Combinatorial Proofs of Bivariate Generating Functions on Coxeter Groups

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Abstract

MacMahon's result that states the *length* and *major index* statistics are equidistributed on the symmetric group S_n has generalizations to other Coxeter groups. Adin, Brenti and Roichman have defined analogues of those statistics for the hyperoctahedral group, B_n and proved equidistribution theorems. Using combinatorial interpretations of Regev and Roichman's statistics *length* and *delent*, we consider a specialization of the bimahonian generating function for the alternating group A_n . We define a new delent statistic, del_L , for the alternating subgroup of the hyperoctahedral group, $L_n \subset B_n$ and give a combinatorial proof of the bimahonian generating function $\sum_{\pi \in L_n} q^{l_L(\pi)} t^{del_L(\pi)}$.

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1 Introduction

MacMahon's classic theorem states that the *inversion* and *major index* statistics are equidistributed over the symmetric group S_n . Generalizations of this theorem have been given for the hyperoctahedral group, the group of signed permutations, by Adin, Brenti and Roichman using the flag-inversion statistic and the flag-major index statistic. Regev and Roichman defined the delent statistic and used it to give a refinement of MacMahon's result for the alternating group A_n . Bernstein has given an equidistribution result for statistics on

both the group of signed, even permutations L_n and the group of even-signed even permutations. The main results of our paper are to specialize $q = t = -1$ in two bimahonian distribution theorems of Regev and Roichman and to give a sign-reversing involution to prove the results. We then define a delent statistic for L_n and give a bimahonian generating function for the length and delent statistics.

In Section 2, we give the relevant background and definitions for the results on the symmetric group and the alternating group. In Section 3 we specialize $q = t = -1$ in the two theorems of Regev and Roichman and construct a sign-reversing involution. We describe statistics on signed permutations in Section 4 and give a new definition of the delent statistic on L_n . In Section 5 we give a closed formula for the distribution of the length and delent statistics on L_n .

2 Background and Definitions

The symmetric group S_n can be generated by the set of Coxeter generators $\{s_1, s_2, \dots, s_{n-1}\}$ where $s_i = (i \ i+1)$. This means that for $\sigma \in S_n$, σs_i interchanges the elements in positions i and $i+1$ and $s_i \sigma$ interchanges the elements i and $i+1$. (See [5] for a thorough exposition of Coxeter generators.) Every permutation in S_n can be written as a product of the s_i 's and the minimum number of generators required to express a permutation π is called the **length** of π and is written $l(\pi)$. The canonical presentation for a permutation π is the expression for π in terms of the minimum number of Coxeter generators. The alternating group A_n is the group of even permutations in S_n , i.e. those permutations which can be written as a product of an even number of s_i 's.

Definition 2.1. For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, define an **inversion** to be a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. The **inversion statistic**, $\text{inv}(\pi)$, is defined as the total number of inversions in π .

It is well-known that for any permutation π in S_n , $l(\pi) = \text{inv}(\pi)$.

Regev and Roichman [9] define the **delent** statistic for a permutation π in S_n as the number of times the generator s_1 appears in the canonical presentation of π , written $\text{dels}(\pi)$. They go on to give the following combinatorial interpretation of this statistic:

Definition 2.2. Let $\pi \in S_n$. Then j is a **left-to-right minimum** of π if $\pi_i > \pi_j$ for all $1 \leq i < j$. Let $\text{Dels}(\pi) = \{1 < j \leq n \mid j \text{ is a left-to-right minimum}\}$. Then the **delent** statistic $\text{dels}(\pi) = |\text{Dels}(\pi)|$.

For example, for the permutation $\pi = 5 \ 4 \ 3 \ 8 \ 2 \ 9 \ 1 \ 7 \ 6$, $\text{inv}(\pi) = 18$ and $\text{dels}(\pi) = 4$.

Mitsuhashi [8] described a useful generating set for the alternating group A_n . Let $a_i = s_1 s_{i+1}$ so $a_i = a_i^{-1}$ for $i \geq 2$ and $a_1^2 = a_1^{-1}$. As in S_n , for a

permutation $\sigma \in A_n$, the **length in A_n** of σ , written $l_A(\sigma)$, is equal to the number of generators in the canonical presentation for σ using the generating set $\{a_1, a_2, \dots, a_{n-1}\}$. In addition Regev and Roichman show that $l_A(\sigma) = l_S(\sigma) - \text{del}_S(\sigma) = \text{inv}(\sigma) - \text{del}_S(\sigma)$.

For $\sigma \in A_n$, the **A-delent statistic** of σ , written $\text{del}_A(\sigma)$, is the number of times a_1 or a_1^{-1} appears in the canonical presentation for σ . The del_A statistic also has a combinatorial interpretation:

Definition 2.3. Let $\sigma \in A_n$. Then $2 < j \leq n$ is an **almost left-to-right minimum** of σ if $\sigma_i < \sigma_j$ for at most one $i < j$. Let $\text{Del}_A(\pi) = \{2 < j \leq n \mid j \text{ is an almost left-to-right minimum}\}$. Then the **A-delent statistic** $\text{del}_A(\pi) = |\text{Del}_A(\pi)|$.

For example, for the permutation $\pi = 5 \ 4 \ 2 \ 8 \ 3 \ 9 \ 1 \ 6 \ 7$ the left-to-right minima are 2, 3, 1, so $\text{del}_A(\pi) = 3$.

Let O_n be the set of odd permutations in S_n . The odd permutations do not form a group, since they do not contain the identity and are not closed. For any element $\omega \in O_n$ define the **length in O_n** of ω , $l_O(\omega) = l_S(\omega) - \text{del}_S(\omega)$.

Definition 2.4. Let $\omega \in O_n$. Then the **O-delent statistic** of ω , $\text{del}_O(\omega)$, is the number of almost left-to-right minima in ω .

3 Bivariate Generating Functions for S_n , A_n and O_n

Regev and Roichman proved the following two theorems:

Theorem 3.1. (Regev and Roichman, 2004)

$$\sum_{\pi \in S_n} q^{l_S(\pi)} t^{\text{del}_S(\pi)} = (1 + qt)(1 + q + q^2t) \cdots (1 + q + q^2 + \cdots + q^{n-1}t).$$

Theorem 3.2. (Regev and Roichman, 2004)

$$\sum_{\sigma \in A_{n+1}} q^{l_A(\sigma)} t^{\text{del}_A(\sigma)} = (1 + 2qt)(1 + q + 2q^2t) \cdots (1 + q + q^2 + \cdots + 2q^{n-1}t).$$

In this section we specialize $q = t = -1$ in Theorems 3.1 and 3.2 and give a combinatorial proof of the results.

Theorem 3.3.

$$\sum_{\pi \in S_{2n}} (-1)^{\text{inv}(\pi)} (-1)^{\text{del}_S(\pi)} = (-1)^{n-1} 2^n$$

and

$$\sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)} (-1)^{del_S(\pi)} = (-1)^n 2^n$$

Proof. We will first prove the result for $\pi \in S_{2n}$. We will use induction on n and construct a sign-reversing involution ϕ which will have 2^n signed fixed points.

For $n = 1$, there are only two permutations in S_2 , $\pi_1 = 12$ and $\pi_2 = 21$. Since $inv(\pi_1) = 0$ and $del_S(\pi_1) = 0$, $(-1)^{inv(\pi_1)}(-1)^{del_S(\pi_1)} = 1$. Also, since $inv(\pi_2) = 1$ and $del_S(\pi_2) = 1$, $(-1)^{inv(\pi_2)}(-1)^{del_S(\pi_2)} = 1$, thus the sum is $2 = (-1)^{(1-1)}2^1$.

Now we assume the result is true for n and we will prove it is true for $n+1$. Let $\pi \in S_{2n+2}$. Consider the two disjoint cases based on the location of $2n+1$ and $2n+2$ in π .

Case 1: Suppose $2n+1$ and $2n+2$ are in non-adjacent positions and without loss of generality assume $2n+1$ appears before $2n+2$ in π . Thus

$$\pi = \pi_1 \ \pi_2 \ \cdots \ \pi_{i-1} \ 2n+1 \ \pi_{i+1} \ \cdots \ \pi_{j-1} \ 2n+2 \ \pi_{j+1} \ \cdots \ \pi_{2n+2}.$$

Now form $\phi(\pi)$ by interchanging $2n+1$ and $2n+2$ in π so

$$\phi(\pi) = \pi_1 \ \cdots \ \pi_{i-1} \ 2n+2 \ \pi_{i+1} \ \cdots \ \pi_{j-1} \ 2n+1 \ \pi_{j+1} \ \cdots \ \pi_{2n+2}.$$

Any inversion in π between the elements 1 through $2n$ also exists as an inversion in $\phi(\pi)$ since these elements appear in the same order in both permutations. In π , both $2n+1$ and $2n+2$ form an inversion with each of the elements π_{j+1} through π_{2n+2} . In addition, $2n+1$ forms an inversion with each of the elements π_{i+1} through π_{j-1} . In $\phi(\pi)$, both $2n+1$ and $2n+2$ form an inversion with each of the elements π_{j+1} through π_{2n+2} . The number $2n+1$ no longer forms inversions with the elements π_{i+1} through π_{j-1} , but $2n+2$ now forms an inversion with each of these elements. There is one additional inversion in $\phi(\pi)$ formed between $2n+1$ and $2n+2$, thus the total number of inversions in $\phi(\pi)$ is one greater than the number of inversions in π .

Since $2n+1$ and $2n+2$ are larger than all other elements in π and $\phi(\pi)$, they will not affect any left-to-right minima that exist among π_1 through π_{i-1} , π_{i+1} through π_{j-1} , or π_{j+1} through π_{2n+2} . In addition, since $2n+1$ and $2n+2$ are non-adjacent, neither can be a left-to-right minimum so the total number of left-to-right minima in π and $\phi(\pi)$ is unchanged. Thus $del_S(\pi) = del_S(\phi(\pi))$.

Thus it is clear that ϕ is a sign-reversing involution on permutations with $2n+1$ and $2n+2$ in non-adjacent positions.

Case 2: Suppose that $2n+1$ and $2n+2$ are in adjacent positions in S_{2n+2} and without loss of generality assume $2n+1$ occurs before $2n+2$. Then we

will construct $\phi(\pi)$ inductively. Let π be as follows.

$$\pi = \pi_1 \cdots \pi_{j-1} \ 2n+1 \ 2n+2 \ \pi_{j+2} \cdots \pi_{2n+2}.$$

Form $\phi(\pi)$ by swapping $2n+1$ and $2n+2$ so

$$\phi(\pi) = \pi_1 \cdots \pi_{j-1} \ 2n+2 \ 2n+1 \ \pi_{j+2} \cdots \pi_{2n+2}.$$

Any inversion formed between elements 1 through $2n$ that occur in π also occurs in $\phi(\pi)$ since the order of these elements remains the same in both permutations. The elements $2n+1$ and $2n+2$ form inversions with all of the elements π_{j+2} through π_{2n+2} in both π and $\phi(\pi)$. However, in $\phi(\pi)$ there is one additional inversion between $2n+2$ and $2n+1$ so the total number of inversions in $\phi(\pi)$ is one greater than the number of inversions in π .

We now consider two subcases:

Subcase 2a: If $j \neq 1$, there is at least one element π_{j-1} that is less than both $2n+1$ and $2n+2$ and to the left of both, so neither $2n+1$ nor $2n+2$ can be a left-to-right minimum. Since $2n+1$ and $2n+2$ are larger than all other elements in π , any left-to-right minima that exist in π also exist in $\phi(\pi)$. Then in this case, the number of left-to-right minima in π and $\phi(\pi)$ are the same, so $del_S(\pi) = del_S(\phi(\pi))$ and thus ϕ is a sign-reversing involution on permutations of this type.

Subcase 2b: If $j = 1$, then $2n+1$ is a new left-to-right minima in σ that did not exist in π so $del_S(\pi) + 1 = del_S(\sigma)$.

In this case, let $\omega = \pi_3 \pi_4 \cdots \pi_{2n+2}$. Note that $\omega \in S_{2n}$ so we can apply ϕ to ω by induction. If ω is not a fixed point of ϕ , then $\phi(\omega) \in S_{2n}$ is the element of opposite parity that is paired with ω . Then $\pi = 2n+1 \ 2n+2 \ \omega$ and $\phi(\pi) = 2n+1 \ 2n+2 \ \phi(\omega)$ are in S_{2n+2} and have opposite signs. In addition, $\sigma = 2n+2 \ 2n+1 \ \omega$ and $\phi(\sigma) = 2n+2 \ 2n+1 \ \phi(\omega)$ are in S_{2n+2} and also have opposite signs.

If ω is a fixed point of the sign-reversing involution on S_{2n} , then both $\pi = 2n+1 \ 2n+2 \ \omega$ and $\sigma = 2n+2 \ 2n+1 \ \omega$ will be defined as fixed points of the sign-reversing involution on S_{2n+2} . Both $2n+1$ and $2n+2$ will form inversions with all of the $2n$ elements in ω so the number of inversions in π will be $2n$ greater than the number of inversions in ω . The number of left-to-right minima in π will be one greater than the number of left-to-right minima in ω since there is a new left-to-right minimum in position 3. Thus π and ω have opposite parity (and therefore σ and ω also have opposite parity since π and σ have the same parity). The number of fixed points of ϕ in S_{2n} is 2^n by induction with sign $(-1)^{n-1}$ and we have shown that each fixed point in S_{2n} gives rise to two fixed points in S_{2n+2} , of opposite parity as those in S_{2n} so the number of fixed points in S_{2n+2} is $(-1)^{(n-1)}2^n(-1)2 = (-1)^n2^{(n+1)}$.

The proof for elements in S_{2n+1} is similar and we omit the details. $\square$

We now specialize $q = t = -1$ in Theorem 3.2 and construct a sign-reversing involution ϕ that proves the result.

Theorem 3.4.

$$\sum_{\pi \in A_{2n}} (-1)^{l_A(\pi)} (-1)^{del_A(\pi)} = (-1)^{n-1} 2^{(n-1)} 3^{(n-1)}$$

and

$$\sum_{\pi \in A_{2n+1}} (-1)^{l_A(\pi)} (-1)^{del_A(\pi)} = (-1)^{(n-1)} 2^{(n-1)} 3^n$$

Proof. We will again prove the result by induction on n . If $n = 2$, there are 12 permutations in A_4 . These 12 permutations are 1234, 1342, 1423, 2143, 2314, 2431, 3124, 3241, 3412, 4132, 4213, 4321. We will construct ϕ by swapping the two largest (3 and 4) and the two smallest (1 and 2) elements in the permutation, except in those cases where this does not result in a sign change. Then $\phi(1234) = 2143$, $\phi(2314) = 1423$, and $\phi(3124) = 4213$. The fixed points are then defined as 1342, 2431, 3241, 3412, 4132 and 4321, each with negative sign, thus our sum is equal to -6 .

Let $\pi \in A_{2n+2}$ then

$$\pi = \omega_1 \quad \omega_2 \quad \cdots \quad \omega_{i-1} \quad 2n+1 \quad \omega_i \quad \cdots \quad \omega_{j-2} \quad 2n+2 \quad \omega_{j-1} \quad \cdots \quad \omega_{2n}$$

and let

$$\sigma = \tilde{\omega}_1 \quad \tilde{\omega}_2 \quad \cdots \quad \tilde{\omega}_{i-1} \quad 2n+2 \quad \tilde{\omega}_i \quad \cdots \quad \tilde{\omega}_{j-2} \quad 2n+1 \quad \tilde{\omega}_{j-1} \quad \cdots \quad \tilde{\omega}_{2n}$$

where $\tilde{\omega}$ is the permutation in S_{2n} formed by interchanging 1 and 2 in ω , i.e. $\tilde{\omega} = (12)\omega$. Since σ differs from π by two transpositions, $\sigma \in A_{2n+2}$.

If π and σ have opposite sign, then define $\phi(\pi) = \sigma$. It is not difficult to check that there are three cases in which π and σ have the same sign.

Case 1: Suppose π and σ have $2n+1$ and $2n+2$ in positions 1 and 3. Without loss of generality assume

$$\pi = 2n+1 \quad \omega_1 \quad 2n+2 \quad \omega_2 \quad \cdots \quad \omega_{2n}$$

and

$$\sigma = 2n+2 \quad \tilde{\omega}_1 \quad 2n+1 \quad \tilde{\omega}_2 \quad \cdots \quad \tilde{\omega}_{2n}.$$

Since $\tilde{\omega} = (12)\omega$ either $\omega \in A_{2n}$ or $\tilde{\omega} \in A_{2n}$. Suppose $\omega \in A_{2n}$. If ω was not a fixed point of ϕ on A_{2n} then let $\mu = \phi(\omega)$ and let $\tilde{\mu} = (12)\mu$. If $\tilde{\omega} \in A_{2n}$ and is not a fixed point of ϕ on A_{2n} then let $\tilde{\mu} = \phi(\tilde{\omega})$ and let $\mu = (12)\tilde{\mu}$. Then we can define $\phi(\pi) = 2n+1 \mu_1 \ 2n+2 \mu_2 \ \cdots \ \mu_{2n}$ and $\phi(\sigma) = 2n+2 \tilde{\mu}_1 \ 2n+1 \ \tilde{\mu}_2 \ \cdots \ \mu_{2n}$.

If either ω or $\tilde{\omega}$ is a fixed point of the sign reversing involution on A_{2n} then π and σ will be two new fixed points of ϕ on A_{2n+2} . In π , $2n+1$ and $2n+2$ each form inversions with $\omega_2, \omega_3, \dots, \omega_{2n}$ and $2n+1$ also forms an inversion with ω_1 so π has $2(2n-1)+1 = 4n-1$ more inversions than ω . As compared to ω , π has an additional left-to-right minima at ω_1 and a new almost left-to-right minima at ω_2 . Thus the parity of $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ (which is the same as the parity of $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$) is different than the parity of $(-1)^{l_A(\omega)}(-1)^{del_A(\omega)}$, thus the two new fixed points of the sign reversing involution on A_{2n+2} have opposite parity as ω .

Case 2: Suppose π and σ have $2n+1$ and $2n+2$ in positions 1 and 2. Without loss of generality assume

$$\pi = 2n+1 \quad 2n+2 \quad \omega_1 \quad \omega_2 \quad \cdots \quad \omega_{2n}$$

and

$$\sigma = 2n+2 \quad 2n+1 \quad \tilde{\omega}_1 \quad \tilde{\omega}_2 \quad \cdots \quad \tilde{\omega}_{2n}.$$

where $\tilde{\omega}$ is the permutation in S_{2n} formed by interchanging 1 and 2 in ω , i.e. $\tilde{\omega} = (12)\omega$. Since σ differs from π by two transpositions, $\sigma \in A_{2n+2}$.

Since $\tilde{\omega} = (12)\omega$ either $\omega \in A_{2n}$ or $\tilde{\omega} \in A_{2n}$. Suppose $\omega \in A_{2n}$. If ω was not a fixed point of ϕ on A_{2n} then let $\mu = \phi(\omega)$ and let $\tilde{\mu} = (12)\mu$. If $\tilde{\omega} \in A_{2n}$ and is not a fixed point of ϕ on A_{2n} then let $\tilde{\mu} = \phi(\tilde{\omega})$ and let $\mu = (12)\tilde{\mu}$. Then we can define $\phi(\pi) = 2n+1 \quad 2n+2 \quad \mu_1 \quad \mu_2 \quad \cdots \quad \mu_{2n}$ and $\phi(\sigma) = 2n+2 \quad 2n+1 \quad \tilde{\mu}_1 \quad \tilde{\mu}_2 \quad \cdots \quad \tilde{\mu}_{2n}$.

If either ω or $\tilde{\omega}$ is a fixed point of ϕ on A_{2n} , then both $\pi = 2n+1 \quad 2n+2 \quad \omega$ and $\sigma = 2n+2 \quad 2n+1 \quad \tilde{\omega}$ will be fixed points of the sign-reversing involution on A_{2n+2} . Both $2n+1$ and $2n+2$ will form inversions with all of the $2n$ elements in ω so the number of inversions in π will be $2n$ greater than the number of inversions in ω . The number of left-to-right minima in π will be one greater than the number of left-to-right minima in ω since there is a new left-to-right minimum in position 3. The number of almost left-to-right minima in π will be two greater than the number of almost left-to-right minima in ω since ω_1 and ω_2 are new almost left-to-right minima in π . So, $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\omega)}(-1)^{del_A(\omega)}$ have opposite signs.

In σ , both $2n+1$ and $2n+2$ will form inversions with all of the $2n$ elements in $\tilde{\omega}$ plus there is an additional inversion between $2n+1$ and $2n+2$, so the number of inversions in σ will be $2n+1$ greater than the number of inversions in $\tilde{\omega}$. The number of left-to-right minima in σ will be two greater than the number of left-to-right minima in $\tilde{\omega}$ since there are two new left-to-right minima in positions 2 and 3. The number of almost left-to-right minima in σ will be two greater than the number of almost left-to-right minima in $\tilde{\omega}$ since $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are new almost left-to-right minima in σ . So, $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ and $(-1)^{l_A(\tilde{\omega})}(-1)^{del_A(\tilde{\omega})}$ have opposite signs.

We have shown that in this case, each fixed point in A_{2n} gives rise to two fixed points in A_{2n+2} , of opposite parity.

Case 3: Suppose π and σ have $2n+1$ and $2n+2$ in positions 2 and 3. Without loss of generality assume

$$\pi = \omega_1 \quad 2n+1 \quad 2n+2 \quad \omega_2 \quad \cdots \quad \omega_{2n}$$

and

$$\sigma = \tilde{\omega}_1 \quad 2n+2 \quad 2n+1 \quad \tilde{\omega}_2 \quad \cdots \quad \tilde{\omega}_{2n}.$$

where $\tilde{\omega}$ is the permutation in S_{2n} formed by interchanging 1 and 2 in ω , i.e. $\tilde{\omega} = (12)\omega$. Since σ differs from π by two transpositions, $\sigma \in A_{2n+2}$.

Since $\tilde{\omega} = (12)\omega$ either $\omega \in A_{2n}$ or $\tilde{\omega} \in A_{2n}$. Suppose $\omega \in A_{2n}$. If ω was not a fixed point of ϕ on A_{2n} then let $\mu = \phi(\omega)$ and let $\tilde{\mu} = (12)\mu$. If $\tilde{\omega} \in A_{2n}$ and is not a fixed point of ϕ on A_{2n} then let $\tilde{\mu} = \phi(\tilde{\omega})$ and let $\mu = (12)\tilde{\mu}$. Then we can define $\phi(\pi) = \mu_1 \quad 2n+1 \quad 2n+2 \quad \mu_2 \quad \cdots \quad \mu_{2n}$ and $\phi(\sigma) = \tilde{\mu}_2 \quad 2n+2 \quad 2n+1 \quad \tilde{\mu}_2 \quad \cdots \quad \mu_{2n}$.

If either ω or $\tilde{\omega}$ is a fixed point of ϕ on A_{2n} , then both

$\pi = \omega_1 \quad 2n+1 \quad 2n+2 \quad \omega_2 \cdots \omega_{2n}$ and $\sigma = \tilde{\omega}_1 \quad 2n+2 \quad 2n+1 \quad \tilde{\omega}_2 \cdots \tilde{\omega}_{2n}$ will be fixed points of the sign-reversing involution on A_{2n+2} . Both $2n+1$ and $2n+2$ will form inversions with all of the $2n-1$ elements $\omega_2, \omega_3 \cdots \omega_{2n}$ so the number of inversions in π will be $2(2n-1) = 4n-2$ greater than the number of inversions in ω . The number of left-to-right minima in π will be the same as the number of left-to-right minima in ω but the number of almost left-to-right minima in π will be one greater than the number of almost left-to-right minima in ω since ω_2 is a new almost left-to-right minimum in π . So, $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\omega)}(-1)^{del_A(\omega)}$ have opposite signs.

In σ , both $2n+1$ and $2n+2$ will form inversions with all of the $2n-1$ elements $\tilde{\omega}_2, \tilde{\omega}_3 \cdots \tilde{\omega}_{2n}$ plus there is an additional inversion between $2n+1$ and $2n+2$, so the number of inversions in σ will be $2(2n-1) + 1 = 4n-1$ greater than the number of inversions in $\tilde{\omega}$. The number of left-to-right minima in σ will be the same as the number of left-to-right minima in $\tilde{\omega}$. The number of almost left-to-right minima in σ will be two greater than the number of almost left-to-right minima in $\tilde{\omega}$ since $\tilde{\omega}_2$ and $2n+1$ are new almost left-to-right minima in σ . So, $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ and $(-1)^{l_A(\tilde{\omega})}(-1)^{del_A(\tilde{\omega})}$ have opposite signs.

We have shown that in this case, each fixed point in A_{2n} gives rise to two fixed points in A_{2n+2} , of opposite parity.

Finally, since the number of fixed points in A_{2n} is equal to $(-1)^{(n-1)}6^{(n-1)}$, the number of fixed points of A_{2n+2} is equal to $(-1)^{(n-1)}6^{(n-1)}(2+2+2)(-1) = (-1)^n 6^n$.

The proof for elements in A_{2n+1} is similar and we omit the details. $\square$

Surprisingly, even though O_n is not a group, we obtain the following interesting results about the generating function for the length and delent statistics on this set of permutations.

Theorem 3.5.

$$\begin{aligned} \sum_{\sigma \in O_n} q^{l_O(\sigma)} t^{del_O(\sigma)} &= \sum_{\sigma \in A_n} q^{l_A(\sigma)} t^{del_A(\sigma)} \\ &= (1 + 2qt)(1 + q + 2q^2t) \cdots (1 + q + q^2 + \cdots + 2q^{n-1}t) \end{aligned}$$

Proof. Let $\omega \in O_n$, then $s_1\omega = \sigma \in A_n$. We can write ω as $s_1\sigma$. The permutation ω differs from σ in that 1 and 2 are interchanged. If 1 appeared to the left of 2 in $\omega = s_1\sigma$, then the number of inversions in σ is one greater than the number of inversions in ω . In addition, the number of left to right minima in σ is one greater than the number in $\omega = s_1\sigma$.

Then

$$\begin{aligned} l_O(\omega) &= l_S(\omega) - del_S(\omega) \\ &= inv(\omega) - del_S(\omega) \\ &= inv(s_1\sigma) - del_S(s_1\sigma) \\ &= (inv(\sigma) - 1) - (del_S(\sigma) - 1) \\ &= inv(\sigma) - del_S(\sigma) \\ &= l_S(\sigma) - del_S(\sigma) \\ &= l_A(\sigma). \end{aligned}$$

If 1 appeared to the right of 2 in $\omega = s_1\sigma$, then the number of inversions in σ is one fewer than the number of inversions in ω . In addition, the number of left to right minima in σ is one fewer than the number in $\omega = s_1\sigma$.

Then

$$\begin{aligned} l_O(\omega) &= l_S(\omega) - del_S(\omega) \\ &= inv(\omega) - del_S(\omega) \\ &= inv(s_1\sigma) - del_S(s_1\sigma) \\ &= (inv(\sigma) + 1) - (del_S(\sigma) + 1) \\ &= inv(\sigma) - del_S(\sigma) \\ &= l_S(\sigma) - del_S(\sigma) \\ &= l_A(\sigma). \end{aligned}$$

Therefore $l_O(\omega) = l_A(\sigma)$. In either case swapping 1 and 2 in ω does not change the number of almost left to right minima in ω so $del_O(\omega) = del_A(\sigma)$. Thus $q^{l_O(\omega)} t^{del_O(\omega)} = q^{l_A(\sigma)} t^{del_A(\sigma)}$.

□

4 Permutation Statistics for B_n

The hyperoctahedral group B_n is the group of all signed permutations of order n . The elements of B_n can be viewed as elements $\pi = \pi_1\pi_2\cdots\pi_n$ of S_n where each π_i can be positive or negative. If π_i is negative then we will denote it by $\overline{\pi_i}$.

Definition 4.1. Let $\pi \in B_n$.

$$Neg(\pi) = \{i \in [n] | \pi(i) < 0\}$$

and

$$Neg(\pi^{-1}) = \{|\pi(i)| | i \in [n], \pi(i) < 0\}$$

Let $neg(\pi) = |Neg(\pi)|$. Given the ordering

$$-n < -(n-1) < \cdots < -1 < 1 < 2 < \cdots < n$$

the definition of the inversion statistic is the same for B_n as for S_n . The Coxeter generators for B_n are the set $\{s_1, s_2, \dots, s_{n-1}\}$ of generators for S_n plus the addition of s_0 which takes π_1 to $\overline{\pi_1}$ or $\overline{\pi_1}$ to π_1 . The **length** function for π in B_n , $l_B(\pi)$, is the minimum length in terms of the Coxeter generators for B_n . The usual inversion statistic on B_n is not the same as the length function in terms of the Coxeter generators for type B_n . Define

$$\overline{inv(\pi)} = \sum_{1 \leq j \leq n} \sum_{\substack{i < j \\ \overline{\pi_i} > \pi_j}} 1$$

Definition 4.2. The **flag-inversion** statistic for B_n is defined as

$$finv(\pi) = inv(\pi) + \overline{inv(\pi)} + neg(\pi).$$

It is also known that

$$finv(\pi) = inv(\pi) + \sum_{i \in Neg(\pi^{-1})} i.$$

The flag-inversion statistic for π is equivalent to $l_B(\pi)$. For example, for

$$\pi = \overline{6} \ 1 \ 3 \ 2 \ \overline{9} \ 7 \ 4 \ \overline{8} \ \overline{5}$$

we have $inv(\pi) = 17$, $\overline{inv(\pi)} = 24$ and $neg(\pi) = 4$ so $finv(\pi) = 45$. Bernstein [3] gave the following definition of the **delent** statistic for B_n .

Definition 4.3. Let $\pi \in B_n$. Then j is a **left-to-right minimum** of π if $\pi_i > \pi_j$ for all $1 \leq i < j$. Let $\mathbf{Del}_B(\pi) = \{1 < j \leq n \mid j \text{ is a left-to-right minimum}\}$. Then the **delent** statistic $\mathbf{del}_B(\pi) = |\mathbf{Del}_B(\pi)|$.

Bernstein also gave a generating set for L_n , the subgroup of B_n consisting of the signed, even permutations. The generating set is the set $\{a_1, a_2, \dots, a_{n-1}\}$ together with s_0 .

The **L-length** of $\pi \in L_n$ is defined as

$$l_L(\pi) := l_B(\pi) - \mathbf{del}_B(\pi) = \mathbf{inv}(\pi) - \mathbf{del}_B(\pi) + \sum_{i \in \mathbf{Neg}(\pi^{-1})} i.$$

This function l_L is NOT a length function with respect to any set of generators.

We define a new delent statistic on L_n in the following way:

Definition 4.4. Let $\pi \in L_n$. The **delent** statistic of L_n , $\mathbf{del}_L(\pi)$ is the number of almost left-to-right minima in π .

In addition, we will let M_n denote the set of signed odd permutations in B_n . Note that this set of permutations does not form a group. Any element π of M_n can be written as $(a \ b)\sigma$ where a and b are the two smallest elements of π and where σ is an element of L_n .

Definition 4.5. The length of an element of M_n , $l_M = \mathbf{inv}(\pi) - \mathbf{del}_B(\pi) + \sum_{i \in \mathbf{Neg}(\pi^{-1})} i$.

5 Bivariate Generating Functions for B_n , L_n and M_n

We can now give a nice closed formula for the joint distribution of the l_L and $\mathbf{del}_L$ statistics.

Theorem 5.1.

$$\sum_{\sigma \in L_n} q^{l_L(\sigma)} t^{\mathbf{del}_L(\sigma)}$$

$$= (1+q) \cdots (1+q^n)(1+2qt)(1+q+2q^2t) \cdots (1+q+\cdots+2q^{n-2}t)$$

Proof. We will prove the result by induction. It is straightforward to check that for $n = 1$, $n = 2$, and $n = 3$ the result is true. Now we assume the result is true for L_{n-1} and prove that the generating function for L_n is equal to $(1+q^n)(1+q+\cdots+q^{n-3}+2q^{n-2}t)$ times the generating function for L_{n-1} .

Case 1: First consider permutations in L_n where n is unbarred.

Let $\pi \in L_{n-1}$. To create a permutation σ in L_n we will insert n into position $n - 2j$ of σ where $0 \leq j \leq \lfloor \frac{n-3}{2} \rfloor$ i.e. in positions $n, n-2, n-4, \dots$ of σ but not in position 1 or 2 (depending on the parity of n). If n is inserted so that there are $2j$ elements to the right of it, then inv will change by $2j$ which means the new permutation will be in L_n . Since n will be the largest element in σ and is not in position 1 or 2 it will not affect any existing left-to-right minima or almost left-to-right minima in π . In addition, since n is unbarred, the addition of n will not affect $\sum_{i \in Neg(\pi^{-1})} i$, thus contributing $\sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} q^{2j}$. If n is odd, then inserting n into π in position 1 creates $n-1$ inversions, a new left-to-right minimum in position 2 and a new almost left-to-right minimum in position 3, thus contributing $q^{n-2}t$. If n is even, then inserting n into π in position 2 creates $n-2$ inversions and a new almost left-to-right minimum in position 3, thus contributing $q^{n-2}t$.

Let $\pi \in L_{n-1}$. The permutation $(x_1 x_2)\pi$ where x_1 and x_2 are the two smallest elements in π is a permutation in O_{n-1} . To create a permutation σ in L_n we will insert n into position $n - 2j - 1$ of $(x_1 x_2)\pi$ where $0 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$ i.e. in positions $n-1, n-3, n-5, \dots$ of $(x_1 x_2)\pi$ but not in position 1 or 2 (depending on the parity of n). If n is inserted so that there are $2j-1$ elements to the right of it, then inv will change by $2j-1$ which means the new permutation will be in L_n since $(x_1 x_2)\pi \in O_n$. Since n will be the largest element in σ and is not in position 1 or 2 it will not affect any existing left-to-right minima or almost left-to-right minima in $(x_1 x_2)\pi$. In addition, since n is unbarred, the addition of n will not affect $\sum_{i \in Neg(\pi^{-1})} i$, thus contributing $\sum_{j=0}^{\lfloor \frac{n-4}{2} \rfloor} q^{2j-1}$. If n is even, then inserting n into $(x_1 x_2)\pi$ in position 1 creates $n-1$ inversions, a new left-to-right minimum in position 2 and a new almost left-to-right minimum in position 3, thus contributing $q^{n-2}t$. If n is odd, then inserting n into $(x_1 x_2)\pi$ in position 2 creates $n-2$ inversions and a new almost left-to-right minimum in position 3, thus contributing $q^{n-2}t$.

Thus the permutations in L_n that contain an unbarred n have a generating function that is $(1+q+q^2+\dots+q^{n-3}+2q^{n-2}t)$ times the generating function for L_{n-1} . Now we show that the contribution to the generating function of those permutations that contain a barred n is $q^n(1+q+q^2+\dots+q^{n-3}+2q^{n-2}t)$ times the generating function for L_{n-1} .

Case 2: Consider permutations in L_n where n is barred.

Let π be an element of L_{n-1} and let $x_1, x_2, \dots, x_{n-1}$ be the elements that make up π with $x_1 < x_2 < \dots < x_{n-1}$. (For example for the permutation $\pi = \bar{2}13\bar{5}4 \in L_5$, $x_1 = \bar{5}, x_2 = \bar{2}, x_3 = 1, x_4 = 3, x_5 = 4$.) Now replace x_1 in π with $\bar{n}$, replace x_2 with x_1 and so on until replacing x_{n-1} with x_{n-2} . Since the elements in this new word contain elements in the same relative order as in π , the new word has the same length and delent statistics as π . Call this new word $\tilde{\pi}$.

Now create a permutation σ in L_n by inserting the element x_{n-1} into position $n - 2j$ of $\tilde{\pi}$ where $0 \leq j \leq \lfloor \frac{n-3}{2} \rfloor$ i.e. in positions $n, n-2, n-4, \dots$ of $\tilde{\pi}$ but not in position 1 or 2 (depending on the parity of n). If x_{n-1} is inserted so that there are $2j$ elements to the right of it, then inv will change by $2j$ which means the new permutation will be in L_n . Since x_{n-1} will be the largest element in σ and is not in position 1 or 2 it will not affect any existing left-to-right minima or almost left-to-right minima in $\tilde{\pi}$.

In addition, since $\bar{n}$ is really the new addition to π then $\sum_{i \in Neg(\pi^{-1})} i$ changes by n , thus contributing $q^n(\sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} q^{2j})$. If n is odd, then inserting x_{n-1} into $\tilde{\pi}$ in position 1 creates $n-1$ inversions, a new left-to-right minimum in position 2 and a new almost left-to-right minimum in position 3, thus contributing $q^n(q^{n-2}t)$. If n is even, then inserting x_{n-1} into $\tilde{\pi}$ in position 2 creates $n-2$ inversions and a new almost left-to-right minimum in position 3, thus contributing $q^n(q^{n-2}t)$.

For any $\pi \in L_{n-1}$, the permutation $(x_1 x_2)\pi$ where x_1 and x_2 are the two smallest elements in π is a permutation in O_{n-1} . Again, let $x_1, x_2, \dots, x_{n-1}$ be the elements that make up $(x_1 x_2)\pi$ with $x_1 < x_2 < \dots < x_{n-1}$. Now replace x_1 in π with $\bar{n}$, replace x_2 with x_1 and so on until replacing x_{n-1} with x_{n-2} . Since the elements in this new word contain elements in the same relative order as in π , the new word has the same length and delent statistics as $(x_1 x_2)\pi$. Now create a permutation σ in L_n by inserting the element x_{n-1} into position $n - 2j - 1$ of the new word where $0 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$ i.e. in positions $n-1, n-3, n-5, \dots$ but not in position 1 or 2 (depending on the parity of n). If x_{n-1} is inserted so that there are $2j-1$ elements to the right of it, then inv will change by $2j-1$ which means the new permutation will be in L_n . Since x_{n-1} will be the largest element in σ and is not in position 1 or 2 it will not affect any existing left-to-right minima or almost left-to-right minima.

In addition, since $\bar{n}$ is really the new addition to π then $\sum_{i \in Neg(\pi^{-1})} i$ changes by n , thus contributing $q^n(\sum_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} q^{2j})$. If n is even, then inserting x_{n-1} in position 1 creates $n-1$ inversions, a new left-to-right minimum in position 2 and a new almost left-to-right minimum in position 3, thus contributing $q^n(q^{n-2}t)$. If n is odd, then inserting x_{n-1} in position 2 creates $n-2$ inversions and a new almost left-to-right minimum in position 3, thus contributing $q^n(q^{n-2}t)$.

Thus the permutations in L_n that contain a barred n have a generating function that is $q^n(1 + q + q^2 + \dots + q^{n-3} + 2q^{n-2}t)$ times the generating function for L_{n-1} .

Together with those permutations in L_n that contain an unbarred n we have that the generating function for permutations in L_n is $(1 + q^n)(1 + q + q^2 + \dots + q^{n-3} + 2q^{n-2}t)$ times the generating function for L_{n-1} .

□

Once again, even though M_n is not a group, we obtain for following interesting results about the generating function for the length and delent statistics on this set of permutations.

Theorem 5.2.

$$\sum_{\sigma \in M_n} q^{l_M(\sigma)} t^{del_M(\sigma)}$$

$$= (1 + q) \cdots (1 + q^n) (1 + 2qt) (1 + q + 2q^2t) \cdots (1 + q + \cdots + 2q^{n-2}t)$$

Proof. Let $\pi \in M_n$, then $(a \ b)\pi = \sigma \in L_n$, where a and b are the two smallest elements in π . In addition, we can write π as $(a \ b)\sigma$. The permutation $(a \ b)\sigma$ differs from σ in that a and b are interchanged. If a appeared to the left of b in π , then the number of inversions in σ is one greater than the number of inversions in π . In addition, the number of left to right minima in σ is one greater than the number in $\pi = (a \ b)\sigma$. By definition $l_M(\pi) = inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i$. Thus

$$\begin{aligned} l_M(\pi) &= inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i \\ &= (inv(\sigma) + 1) - (del_B(\sigma) - 1) + \sum_{i \in Neg(\pi^{-1})} i \\ &= inv(\sigma) - del_B(\sigma) + \sum_{i \in Neg(\sigma^{-1})} i \\ &= l_L(\sigma). \end{aligned}$$

If a appeared to the right of b in π , then the number of inversions in σ is one less than the number of inversions in π . In addition, the number of left to right minima in σ is one less than the number in π . Thus

$$\begin{aligned} l_M(\pi) &= inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i \\ &= (inv(\sigma) - 1) - (del_B(\sigma) + 1) + \sum_{i \in Neg(\pi^{-1})} i \\ &= inv(\sigma) - del_B(\sigma) + \sum_{i \in Neg(\sigma^{-1})} i \\ &= l_L(\sigma). \end{aligned}$$

Therefore $l_M(\omega) = l_L(\sigma)$. Swapping a and b in π does not change the number of almost left to right minima in π so $del_M(\pi) = del_L(\sigma)$. Thus $q^{l_M(\pi)} t^{del_M(\pi)} = q^{l_L(\sigma)} t^{del_L(\sigma)}$ and by Theorem 5.1 we have the result. $\square$

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