Z}_2 \times R_1$, for each prime ring $R_1$. Thus $S(M_n(R)) = Ann_e(\langle A \rangle) \cup Ann_r(\langle A \rangle)$ for some $0 \neq A \in M_n(R)$.

($\Leftarrow$), by Theorem 2.9 is clear. □
Lemma 3.6. Let $R$ be a commutative ring with identity and $n \in \mathbb{N}$. If there is a vertex of $\Gamma(M_n(R))$ which is adjacent to every other vertex and $R \not\cong \mathbb{Z}_2 \times A$, for all prime rings $A$, then there are at least $2^{n^2} - 1$ vertexes in $\Gamma(M_n(R))$ with this property.

Proof. Since $R$ is a commutative ring, by Theorem 2.9, $S(R) = Z(R) = \text{Ann}_r(a)$ for some $0 \neq a \in R$. Thus $aS(R) = 0$ i.e., $\langle a \rangle S(R) = 0$. It follows that $M_n(\langle a \rangle S(R)) = M_n(\langle a \rangle)M_n(S(R)) = 0$. By Lemma 3.2, $S(M_n(R)) \subseteq M_n(S(R))$. Thus $M_n(\langle a \rangle)S(M_n(R)) = 0$ i.e., every nonzero element $A$ of $M_n(\langle a \rangle)$ is a vertex in $\Gamma(M_n(R))$ which is adjacent to every other vertexes. Clearly, $|M_n(\langle a \rangle)| \geq 2^{n^2} - 1$. □

Remark 3.7. Let $R_1$ be a prime ring with identity and $R = \mathbb{Z}_2 \times R_1$. Then the vertex $(1, 0)$ in $\Gamma(R)$ is adjacent to every other vertex. But there is not any vertex in $\Gamma(M_n(R))$ ($n \geq 2$) which is adjacent to every other vertex, for if not, then by Theorem 2.9, either $M_n(R) \cong \mathbb{Z}_2 \times R_1$, where $R_1$ is a prime ring or $S(M_n(R)) = \text{Ann}_t(\langle A \rangle) \cup \text{Ann}_r(\langle A \rangle)$ for some $0 \neq A \in M_n(R)$. Since for each ring $R_1$, the ring $R = \mathbb{Z}_2 \times R_1$ has an ideal with two elements $(I = \langle (1, 0) \rangle)$ but, $S(M_n(R))$ has no any ideal of order 2, $M_n(R) \not\cong \mathbb{Z}_2 \times R_1$, for every prime ring $R_1$. Thus $S(M_n(R)) = \text{Ann}_t(\langle A \rangle) \cup \text{Ann}_r(\langle A \rangle)$ for some $0 \neq A \in M_n(R)$. Since $R$ is not a prime ring, $M_n(R)$ is not prime, i.e., $S(M_n(R)) \neq \{0\}$. It follows that $A$ is a strong zero divisor in $M_n(R)$. Thus $A \in S(M_n(R))$ i.e., $\langle A \rangle^2 = 0$. Since $R$ is a semiprime ring, $M_n(R)$ is also semiprime, and so $A = 0$, a contradiction. Furthermore, in fact by Theorem 2.17, $\Gamma(\mathbb{Z}_2 \times A)$ is star graph, but $\Gamma(M_n(\mathbb{Z}_2 \times A))$ is not a star graph for all $n \geq 2$.

Theorem 3.8. Let $R$ be a commutative ring with identity. Then following statements are equivalent:

(1) For any integer $n \geq 1$ there is a vertex of $\Gamma(M_n(R))$ which is adjacent to every other vertex;

(2) For some integer $n \geq 2$ there is a vertex of $\Gamma(M_n(R))$ which is adjacent to every other vertex;

(3) $S(M_n(R)) = \text{Ann}_t(\langle A \rangle)$ for each integer $n \geq 1$ and for some $0 \neq A \in M_n(R)$;

(4) $S(M_n(R)) = \text{Ann}_t(\langle A \rangle)$ for some integer $n \geq 2$ and for some $0 \neq A \in M_n(R)$;

(5) $R \not\cong \mathbb{Z}_2 \times A$, for all domains $A$ and there is a vertex of $\Gamma(R)$ which is adjacent to every other vertex.
Proof. (1) ⇒ (2) and (3) ⇒ (4), are clear.

(2) ⇒ (3). Let \( n \) be an integer \( \geq 2 \) and there is a vertex of \( \overline{\Gamma}(M_n(R)) \) which is adjacent to every other vertex. By Theorem 2.9, either \( M_n(R) \cong \mathbb{Z}_2 \times R_1 \), where \( R_1 \) is a prime ring or \( S(M_n(R)) = \text{Ann}_R(\langle A \rangle) \cup \text{Ann}_r(\langle A \rangle) \) for some \( 0 \neq A \in M_n(R) \). By Remark 3.7, \( M_n(R) \not\cong \mathbb{Z}_2 \times R_1 \), for each prime ring \( R_1 \).

Thus \( S(M_n(R)) = \text{Ann}_R(\langle A \rangle) \cup \text{Ann}_r(\langle A \rangle) \) for some \( 0 \neq A \in M_n(R) \). Since \( \text{Ann}_R(\langle A \rangle) \) is an ideal of \( M_n(R) \), \( \text{Ann}_R(\langle A \rangle) = M_n(U) \) for some ideal \( U \) of \( R \).

Thus, \( M_n(U)M_n(U_A) = 0 \) i.e., \( UU_A = U_AU = 0 \), i.e., \( M_n(U_A)M_n(U) = 0 \). This follows that \( \langle A \rangle M_n(U) = 0 \) and so \( \text{Ann}_R(\langle A \rangle) \subseteq \text{Ann}_r(\langle A \rangle) \). Therefore, \( S(M_n(R)) = \text{Ann}_r(\langle A \rangle) \) for some \( A \in M_n(R) \) and this yields that \( A \) is a vertex of \( \overline{\Gamma}(M_n(R)) \) which is adjacent to every other vertex.

(4) ⇒ (5). By Remark 3.7, \( R \not\cong \mathbb{Z}_2 \times R_1 \), for all domains \( R_1 \). Since \( A \neq 0 \), \( U_A \neq 0 \). We claim that every nonzero elements \( a \) of \( U_A \) is a vertex of \( \overline{\Gamma}(R) \) which is adjacent to every other vertex. Let \( 0 \neq a \in U_A \) and \( s \in S(R) \). Clearly, \( aE_{11} \in \langle A \rangle \) and \( sE_{11} \in S(M_n(R)) \). By our hypothesis in (4), \( sE_{11}aE_{11} = 0 \) and so \( sa = 0 \). This means that \( a \) is a vertex of \( \overline{\Gamma}(R) \) which is adjacent to every other vertex.

(5) ⇒ (1). By Theorem 2.9, \( S(R) = \text{Ann}_R(\langle a \rangle) \) for some \( 0 \neq a \in R \) i.e., \( S(R)\langle a \rangle = 0 \) i.e., \( M_n(S(R))M_n(\langle a \rangle) = 0 \). By Lemma 3.2, \( S(M_n(R)) \subseteq M_n(S(R)) \), and hence, \( S(M_n(R))M_n(\langle a \rangle) = 0 \). This means that every nonzero matrix \( A \) of \( M_n(\langle a \rangle) \) is a vertex in \( \overline{\Gamma}(M_n(R)) \) which is adjacent to every other vertex. \( \square \)

Corollary 3.9. Let \( R \) be an Artinian commutative ring with identity. Then following statements are equivalent:

1. For any integer \( n \geq 1 \) there is a vertex of \( \overline{\Gamma}(M_n(R)) \) which is adjacent to every other vertex;
2. For some integer \( n \geq 2 \) there is a vertex of \( \overline{\Gamma}(M_n(R)) \) which is adjacent to every other vertex;
3. \( S(M_n(R)) = M_n(S(R)) \);
4. \( S(R) \) is an annihilator ideal of \( R \);
5. \( R \not\cong \mathbb{Z}_2 \times F \), for all fields \( F \) and there is a vertex of \( \overline{\Gamma}(R) \) which is adjacent to every other vertex.

Theorem 3.10. Let \( R \) be a ring with identity. Then following statements are equivalent:

1. \( \overline{\Gamma}(R) \) is a complete graph and \( R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \);
(2) $S(R)$ is an ideal of $R$ and $\tilde{\Gamma}(R)$ is a complete graph;
(3) $\tilde{\Gamma}(M_n(R))$ is a complete graph for each integer $n \geq 1$.
(4) $\tilde{\Gamma}(M_n(R))$ is a complete graph for some integer $n \geq 2$.

Proof. (1) $\Rightarrow$ (2), by Theorem 2.11.
(2) $\Rightarrow$ (3). Let $S(R)$ be an ideal of $R$ and $\tilde{\Gamma}(R)$ is a complete graph. By Theorem 2.11, $S(R)^2 = 0$. Let $n \geq 1$ be an integer and $A, B \in S(M_n(R))$. Then without loss of generality we can assume that $\langle A \rangle \langle A' \rangle = 0$ and $\langle B \rangle \langle B' \rangle = 0$, for some nonzero $A', B' \in M_n(R)$. Thus $M_n(U_A)M_n(U_A') = 0$ and $M_n(U_B)M_n(U_B') = 0$ and so $U_AU_A' = 0$ and $U_BU_B' = 0$. It follows that $U_A, U_B \subseteq S(R)$ and since $S(R)^2 = 0$, $U_AU_B = 0$. Therefore $M_n(U_A)M_n(U_B) = 0$ and this means that $A \sim B$ is an edge in $\tilde{\Gamma}(M_n(R))$.

(3) $\Rightarrow$ (4), is clear.
(4) $\Rightarrow$ (1). Let $\tilde{\Gamma}(M_n(R))$ be a complete graph for some integer $n \geq 2$. Since the ring $M_n(R)$ is not commutative, by Theorem 2.11, $S(M_n(R))$ is an ideal of $M_n(R)$ and $S(M_n(R))^2 = 0$. Assume that $s_1, s_2 \in S(R)$. By Lemma 3.4, $M_n(\langle s_i \rangle) \subseteq S(M_n(R))$ ($1 \leq i \leq 2$) and since $\tilde{\Gamma}(M_n(R))$ is complete, $M_n(\langle s_1 \rangle)M_n(\langle s_2 \rangle) = 0$ i.e., $\langle s_1 \rangle \langle s_2 \rangle = 0$ and so $s_1s_2 = 0$. Thus $S(R)^2 = 0$ and so by Theorem 2.11, $S(R)$ is an ideal of $R$ and $\tilde{\Gamma}(R)$ is a complete graph. $\square$

Remark 3.11. Previous theorem shows that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the only ring for which $\tilde{\Gamma}(R)$ is a complete graph but $\tilde{\Gamma}(M_n(R))$ is not a complete graph. In fact, since $S(R)$ is not an ideal of $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, by Theorem 3.10, $\tilde{\Gamma}(M_n(R))$ is not complete.

Theorem 3.12. Let $R$ and $S$ be two finite semisimple rings which are non-simple. Then $\tilde{\Gamma}(R) \cong \tilde{\Gamma}(S)$ if and only if $R \cong M_{n_1}(F_{q_1}) \times \cdots \times M_{n_r}(F_{q_r})$, and $S \cong M_{n'_1}(F_{q'_1}) \times \cdots \times M_{n'_r}(F_{q'_r})$, where $q_i$'s and $q'_i$'s are prime powers and $r, n_1, n_2, \cdots, n_r$ and $n'_1, n'_2, \cdots, n'_r$ are natural numbers such that $r \geq 2$ and $|M_{n_i}(F_{q_i})| = |M_{n'_i}(F_{q'_i})|$ (i.e., $q_i^{n_i^2} = q'_i^{n'_i^2}$), for all $1 \leq i \leq r$.

Proof. For one direction, the proof is straightforward. For the other direction, let $R_t := M_{n_t}(F_{q_t})$, $S_i := M_{n'_i}(F_{q'_i})$ and $t$ be the number of components of $S$. We argue induction on $r$. Assume that $r = 2$. Since $\tilde{\Gamma}(R) \neq \emptyset$ and is a complete 2-partite graph, $t \geq 2$. If $t \geq 3$, then $\tilde{\Gamma}(S)$ is not complete 2-partite, therefore $t = 2$. Now assume that $r > 2$ and $X = (x_1, x_2, \ldots, x_r)$ is a vertex in $\tilde{\Gamma}(R)$ with maximum degree. Since $(x_1, x_2, \ldots, x_r)$ has maximum degree, exactly one of the components $x_1, x_2, \ldots, x_r$ is non-zero. Without loss of generality, we can assume
that $x_1 \neq 0$. Let $Y = (y_1, y_2, ..., y_t)$ be a vertex corresponding to $X$, since $Y$
also is a vertex with maximum degree, we may assume that $Y = (y_1, 0, ..., 0)$, where $y_1$
is a non-zero element of $S_1$. Since the rings $R_1, R_2, ..., R_r, S_1, S_2, ..., S_t$
are prime, for any non-zero element $X' = (x', 0, ..., 0)$ in $\tilde{\Gamma}(R)$ there is only a
non-zero element $Y' = (y', 0, ..., 0)$ in $\tilde{\Gamma}(S)$ such that corresponding to $X'$
and vice versa. Therefore $|R_1| = |S_1|$. Clearly $\tilde{\Gamma}(R_2 \times \cdots \times R_r) \cong \tilde{\Gamma}(S_2 \times \cdots \times S_t)$.
By induction hypothesis, $r - 1 = t - 1$, $|R_i| = |S_i|$ for all $i$ ($2 \leq i \leq r - 1$), on
the other hand, we have already $|R_1| = |S_1|$, thus the proof is complete. □

Acknowledgments

This work was partially supported by IUT (CEAMA). The authors express
their gratitude to Prof. A. Haghany and Dr. H. Khabazian for the valuable
advice and encouragement given during the preparation of this paper.

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Received: June 27, 2007