The Calculation of Represenations of the Group of Overhand Knot

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Abstract

This paper is a continuation of earlier Works of the author [1,2]. The Group of overhand knot is given [2] by

$$G = |x, y : xyx = yxy|$$

The symmetric group of degree four is denoted by S_4 and is given as follows [4] :

$$S_4 = |a, b : a^3 = b^4 = (ab)^2 = 1|$$

where $a = (234)$ and $b = (243)(12)$ Here, the complete classification of G in S_4 is given. These calculations are mostly obtained in [6]. These representations are divided into eight non-equivalent classes. For each class a representative homomorphism is considered and the corresponding fundamental group of 4-sheeted covering space of the overhand knot has been calculated by the algorithm of Professor Fox.

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1. Preliminaries

A planer normal projection of the overhand knot K is given [6]. The fundamental group of the complement of K in 3-sphere, i.e. $G = \pi(S^3 - K)$, is called the group of overhand knot. By means of Wirtinger's method G can be calculated as

$$G = |x, y : x^3 = y^4 = (xy)^2 = 1|$$

As it is well known the group G is an infinite group of deficiency 1. The symmetric group of order 24 is given as [4]

$$S_4 = |x, y : x^3 = y^4 = (xy)^2 = 1|$$

where $x = (234)$ and $y = (243)(12)$. To find representations of G in S_4 depends on the following theorem [9, p.11]

Theorem. If $|a, b, c, \dots : P, Q, R, \dots|$ is a presentation of a group G and if $x, y, z, \dots$ are elements of a group H such that $P(x, y, z, \dots), Q(x, y, z, \dots),$

$R(x, y, z, \dots), \dots$ is each the identity element in H , then the group G can be mapped homomorphically into H under the obvious homomorphism.

Two homomorphisms $f, g : G \rightarrow S_4$ are called equivalent if there exists a permutation $p \in S_4$ such that for all $w \in G$, $f(w) = p.g(w).p^{-1}$.

2. Representation of G in S_4 , $h : G \rightarrow S_4$

The group of overhand knot is given [2] by

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Class 1: Cyclic homomorphisms[6]

$$h(x) = h(y) = (1234)$$

$$h(x) = h(y) = (1243)$$

$$h(x) = h(y) = (1324)$$

$$h(x) = h(y) = (1342)$$

$$h(x) = h(y) = (1423)$$

$$h(x) = h(y) = (1432)$$

Class 2: Almost cyclic homomorphisms[6]

$$h(x) = (1234), h(y) = (1243)$$

$$h(x) = (1243), h(y) = (1234)$$

$$h(x) = (1243), h(y) = (1432)$$

$$h(x) = (1432), h(y) = (1243)$$

$$h(x) = (1234), h(y) = (1342)$$

$$h(x) = (1342), h(y) = (1234)$$

$$h(x) = (1234), h(y) = (1423)$$

$$h(x) = (1423), h(y) = (1234)$$

$$h(x) = (1234), h(y) = (1324)$$

$$h(x) = (1324), h(y) = (1234)$$

$$h(x) = (1243), h(y) = (1324)$$

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$$h(x) = (1243), h(y) = (1423)$$

$$h(x) = (1423), h(y) = (1243)$$

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$$h(x) = (1342), h(y) = (1432)$$

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$$h(x) = (1423), h(y) = (1342)$$

$$h(x) = (1324), h(y) = (1432)$$

$$h(x) = (1432), h(y) = (1324)$$

$$h(x) = (1432), h(y) = (1423)$$

$$h(x) = (1423), h(y) = (1432)$$

Class 3 : Interesting homomorphisms[6]

$$h(x) = (123)(4), h(y) = (142)(3)$$

$$h(x) = (123)(4), h(y) = (134)(2)$$

$$h(x) = (123)(4), h(y) = (243)(1)$$

$$h(x) = (132)(4), h(y) = (124)(3)$$

$$h(x) = (132)(4), h(y) = (124)(3)$$

$$h(x) = (132)(4), h(y) = (143)(2)$$

$$h(x) = (124)(3), h(y) = (234)(1)$$

$$h(x) = (124)(3), h(y) = (143)(2)$$

$$h(x) = (124)(3), h(y) = (132)(4)$$

$$h(x) = (142)(3), h(y) = (243)(1)$$

$$h(x) = (142)(3), h(y) = (134)(2)$$

$$h(x) = (142)(3), h(y) = (132)(4)$$

$$h(x) = (134)(2), h(y) = (243)(1)$$

$$h(x) = (134)(2), h(y) = (142)(3)$$

$$h(x) = (134)(2), h(y) = (123)(4)$$

$$h(x) = (143)(2), h(y) = (243)(1)$$

$$h(x) = (143)(2), h(y) = (124)(3)$$

$$h(x) = (143)(2), h(y) = (132)(4)$$

$$h(x) = (234)(1), h(y) = (143)(2)$$

$$h(x) = (234)(1), h(y) = (124)(3)$$

$$h(x) = (234)(1), h(y) = (132)(4)$$

$$h(x) = (243)(1), h(y) = (134)(2)$$

$$h(x) = (243)(1), h(y) = (142)(3)$$

$$h(x) = (243)(1), h(y) = (123)(4)$$

Note : $h(x) = (1)(234), h(y) = (2)(134)$ is not a representation of G in S_4 .
For, $(xyx\bar{y}\bar{x}\bar{y})$ can not be map onto identity permutation under h [6].

Class 4 : The trivial homomorphism[6]

$$h(x) = h(y) = (1)(2)(3)(4)$$

Class 5[6]

$$h(x) = h(y) = (12)(34)$$

$$h(x) = h(y) = (13)(24)$$

$$h(x) = h(y) = (14)(23)$$

Class 6 : Three cyclic homomorphisms[6]

$$h(x) = h(y) = (123)(4)$$

$$h(x) = h(y) = (132)(4)$$

$$h(x) = h(y) = (124)(3)$$

$$h(x) = h(y) = (142)(3)$$

$$h(x) = h(y) = (134)(2)$$

$$h(x) = h(y) = (143)(2)$$

$$h(x) = h(y) = (234)(1)$$

$$h(x) = h(y) = (243)(1)$$

Class 7 : Two cyclic homomorphisms[6]

$$h(x) = h(y) = (12)$$

$$h(x) = h(y) = (13)$$

$$h(x) = h(y) = (14)$$

$$h(x) = h(y) = (23)$$

$$h(x) = h(y) = (24)$$

$$h(x) = h(y) = (34)$$

Class 8 : Irregular homomorphisms. The corresponding branched covering space of overhand knot is always the three sphere [6].

$$h(x) = (12)(3), h(y) = (13)(2)$$

$$h(x) = (13)(2), h(y) = (12)(3)$$

$$h(x) = (13)(2), h(y) = (23)(1)$$

$$h(x) = (23)(1), h(y) = (13)(2)$$

$$h(x) = (23)(1), h(y) = (12)(3)$$

$$h(x) = (12)(3), h(y) = (23)(1)$$

$$h(x) = (12)(4), h(y) = (14)(2)$$

$$h(x) = (14)(2), h(y) = (12)(4)$$

$$h(x) = (14)(2), h(y) = (24)(1)$$

$$h(x) = (24)(1), h(y) = (14)(2)$$

$$h(x) = (24)(1), h(y) = (12)(4)$$

$$h(x) = (12)(4), h(y) = (24)(1)$$

$$h(x) = (13)(4), h(y) = (34)(1)$$

$$h(x) = (34)(1), h(y) = (13)(4)$$

$$h(x) = (13)(4), h(y) = (14)(3)$$

$$h(x) = (14)(3), h(y) = (13)(4)$$

$$h(x) = (14)(3), h(y) = (34)(1)$$

$$h(x) = (34)(1), h(y) = (14)(3)$$

$$h(x) = (24)(3), h(y) = (23)(4)$$

$$h(x) = (23)(4), h(y) = (24)(3)$$

$$h(x) = (34)(2), h(y) = (24)(3)$$

$$h(x) = (24)(3), h(y) = (34)(2)$$

$$h(x) = (34)(2), h(y) = (23)(4)$$

$$h(x) = (23)(4), h(y) = (34)(2)$$

3. Examples

In the above classification only the examples of classes 1,2,3 are considered here. The others are related to disconnected coverings and some of them are already studied in [6].

a. Cyclic four sheeted covering space of Overhand Knot

$$h : G \rightarrow S_4, h(x) = h(y) = (1234),$$

$$\begin{array}{cccccc} x & y & x & , & \bar{y} & \bar{x} & \bar{y} \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 1 & 4 & 3 & 2 \\ 3 & 4 & 1 & 2 & 1 & 4 & 3 \\ 4 & 1 & 2 & 3 & 2 & 1 & 4 \end{array}$$

Let $\sum$ be the branched covering related to this homomorphism. If L is the branch curve over K in $\sum$, then $\sum - L$ is a four sheeted cyclic covering space of $S^3 - K$. By Fox algorithm and from the above table the free product group $\pi(\sum - K) * F_3$ is calculated as follows

$$\pi(\sum - K) * F_3 = \left| \begin{array}{l} x_1, y_1 : x_1 y_2 x_3 \bar{y}_3 \bar{x}_2 \bar{y}_1 \\ x_2, y_2 : x_2 y_3 x_4 \bar{y}_4 \bar{x}_3 \bar{y}_2 \\ x_3, y_3 : x_3 y_4 x_1 \bar{y}_1 \bar{x}_4 \bar{y}_3 \\ x_4, y_4 : x_4 y_1 x_2 \bar{y}_2 \bar{x}_1 \bar{y}_4 \end{array} \right|$$

where F_3 is a free group of rank 3.[1,6,7,8].

As generators of F_3 we choose y_2, y_3, y_4 and adjoining $y_2 = y_3 = y_4 = 1$ to the above group we get

$$\pi(\sum -K) = \left| \begin{array}{l} x_1, x_1, x_3 : x_1 x_3 \bar{x}_2 \bar{y}_1, x_2 x_4 \bar{x}_3 \\ x_4, y_1 : x_3 x_1 \bar{y}_1 \bar{x}_4, x_4 y_1 x_2 \bar{x}_1 \end{array} \right|$$

The branch relation is $y_1 y_2 y_3 y_4 = 1$ (or $x_1 x_2 x_3 x_4 = 1$) which reduces to $y_1 = 1$. Upon adjoining $y_1 = 1$ to $\pi(\sum -K)$ we get

$$\pi(\sum) = |x_i : x_{i+2} \bar{x}_{i+1}, i = 1, 2, 3, 4(\text{mod } 4)|.$$

$x_4 = x_3 x_1, x_2 = x_1 x_3, x_1 = a$ and $x_3 = b$ are used to get

$$\pi(\sum) = |a, b : ab^2 a \bar{b}, ba^2 b \bar{a}|$$

Then, we obtain $H_1(\sum) = |a, b : b = a^{-2}, a = b^{-2}|$ or $H_1(\sum) = |a : a^3|$.

b. Cyclic four sheeted almost cyclic covering of Overhand Knot[6,7,8]

$$h : G \rightarrow S_4, h(x) = (1234), h(y) = (1243).$$

$$\begin{array}{cccccc} x & y & x & , & \bar{y} & \bar{x} & \bar{y} \\ 1 & 2 & 4 & 1 & 3 & 2 & 1 \\ 2 & 3 & 1 & 2 & 1 & 4 & 2 \\ 3 & 4 & 3 & 4 & 2 & 1 & 3 \\ 4 & 1 & 2 & 3 & 4 & 3 & 4 \end{array}$$

Let Ω be the branched covering space related to this homomorphism. Let M be the branch curve over Overhand knot. By calculations similar to those above we obtain

$$\Omega(\sum -M) * F_3 = \left| \begin{array}{l} x_1, y_1 : x_1 y_2 x_4 \bar{y}_3 \bar{x}_2 \bar{y}_1 \\ x_2, y_2 : x_2 y_3 x_1 \bar{y}_1 \bar{x}_4 \bar{y}_2 \\ x_3, y_3 : x_3 y_4 x_3 \bar{y}_2 \bar{x}_1 \bar{y}_3 \\ x_4, y_4 : x_4 y_1 x_2 \bar{y}_4 \bar{x}_3 \bar{y}_4 \end{array} \right|$$

Choosing x_1, x_3, x_4 as generators of F_3 and adding $x_1 = x_3 = x_4 = 1$ to the above group we obtain

$$\Omega(\sum -M) = \left| \begin{array}{l} x_1, y_1, y_2 : x_1 y_2 \bar{y}_3 \bar{y}_1, y_3 x_1 \bar{y}_1 \bar{y}_2 \\ y_3, y_4 : y_4 \bar{y}_2 \bar{x}_1 \bar{y}_3, y_1 \bar{y}_4^2 \end{array} \right|$$

The branch relation is $x_1 x_2 x_3 x_4 = 1$, i.e. $x_1 = 1$ (or $y_1 y_2 y_3 y_4 = 1$). When we add $x_1 = 1$ to $\pi(\Omega - M)$, we get [9]

$$\pi(\Omega) = \left| y_1, y_2, y_3, y_4 : y_2 \bar{y}_3 \bar{y}_1, y_3 \bar{y}_1 \bar{y}_2, y_4 \bar{y}_2 \bar{y}_3, y_1 \bar{y}_4^2 \right|.$$

Using $y_2 = \bar{y}_3 y_4$ and $y_1 = y_4^2$

$$\pi(\Omega) = \left| y_3, y_4 : \bar{y}_3 y_4 \bar{y}_3 \bar{y}_4^2, y_3 \bar{y}_4^2 \bar{y}_4 y_3 \right|.$$

or putting $y_4 = a$ and $y_3 = b$ we obtain

$$\pi(\Omega) = \left| a, b : \bar{a} b \bar{a} = b^2, a^2 = b^3 \right|.$$

From this we get [9]

$$H_1(\Omega) = \left| a, b : b = a^{-2}, a^2 = b^3 \right| = \left| a : a^8 \right| = Z_8.$$

c. An interesting four sheeted covering of overhand knot [6,7,8,9,2,3]

$$h : G \rightarrow S_4, h(x) = (123)(4), h(y) = (142)(3).$$

$$\begin{array}{cccccc} x & y & x & , & \bar{y} & \bar{x} & \bar{y} \\ 1 & 2 & 1 & 2 & 4 & 4 & 1 \\ 2 & 3 & 3 & 1 & 2 & 1 & 2 \\ 3 & 1 & 4 & 4 & 1 & 3 & 3 \\ 4 & 4 & 2 & 3 & 3 & 2 & 4 \end{array}$$

Let Ψ and N be the related branched covering and the branch curve over K respectively. Similar calculations lead to the following results:

$$\Pi(\Psi - N) * F_3 = \left| \begin{array}{l} x_1, y_1 : x_1 y_2 x_1 \bar{y}_4 \bar{x}_4 \bar{y}_1 \\ x_2, y_2 : x_2 y_3 x_3 \bar{y}_2 \bar{x}_1 \bar{y}_2 \\ x_3, y_3 : x_3 y_1 x_4 \bar{y}_1 \bar{x}_3 \bar{y}_3 \\ x_4, y_4 : x_4 y_4 x_2 \bar{y}_3 \bar{x}_2 \bar{y}_4 \end{array} \right|$$

$$\Pi(\Psi - N) = \left| \begin{array}{l} x_1, y_4, y_1 : x_1 y_2 x_1 \bar{x}_4 \bar{y}_1, y_3 \bar{y}_2 \bar{x}_1 \bar{y}_2 \\ y_2, y_3 : y_1 x_4 \bar{y}_1 \bar{y}_3, x_4 \bar{y}_3 \end{array} \right|,$$

$$\pi(\Psi) = |x, y : (xy)^3 (yx)^{-3}|.$$

Hence $H_1(\Psi) = 0$. [9]

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