Study of Monotonicity of Trinomial Arcs $M(p, k, r, n)$ when $1 < \alpha < +\infty$

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Abstract
We are interested in this paper in the family of curves $M(p, k, r, n)$, solutions of equation $z^n = \alpha z^k + (1 - \alpha)$, where $z$ is a complex variable, $n$, $k$, $p$ and $r$ are nonzero integers and $\alpha$ is a real number greater than $1$. Expressing each of these curves in polar coordinates $(\rho, \theta)$, we will prove that $\rho(\theta)$ is an increasing function.

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1 Introduction
In [6], Fell studied the trinomial equation

$$\lambda z^n + (1 - \lambda) z^k - 1 = 0,$$

where $z = \rho e^{i\theta}$ is a complex variable, $n$ and $k$ are two integers such that $1 \leq k \leq n - 1$ and $\lambda$ is a real number. She established especially a large description of the trajectories of roots of the trinomial equation (1), called trinomial arcs. These arcs are continuous curves, corresponding to a number $\lambda$ which is restricted whether to $[0, 1]$, or to $[1, +\infty[$, or also to $]-\infty, 0]$. Each of these arcs can be expressed in polar coordinates $(\rho, \theta)$ by a function $\rho(\theta)$. The problem of monotonicity of the function $\rho(\theta)$ for the different families
of trinomial curves is pointed out in [6]. This problem is never solved. The descriptive results of Fell [6] gave us the information about the form and the localization of the trinomial arcs. However, these families of curves are not well-defined, in order to be studied. The original intent of this paper was to study the behavior of one of these different families. In fact, we consider in our work the equivalent trinomial equation

$$z^n = \alpha z^k + (1 - \alpha),$$

(2)

where $z = \rho e^{i\theta}$ is a complex variable, $n$ is an integer larger than 1, $k = 1, 2, ..., n - 1$ and $\alpha$ is a real number. We restrict our attention in this paper to a family of trinomial curves, denoted by $M(p, k, r, n)$, solutions of equation (2) with $1 < \alpha < +\infty$ and $p, k, r$ and $n$ are four integers satisfying some conditions.

The rest of this paper is organized as follows. Section 2 contains some preliminary material concerning the expression of the trinomial equation (2) in polar coordinates. In section 3, we establish the description and the definition of the arcs $M(p, k, r, n)$. Finally, we study in section 4 the monotonicity of the function $\rho(\theta)$ for these trinomial curves and we arrive at the main result stating that $\rho(\theta)$ is an increasing function for the arcs $M(p, k, r, n)$.

2 Basic definitions and formulas

The two trinomial equations (1) and (2) are equivalent. In fact, in order to pass from (2) to (1), we put $\alpha = 1 - 1/\lambda$. Indeed, substituting into equation (1) the expression given for $z^n$ by equation (2) yields $[1 - \lambda (1 - \alpha)](z^k - 1) = 0$. So, $\lambda (1 - \alpha) = 1$ or $z^k = 1$. Because $z$ is a complex number, we obtain $\alpha = 1 - 1/\lambda$. By this relation, we can say that the case $1 \leq \alpha < +\infty$ of equation (2) corresponds to the case $-\infty < \lambda < 0$ of equation (1).

Remark 2.1 The case $\alpha = 1$ is one particular case for the trinomial equation (2). In fact, when $\alpha = 1$, this equation becomes $z^k[z^{n-k} - 1] = 0$. Then, the $n$ roots of equation (2) are exactly the $(n-k)^{th}$ roots of unity, which are simple roots and 0, a root of multiplicity $k$.

In the rest of this paper, we will study the trinomial equation (2) with $1 < \alpha < +\infty$. Putting $z = \rho e^{i\theta}$ in this equation and separating real and imaginary parts, we obtain $\rho^n \sin n\theta = \alpha \rho^k \sin k\theta$ and $\rho^n \cos n\theta = \alpha \rho^k \cos k\theta + (1 - \alpha)$. Then, if $\theta \neq l\pi/n$, where $l$ is an integer, one has

$$\rho^{n-k} = \alpha \sin k\theta / \sin n\theta.$$

(3)
On the other side, divide equation (2) by \( z^n \) and take the imaginary part.

When \( \theta \neq l\pi/(n - k) \) such that \( l \) is an integer, we get

\[
\rho^k = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta. \tag{4}
\]

Lastly, we obtain the following equation for the trajectories of roots of (2):

\[
\rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta = \sin k\theta. \tag{5}
\]

There is one basic remark: as \( 1 < \alpha < +\infty \) and \( \rho > 0 \), from (3) and (4), we are interested in those angles \( \theta \) for which:

\[
\text{sign} (\sin n\theta) = \text{sign} (\sin k\theta) = \text{sign} (\sin(n - k)\theta). \tag{6}
\]

**Definition 2.2** An angle \( \theta \) which fulfills (6) will be called a \((n,k)\)-feasible angle for the trinomial equation (2) with \( 1 < \alpha < +\infty \).

**Remark 2.3** Because the upper and lower half-planes are symmetrical, we restrict our study of the trinomial arcs to the upper half-plane.

### 3 Trinomial arcs \( M(p, k, r, n) \)

By Remark 2.1, when \( \alpha = 1 \), the \( n \) roots of equation (2) are the \((n - k)^{th}\) roots of unity, which are simple roots and 0, a root of multiplicity \( k \). In [6], Fell asserts that when \( \alpha \) moves from 1 to \(+\infty\), the trajectories of these \( n \) roots are continuous arcs, such that some of these arcs are situated inside the unit disk and the others are outside the disk. In fact, inside the unit disk, she considers in [6] the following sets: \( A = \{k^{th} \text{ roots of unity}\} \), \( B = \{(n - k)^{th} \text{ roots of } -1\} \) and \( C = \{n^{th} \text{ roots of unity}\} \). Let \( \gamma \) be in \( C \) and \( \beta \) be the unique nearest neighbor of \( \gamma \) in \( A \cup B \). We are interested in the case \( \beta \in A, \beta \notin B \).

In [6], Fell tells us that in this situation, as \( \alpha \) moves from \(+\infty\) to 1, there is a trajectory starting at \( \beta \) such that \( \rho \) moves from 1 to 0, this trajectory is tangent to the ray \( \theta = \text{arg}(\beta) \). Fell has showed also that, in this case, the feasible angles \( \theta \) belong to intervals of length less than or equal to \( \pi/n \) and bounded by \( \text{arg}(\gamma) \) and \( \text{arg}(\beta) \), such that \( \gamma \) is an \( n^{th} \) root of \(-1\) and \( \beta \) is an \( k^{th} \) root of unity. In this paper, we will consider and restrict our attention to the trinomial arcs such that the feasible angles \( \theta \) belong to an interval of the form \( [(2r + 1)\pi/n, 2\pi p/k] \), where \( p \) and \( r \) are nonzero integers verifying some conditions. In fact, \( \gamma \) such that \( \text{arg}(\gamma) = (2r + 1)\pi/n \) is an \( n^{th} \) root of \(-1\) and \( \beta \) such that \( \text{arg}(\beta) = 2\pi p/k \) is an \( k^{th} \) root of unity. These trinomial curves will be denoted by \( M(p, k, r, n) \). Remark that, when \( n = 2 \) and \( n = 3 \), the trajectories of roots of equation (2) with \( 1 < \alpha < +\infty \) are linear. Moreover, when \( n = 4 \), the roots of (2) don't move in trajectories of the form \( M(p, k, r, n) \).

We define so the family of trinomial arcs \( M(p, k, r, n) \) as follows:
Definition 3.1 If $n$ is an integer greater than or equal to 5, so $M(p, k, r, n)$ is the set of roots of equation (2) with $\alpha > 1$ and the feasible angles belong to the interval $[(2r+1)\pi/n, 2\pi p/k]$, where $p$ and $r$ are nonzero integers verifying $r \geq p$ and $k$ is an integer such that $pn/(r+1) < k < 2pn/(2r+1)$.

The trinomial curves $M(p, k, r, n)$ are illustrated in the figure below. These arcs exist in view of the next lemma.

Lemma 3.2 If $n$ is an integer greater than or equal to 5, $k$ is an integer such that $1 \leq k \leq n-1$ and $\alpha > 1$, so in the trinomial equation (2) with $pn/(r+1) < k < 2pn/(2r+1)$, where $p$ and $r$ are nonzero integers verifying $r \geq p$, any angle of the interval $[(2r+1)\pi/n, 2\pi p/k]$ is feasible.

Proof. We assume that the integer $k$ satisfy $pn/(r+1) < k < 2pn/(2r+1)$. First, notice that because $0 < k < n$, the two nonzero integers $p$ and $r$ verify the condition $2p/(2r+1) \leq 1$, i.e. $r \geq p$. Now, let be $(2r+1)\pi/n < \theta < 2\pi p/k$. We have $(2r+1)\pi < n\theta < 2\pi pn/k$. Since $pn/(r+1) < k$, we obtain that $2\pi pn/k < 2(r+1)\pi$ and that $\sin n\theta < 0$. On the other side, we have $(2r+1)\pi k/n < k\theta < 2\pi p$. From the inequality $r \geq p$ stems that $(2p-1)n/(2r+1) < pn/(r+1) < k$. So, $(2p-1)\pi < (2r+1)\pi k/n$ and so $\sin k\theta < 0$. Finally, we have $(2r+1)\pi (1-k/n) < (n-k)\theta < 2\pi p(n/k-1)$. Because $k < 2pn/(2r+1)$, then $[2(r-p)+1] \pi < (2r+1)\pi (1-k/n)$ and since $pn/(r+1) < k$, so $2\pi p(n/k-1) < [2(r-p)+2] \pi$. It follows that $\sin(n-k)\theta < 0$. Thus, the conditions (6) are fulfilled and the proof is achieved.

Remark 3.3 From the proof of Lemma 3.2, we get $\sin n\theta < 0$, $\sin k\theta < 0$ and $\sin(n-k)\theta < 0$ for any angle $\theta$ in the interval $[(2r+1)\pi/n, 2\pi p/k]$. 

Trinomial arcs $M(p, k, r, n)$ inside the upper half unit disk
4 Behavior of the trinomial arcs $M(p, k, r, n)$

In this section, our objective is to prove that for each trinomial arc $M(p, k, r, n)$, $ho(\theta)$ is a monotonic function, i.e. $d\rho/d\theta$ is never zero. Thus, we begin by proving that the derivative $d\rho/d\theta$ exists and it is well-defined.

**Proposition 4.1** $\rho(\theta)$ is a derivable function for each arc $M(p, k, r, n)$.

**Proof.** Consider a trinomial arc $M(p, k, r, n)$. By equation (4), we have $\rho^k(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta$. From Remark 3.3, the feasible angles $\theta$ are such that $\sin n\theta < 0$ and $\sin(n - k)\theta < 0$. Define the function $g(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta$ on the interval $\lbrack (2r + 1)\pi/n, 2\pi p/k \rbrack$. So, as $\alpha > 1$, the denominator of $g(\theta)$ is never 0. It follows that $g(\theta)$ is well-defined. In addition, $g(\theta)$ is derivable and positive. Hence, the function $\rho(\theta) = g^{1/k}$ is derivable. Therefore, its derivative $d\rho/d\theta$ exists and it is well-defined.

Now, Proposition 4.1 allows us to estimate $d\rho/d\theta$. In fact, differentiating both sides of equation (5) with respect to $\theta$, we get

$$
\left[(n - k) \rho^{n-k-1} \sin n\theta - n \rho^{n-1} \sin(n - k)\theta\right] d\rho/d\theta = k \cos k\theta + (n - k) \rho^n \cos(n - k)\theta - n \rho^{n-k} \cos n\theta.
$$

Supposing that $d\rho/d\theta = 0$ and considering equation (5) of trajectories of roots, we will consider $\rho^n$ and $\rho^{n-k}$ as solutions of the next system:

$$
\begin{cases}
\rho^n \sin n\theta - n \rho^{n-k} \sin n\theta = 0 \\
\rho^{n-k} \sin n\theta - \rho^n \sin(n - k)\theta - \sin k\theta = 0.
\end{cases}
$$

This system is equivalent to the following system:

$$
\begin{cases}
R(\theta) \cdot \rho^{n-k} = N_1(\theta) \\
R(\theta) \cdot \rho^n = N_2(\theta),
\end{cases}
$$

where

$$
R(\theta) = (n - k) \sin k\theta - k \cos n\theta \sin(n - k)\theta
$$

$$
N_1(\theta) = n \sin k\theta \cos(n - k)\theta - k \sin n\theta
$$

$$
N_2(\theta) = (n - k) \sin k\theta \cos n\theta - k \sin(n - k)\theta.
$$

Remark that $N_1(\theta)$ and $N_2(\theta)$ can be expressed as follows: $N_1(\theta) = (n - k) \sin n\theta - n \sin(n - k)\theta \cos k\theta$ and $N_2(\theta) = (n - k) \sin k\theta \cos n\theta - n \sin(n - k)\theta$. Then, the difference of the two equalities of (7) leads to the relation:

$$
R(\theta) [\rho^{n-k} - \rho^n] = S(\theta) [1 - \cos k\theta],
$$

with

$$
S(\theta) = (n - k) \sin n\theta + n \sin(n - k)\theta.
$$
**Remark 4.2** By Remark 3.3, we can deduce, for the arcs $M(p, k, r, n)$, that $S(\theta) < 0$ for any feasible angle $\theta$ in the interval $[2r + 1]\pi/n, 2\pi p/k[.\]

Let us recall that our main interest is to prove that $d\rho/d\theta$ is never zero, for each trinomial arc $M(p, k, r, n)$. So, in what follows, the question is to show that the hypothesis $d\rho/d\theta = 0$ considered above is not possible for this family of arcs. For that, we will need the two following propositions and we will use the remark below.

**Remark 4.3** For any feasible angle $\theta$ in the interval $[2r + 1]\pi/n, 2\pi p/k[$, the sign of $\cos n\theta$ is determined as follows:

Case i): $pn/(r + 1) < k < 4pn/(4r + 3)$. We have $\cos n\theta = 0$ on $[2r + 1]\pi/n, 2\pi p/k[ if and only if $\theta = (4r + 3)\pi/2n$. Moreover, we have $\cos n\theta < 0$ for $\theta < (4r + 3)\pi/2n$ and $\cos n\theta > 0$ for $\theta > (4r + 3)\pi/2n$.

Case ii): $4pn/(4r + 3) \leq k < 2pn/(2r + 1)$. In this situation, we have $\cos n\theta < 0$ for any feasible angle $\theta$ in the interval $[2r + 1]\pi/n, 2\pi p/k[.

**Proposition 4.4** Let $M(p, k, r, n)$ be a trinomial arc. For any integer $k$ such that $pn/(r + 1) < k < 2pn/(2r + 1)$, we have $N_2(\theta) > 0$ for any feasible angle $\theta$ in the interval $[2r + 1]\pi/n, 2\pi p/k[.$

**Proof.** Let $\theta$ be a feasible angle in $[2r + 1]\pi/n, 2\pi p/k[.$ Recall that $N_2(\theta) = (n - k)\sin k\theta \cos n\theta - k\sin(n - k)\theta$. From Remark 3.3, we get $\sin k\theta < 0$ and $\sin(n - k)\theta < 0$. Moreover, according to Remark 4.3, two cases are possible. So, in Case i), we distinguish the two following subcases:

Subcase 1): $\theta$ belongs to $[2r + 1]\pi/n, (4r + 3)\pi/2n[$. In this situation, we have $\cos n\theta \leq 0$. So, it follows that $N_2(\theta) > 0$.

Subcase 2): $\theta$ belongs to $[(4r + 3)\pi/2n, 2\pi p/k[.$ Since $N_2(\theta)$ can be expressed as $N_2(\theta) = n\sin k\theta \cos n\theta - k\sin n\theta \cos k\theta$, we will consider the function $L_2(\theta) = N_2(\theta)/\cos n\theta \cos k\theta = n\tan k\theta - k\tan n\theta$. According to Remark 4.3, in this subcase, we have $\cos n\theta > 0$. On the other side, we have $(2r + 3/2)\pi/k/n < k\theta < 2\pi p$. Because the two integers $p$ and $r$ verify $r \geq p$, we obtain that $(2p - 1/2)/(2r + 3/2) < p/(r + 1) < k/n$ and that $(2p - 1/2)\pi < (2r + 3/2)\pi/k/n$. So, we deduce that $(2p - 1/2)\pi < k\theta < 2\pi p$ and that $\cos k\theta > 0$. Hence, the sign of $N_2(\theta)$ is exactly the sign of $L_2(\theta)$. For that, we study this last function on the interval $[(4r + 3)\pi/2n, 2\pi p/k[.$

$L_2(\theta)$ is derivable with $L_2'(\theta) = nk[\tan^2 k\theta - \tan^2 n\theta]$. As $\tan n\theta < 0$ and $\tan k\theta < 0$, the roots of the equation $L_2'(\theta) = 0$ are those of the equation $\tan n\theta = \tan k\theta$. The unique root of this equation is of the form $\theta = l\pi/(n - k)$ where $l$ is an integer. However, $l\pi/(n - k) \in [(4r + 3)\pi/2n, 2\pi p/k[ if and only if $(2r + 3/2)(1 - k/n) < l < 2p(n/k - 1)$. Because $k < 4pn/(4r + 3)$, so $2(r - p) + 3/2 < (2r + 3/2)(1 - k/n)$ and because $pn/(r + 1) < k$, so $2p(n/k - 1) < 2(r - p + 1)$. The integer $l$ verifies then $2(r - p) + 3/2 < l <
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2(r - p) + 2, which is not possible. We deduce so that \( L_2(\theta) \neq 0 \) for any \( \theta \) in \( ](4r + 3)\pi/2n, 2\pi p/k[ \). In addition, \( L_2(\theta) \) goes to \( +\infty \) as \( \theta \) tends on the right to \( (4r + 3)\pi/2n \) and \( L_2(2\pi p/k) > 0 \). So, it follows that \( L_2(\theta) > 0 \) and that \( N_2(\theta) > 0 \) for any \( \theta \) in \( ](4r + 3)\pi/2n, 2\pi p/k[ \).

On the other hand, Remark 4.3 tells us that in Case ii), we have \( \cos n\theta < 0 \) for any \( \theta \) in \( ](2r + 1)\pi/n, 2\pi p/k[ \). Then, \( N_2(\theta) > 0 \) for any \( \theta \) in this interval.

Consequently, for any integer \( k \) such that \( pn/(r + 1) < k < 2pn/(2r + 1) \), we proved that \( N_2(\theta) > 0 \) for any feasible angle \( \theta \) in \( ](2r + 1)\pi/n, 2\pi p/k[ \).

**Proposition 4.5** Let \( M(p, k, r, n) \) be a trinomial arc. For any integer \( k \) such that \( pn/(r + 1) < k < 2pn/(2r + 1) \), we have the following results:

- If \( pn/(r + 1) < k < 4pn/(4r + 3) \), so there exists \( \theta_0 \) in \( ](2r + 1)\pi/n, 2\pi p/k[ \) such that \( R(\theta_0) = 0 \). Moreover, \( R(\theta) < 0 \) for \( \theta < \theta_0 \) and \( R(\theta) > 0 \) for \( \theta > \theta_0 \).
- If \( 4pn/(4r + 3) \leq k < 2pn/(2r + 1) \), so \( R(\theta) < 0 \) for any angle \( \theta \) in the interval \( ](2r + 1)\pi/n, 2\pi p/k[ \).

**Proof.** Let \( \theta \) be a feasible angle in \( ](2r + 1)\pi/n, 2\pi p/k[ \). Let us recall that \( R(\theta) = (n - k)\sin k\theta - k\cos n\theta \sin(n - k)\theta \). We know by Remark 3.3 that \( \sin k\theta < 0 \) and \( \sin(n - k)\theta < 0 \). Then, in view of Remark 4.3, we distinguish in Case i) the two subcases:

**Subcase 1:** \( \theta \) belongs to \( ](2r + 1)\pi/n, (4r + 3)\pi/2n[ \). Because \( \cos n\theta \leq 0 \) in this subcase, we deduce that \( R(\theta) < 0 \).

**Subcase 2:** \( \theta \) belongs to \( ](4r + 3)\pi/2n, 2\pi p/k[ \). Observing that \( R(\theta) \) can be expressed as \( R(\theta) = (n - k)\sin k\theta \cos(n - k)\theta - n\sin(n - k)\theta \cos n\theta \), we will define \( K(\theta) = R(\theta)/\cos n\theta \cos(n - k)\theta = (n - k)\tan n\theta - n\tan(n - k)\theta \). In this situation, we have \( \cos n\theta > 0 \). In addition, we have \( (2r + 3/2)\pi(1 - k/n) < (n - k)\theta < 2\pi p/(n - k) \). Since \( k < 4pn/(4r + 3) \), so \( (2r - p) + 3/2)\pi < (2r + 3/2)\pi(1 - k/n) \) and as \( pn/(r + 1) < k \), we get \( 2\pi p/(n - k) < (2r - p) + 2)\pi \). It follows that \( (2r - p) + 3/2)\pi < (n - k)\theta < (2r - p) + 2)\pi \) and that \( \cos(n - k)\theta > 0 \). Thus, \( R(\theta) \) has the same sign as \( K(\theta) \). Because \( \tan n\theta < 0 \) and \( \tan(n - k)\theta < 0 \), the zeros of \( K'(\theta) = n(n - k)[\tan^2 n\theta - \tan^2(n - k)\theta] \) are those of the equation \( \tan n\theta = \tan(n - k)\theta \). The unique solution of this last equation is of the form \( \theta = l\pi/k \) with \( l \) is an integer. However, \( l\pi/k \in ](4r + 3)\pi/2n, 2\pi p/k[ \) if and only if \( (2r + 3/2)k/n < l < 2p \). As \( pn/(r + 1) < k \), so \( (2r + 3/2)p/(r + 1) < (2r + 3/2)k/n \) and because the two integers \( p \) and \( r \) verify \( r \geq p \), we get \( (2p - 1/2) < (2r + 3/2)p/(r + 1) \). We obtain so that \( (2p - 1/2) < l < 2p \), which is impossible as \( l \) is an integer. Then, we conclude that \( K'(\theta) \) is never zero. We have also that \( K(\theta) \) goes to \( -\infty \) as \( \theta \) tends on the right to \( (4r + 3)\pi/2n \) and \( K(2\pi p/k) > 0 \). Thus, there exists \( \theta_0 \) in the interval \( ](4r + 3)\pi/2n, 2\pi p/k[ \) such that \( R(\theta_0) = 0 \). Moreover, we deduce that \( R(\theta) < 0 \) for \( \theta < \theta_0 \) and that \( R(\theta) > 0 \) for \( \theta > \theta_0 \).

Consequently, in Case i), i. e. if \( pn/(r + 1) < k < 4pn/(4r + 3) \), there exists \( \theta_0 \) in the interval \( ](2r + 1)\pi/n, 2\pi p/k[ \) such that \( R(\theta_0) = 0 \). Moreover,
we obtain that $R(\theta) < 0$ for $\theta < \theta_0$ and that $R(\theta) > 0$ for $\theta > \theta_0$.

Finally, from Remark 4.3, we have in Case ii) that $\cos n\theta < 0$ for any $\theta$ in $\{(2r + 1)\pi/n, 2\pi p/k\}$. So, $R(\theta) < 0$ for any angle $\theta$ in this interval.

Now, the two propositions above allow us to state the next main result for the trinomial curves $M(p,k,r,n)$.

**Theorem 4.6** For the trinomial curves $M(p,k,r,n)$, the function $\rho(\theta)$ is monotonic on the interval $[(2r + 1)\pi/n, 2\pi p/k]$.

**Proof.** Assume that $\theta$ is a feasible angle in the interval $[(2r + 1)\pi/n, 2\pi p/k]$. Proposition 4.5 and Remark 4.3 tell us that in Case i), there exists $\theta_0$ in this interval such that $R(\theta_0) = 0$. So, we can distinguish the two following subcases:

**Subcase 1:** $\theta$ belongs to $[(2r + 1)\pi/n, \theta_0]$. From Propositions 4.4 and 4.5, we have respectively that $N_2(\theta) > 0$ and $R(\theta) \leq 0$. Then, the equation $R(\theta) \cdot \rho^n = N_2(\theta)$ given by (7) is not possible, because $\rho > 0$.

**Subcase 2:** $\theta$ belongs to $[\theta_0, 2\pi p/k]$. On the one hand, we have $R(\theta) > 0$ by Proposition 4.5. On the other hand, we have $S(\theta) < 0$ by Remark 4.2.

Then, the relation $R(\theta) [\rho^{n-k} - \rho^n] = S(\theta) [\cos k\theta]$ given by (8) implies that $\rho^{n-k} - \rho^n < 0$, which is impossible because $0 < \rho < 1$.

Hence, in Case i), the hypothesis $d\rho/d\theta = 0$ is false. On the other side, Proposition 4.5 tells us that in Case ii), we have $R(\theta) < 0$ for any $\theta$ in the interval $[(2r + 1)\pi/n, 2\pi p/k]$. Since $N_2(\theta) > 0$ by Proposition 4.4, then, the fact that $\rho > 0$ contradicts the equation $R(\theta) \cdot \rho^n = N_2(\theta)$ given by (7).

Consequently, in the two cases i) and ii), the hypothesis $d\rho/d\theta = 0$ is not possible. Thus, we deduce that $d\rho/d\theta$ is never 0, i.e. $\rho(\theta)$ is monotonic on the interval $[(2r + 1)\pi/n, 2\pi p/k]$. Thus, we achieve the proof.

**Remark 4.7** When $k = pn/(r + 1)$, we have $2\pi p/k = 2(r + 1)\pi/n$. By using the notations of Fell [6], we can tell that $2(r + 1)\pi/n = \arg(\gamma)$, such that $\gamma$ belongs to $C = \{n^{th}$ roots of unity$\}$ and that $2\pi p/k = \arg(\beta)$, such that $\beta$ belongs to $A = \{k^{th}$ roots of unity$\}$. According to Fell [6], because $\arg(\beta) = \arg(\gamma)$, the particular case $k = pn/(r + 1)$ corresponds to a linear trinomial arc inside the unit disk.

**Remark 4.8** When $k = 2pn/(2r + 1)$, we have $(2r + 1)\pi/n = 2\pi p/k$. Then, in this case, the trinomial arc $M(p,k,r,n)$ is such that the bounds of the interval of feasible angles are identical. So, the particular case $k = 2pn/(2r + 1)$ corresponds to a linear trinomial arc inside the unit disk.

Lastly, Theorem 4.6 allows us to prove the following main result.

**Theorem 4.9** Let $M(p,k,r,n)$ be a trinomial arc. For any integer $k$ such that $pn/(r + 1) < k < 2pn/(2r + 1)$, $\rho(\theta)$ is an increasing function on the interval $[(2r + 1)\pi/n, 2\pi p/k]$. 
**Proof.** Let $M(p, k, r, n)$ be a trinomial arc. From Theorem 4.6, the function $\rho(\theta)$ is monotonic on the interval $[(2r + 1)\pi/n, 2\pi p/k]$. Let us estimate $\rho(\theta)$ at the bound $(2r + 1)\pi/n$. So, put $\theta = (2r + 1)\pi/n$ in the equation $\rho^k(\theta) = (1 - 1/\alpha) \sin n\theta/\sin(n - k)\theta$ given by (4). Because $(2p - 1)\pi < (2r + 1)\pi k/n < 2\pi p$, we get $\sin [(2r + 1)\pi k/n] \neq 0$. Moreover, as $\alpha > 1$, we obtain that $\rho [(2r + 1)\pi/n] = 0$. Therefore, since $\rho(\theta)$ changes between 0 and 1, it follows that $\rho(\theta)$ is an increasing function on the interval $[(2r + 1)\pi/n, 2\pi p/k]$.

**References**


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