Group Connectivity of Kneser Graphs

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Abstract

Let \(G\) be an undirected graph, \(A\) be an (additive) abelian group and \(A^* = A - \{0\}\). A graph \(G\) is \(A\)-connected if \(G\) has an orientation \(D(G)\) such that for every function \(b : V(G) \rightarrow A\) satisfying \(\sum_{v \in V(G)} b(v) = 0\), there is a function \(f : E(G) \rightarrow A^*\) such that \(\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = b(v)\). For an abelian group \(A\), let \(\langle A \rangle\) be the family of graphs that are \(A\)-connected. The group connectivity number \(\Lambda_g(G) = \min\{n : A\) is an abelian group with \(|A| \geq n\), then \(G \in \langle A \rangle\}\).

Let \(\lfloor n \rfloor = \{1, 2, \cdots, n\}\), and \(\binom{X}{k}\) represents the set of all \(k\)-subsets of \(X\). For \(n > 2k\), the Kneser graph \(KG(n, k)\) has vertex set: \(\binom{\lfloor n \rfloor}{k}\), edge set: \(A \sim B\) if and only if \(A \cap B = \emptyset\). In this paper, the group connectivity of Kneser graph are determined or given its limits.

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1 Intruduction

Graphs in this paper are finite and may have loops and multiple edges. Undefined terms and notations are from [1].

Let \(D = D(G)\) be an orientation of an undirected graph \(G\). If an edge \(e \in E(G)\) is directed from a vertex \(u\) to a vertex \(v\), then let \(\text{tail}(e) = u\) and \(\text{head}(e) = v\). For a vertex \(v \in V(G)\), let

\[E^D_-(v) = \{e \in E(D) : v = \text{tail}(e)\}\], and \(E^D_+(v) = \{e \in E(D) : v = \text{head}(e)\}\).
The subscript $D$ may be omitted when $D(G)$ is understood from the context.

Let $A$ denote an (additive) Abelian group, the additive identity of $A$ is denoted by 0(zero). Let $A^*$ denote the set of nonzero elements of $A$. Define:

$$F(G, A) = \{ f : E(G) \rightarrow A \} \text{ and } F^*(G, A) = \{ f : E(G) \rightarrow A^* \}$$

Given a function $f \in F(G, A)$, let $\partial f : V(G) \mapsto A$ be given by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where “$\sum$” refers to the addition in $A$.

A function $b : V(G) \mapsto A$ is called an $A$-valued zero sum function on $G$ if $\sum_{v \in V(G)} b(v) = 0$ in $G$. The set of all $A$-valued zero sum functions on $G$ is denoted by $Z(G, A)$. Given $b \in Z(G, A)$, if $G$ has an orientation $G'$ and a function $f \in F^*(G, A)$ such that $\partial f = b$, then we called $f$ an $(A, b)$-nowhere zero flow ($(A, b)$-NZF).

A graph $G$ is $A$-connected if for any $b \in Z(G, A)$, $G$ has an $(A, b)$-NZF. For an abelian group $A$, let $\langle A \rangle$ be the family of graphs that are $A$-connected.

The concept of $A$-connectivity was firstly introduced by Jaeger et al in [2]. For a 2-edge-connected graph $G$, define the group connectivity number of $G$ as follows:

$$\Lambda_g(G) = \min \{ k : \text{if } A \text{ is an abelian group with } |A| \geq k, \text{ then } G \in \langle A \rangle \}.$$ 

If $G$ is 2-edge-connected, then $\Lambda_g(G)$ exists as a finite number. In this paper, we will investigate the group connectivity of Kneser graph.

## 2 Preliminaries

In this section, we will introduce some known results which can be used in the proofs.

**Lemma 2.1** ([2] and [3]) Let $C_n$ denote a cycle of $n$ vertices. Then $C_n \in \langle A \rangle$ if and only if $|A| \geq n + 1$.

**Lemma 2.2** ([3]) Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then $\Lambda_g(G) = 2$ if and only if $n = 1$ (and so $G$ has $m$ loops).

**Lemma 2.3** ([4]) Let $A$ be an abelian group with $|A| \geq 3$, and let $G$ be a connected graph. If for each $e \in E(G)$, $G$ has a subgraph $H_e \in \langle A \rangle$, then $G \in \langle A \rangle$.

A wheel $W_n$ is a graph obtained by joining a cycle with $n$ vertices and $K_1$. The vertex of $K_1$ is called the center of $W_n$. 

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Lemma 2.4 ([6]) \( \Lambda_g(W_{2n}) = 3 \) for \( n \geq 1 \).

Lemma 2.5 ([6])

\[
\Lambda_g(K_n) = \begin{cases} 
4 & \text{if } 3 \leq n \leq 4, \\
3 & \text{if } n \geq 5
\end{cases}
\]

Lemma 2.6 ([6]) Let \( m \geq n \geq 2 \) be integers. Then

\[
\Lambda_g(K_{m,n}) = \begin{cases} 
5 & \text{if } n = 2 \\
4 & \text{if } n = 3 \\
3 & \text{if } n \geq 4
\end{cases}
\]

Lemma 2.7 ([5]) Let \( G \) be a loopless graph. If one of the following holds:

(i) \( G \cong P_{10} \), or
(ii) \( G \) has a collapsible subgraph \( H \) with \( |V(H)| \leq 2 \) such that \( G/H \cong K_{2,m} \)

for some \( m \geq 2 \), or
(iii) \( G \cong S_{m,n} \) for some \( m, n \) with \( m > n > 0 \), then

\( \Lambda_g(G) = 5 \).

3 Group connectivity of Kneser graph \( KG(n, k) \)

Let \([n] = \{1, 2, \cdots, n\}\), and \( \binom{X}{k} \) represents the set of all \( k \)-subsets of \( X \).

For \( n > 2k \), the Kneser graph \( KG(n, k) \) has vertex set: \( \binom{[n]}{k} \), edge set: \( A \sim B \) if and only if \( A \cap B = \emptyset \). For example, \( KG(5, 2) \) is the Petersen graph (See Figure 2.1).

![Figure 2.1: Petersen graph](image)

**Theorem 3.1** If \( n \geq 2k + 2 \), then

\[
\Lambda_g(KG(n, k)) = \begin{cases} 
4 & \text{if } k = 1 \text{ and } 3 \leq n \leq 4 \\
3 \text{ or } 4 & \text{if } k = 2 \text{ and } n = 6 \\
3 & \text{others}
\end{cases}
\]
Proof:

Case one: $k = 1$

It is obviously that $KG(n, 1) \cong K_n(n \geq 2)$. So by Lemma 2.5 we have

$$\Lambda_g(KG(n, 1)) = \begin{cases} 
4 & \text{if } 3 \leq n \leq 4, \\
3 & \text{if } n \geq 5
\end{cases}$$

Case two: $k \geq 2$

Firstly we will proof that if $n \geq 2k + 2$, $k \geq 2$, then every edge of $KG(n, k)$ lies in a subgraph of $KG(n, k)$ that isomorphic to $K_{k+1, k+1}$. Let $e$ be an edge of $KG(n, k)$. Without less the universality, we can suppose that the two vertices incident with $e$ are $u = (1, 2, \cdots, k)$ and $v = (k + 2, k + 2, \cdots, 2k + 1)$. Let

$$u_i = \{1, 2, \cdots, k, k + 1\} \setminus \{v_i\}, i = 1, 2, \cdots, k + 1$$

$$v_j = \{k + 2, k + 3, \cdots, 2k + 1, 2k + 2\} \setminus \{v_{j+k+1}\}, j = 1, 2, \cdots, k + 1$$

Denote

$$S = \{u_1, u_2, \cdots, u_{k+1}\}$$

and

$$T = \{v_1, v_2, \cdots, v_{k+1}\}$$

Then any two vertices of $S$ or $T$ are nonadjacent because they have the same number. At the same time any vertex of $S$ is adjacent with any vertex of $T$. So the induced subgraph by $S \cup T$ is isomorphic to $K_{k+1, k+1}$. Notice $u_{k+1} = u$ and $v_{k+1} = v$. So $e$ is contained in the induced subgraph by $S \cup T$.

Subcase 1: If $k \geq 3$, we will know that every edge of $KG(n, k)$ lies in a subgraph of $KG(n, k)$ that isomorphic to $K_{4,4}$. then the result can be easily obtained by combining Lemma 2.2, Lemma 2.3, Lemma 2.6.

Subcase 2: If $k = 2$ and $n \geq 8$. Let $e$ be an edge of $KG(n, k)$, without loss of the generality we may assume the two vertices incident with $e$ are $u_1 = (1, 2)$ and $v_1 = (5, 6)$. Let $u_2 = (2, 3), u_3 = (3, 4), u_4 = (4, 1), v_2 = (6, 7), v_3 = (7, 8), v_4 = (8, 5)$, then every $u_i(i = 1, 2, 3, 4)$ is adjacent with every $v_j(j = 1, 2, 3, 4)$. So $e$ must be in a subgraph that isomorphic to $K_{4,4}$. Then by Lemma 2.2, Lemma 2.3 and Lemma 2.6 we obtain $\Lambda_g(KG(n, k)) = 3$.

Subcase 3: If $k = 2$ and $n = 6$, we will know then every edge of $KG(n, k)$ lies in a subgraph of $KG(n, k)$ that isomorphic to $K_{3,3}$. then we will obtain we obtain $\Lambda_g(KG(n, k)) = 3$ or 4 by combining Lemma 2.3 and Lemma 2.6.

Subcase 4: If $k = 2$ and $n = 7$, let $e$ be an edge of $KG(7, 2)$, without loss of the generality we may assume the two vertices incident with $e$ are
v_1 = (1, 2) and v_2 = (3, 4). Let v_3 = (5, 6), v_4 = (7, 3), v_5 = (4, 6), then the induced subgraph by v_1, v_2, v_3, v_4, v_5 is a W_4. So we will know every edge of KG(n, k) lies in a subgraph of KG(n, k) that isomorphic to W_4. then we will obtain by combining Lemma 2.2, Lemma 2.3 and Lemma 2.4. □

It just remain the circumstance n = 2k + 1. In this case, we obtain the following conclusion:

Theorem 3.2 If n = 2k + 1, then 3 ≤ Λ_g(KG(n, k)) ≤ 7.

Proof: Firstly we will proof that If n = 2k + 1, then every edge of KG(n, k) lies in a cycle of length 6. Let e be an edge of KG(n, k) and the two vertices incident with e are v_1 = (1, 2, ⋯, k) and v_2 = (k + 1, k + 2, ⋯, 2k). Let v_3 = (1, 2, ⋯, k−1, 2k+1), v_4 = (k+1, k+2, ⋯, 2k−1, k), v_5 = (1, 2, ⋯, k−1, k+1) and v_6 = (k+2, ⋯, 2k, 2k+1). It can be seen that v_1v_2v_3v_4v_5v_6v_1 be a 6-cycle that contain edge e.

The result can be easily obtained by combining Lemma 2.1, Lemma 2.2 and Lemma 2.3. □

Example: It has been noted that KG(5, 2) is the Peterson graph, so by Lemma 2.7 we know Λ_g(K(5, 2)) = 5.

References


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