Tot($M$) and Modules of Generalized Fractions

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1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with non-zero identity and $M$ is a finitely generated $R$-module. This paper is concerned with a certain complex $C(U(x), M)$ of $R$-modules and $R$-homomorphism which involves modules of generalized fractions derived from $M$ and the sequence $x_1, x_2, \ldots, x_n$. The complex is described as follows:

for each $i \in \mathbb{N}$ (throughout we use $\mathbb{N}$ to denote the set of positive integers). Set $U(x)_i = \{(x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_i^{\alpha_i}) \mid \exists j, 0 \leq j \leq i \text{ s.t. } \alpha_1, \ldots, \alpha_j \in \mathbb{N}, \text{and } \alpha_{j+1} = \cdots = \alpha_i = 0\}$, where $x_r$ is interpreted as $x_n$ whenever $r > n$. Then, for each $i \in \mathbb{N}$, $U(x)_i$ is a triangular subset [4,2.1] of $R^i$. We can [2, page 420] from the complex $0 \rightarrow M \xrightarrow{e_0} U(x)^{-1}_1 M \xrightarrow{e_1} \cdots \xrightarrow{e_i} U(x)^{-i+1}_{i+1} M \xrightarrow{e_{i+1}} \cdots$ of $R$-modules and $R$-homomorphisms, which we denote by $C(U(x), M)$.

Here $U(x)^{-i}_i M$ is module of generalized fractions of $M$ with respect to the triangular subset $U(x)_i$ of $R^i$, and the homomorphisms $e^i$ ($i \geq 0$) are given by the following formulas. $e^0(b) = \frac{b}{(1)}$ for all $b \in M$ and, for each $i \in \mathbb{N}$,

$$e^i\left(\left(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}\right)^b\right) = \frac{(-1)^b}{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}, 1)}$$

for all $b \in M$ and $(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}) \in U(x)_i$.

let

$$F: \cdots \xrightarrow{f_{i+1}} F_{i} \xrightarrow{f_{i}} F_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0$$

be a fix free resolution of $M$ such that $F_i = 0$ if $i < 0$.

We define a bigraded module $M = \{M_{p,q}\}$ by $M_{p,q} = U_p^{-p}F_q$. therefore we
have the total complex \( \text{Tot}(M) \) where \( \text{Tot}(M)_n = \bigoplus_{q-p=n} M_{p,q} \) and \( p > 0, \ q \geq 0 \).

We show \( \text{Tot}(M) \) is exact sequence either \( x_1, \ldots, x_n \) is \( M \)-sequence or \( \text{pd}(M) \) is finite.

1. The result

Throughout this paper \( \alpha \) is an ideal of \( R \). For a submodoule \( N \) of \( M \), we set \( N: M < \alpha > = \{ m \in M \mid \alpha^r m \subset N \text{ for some } r \in \mathbb{N} \} \). We recall the definition of a filter regular sequence. A sequence \( x_1, \ldots, x_n \) of elements of \( \alpha \) is called an \( \alpha \)-filter regular on \( M \), if \( (x_1, \ldots, x_n)M: M x_{i+1} \subset (x_1, \ldots, x_i)M: M < \alpha > \) for all \( 0 \leq i < n \).

When such property for \( x_1, \ldots, x_n \) holds in any order, we will say that the sequence \( x_1, \ldots, x_n \) forms an unconditioned \( \alpha \)-filter sequence on \( M \). Note that \( x_1, \ldots, x_n \) is a poor \( M \)-sequence if and only if it is an \( R \)-filter regular sequence on \( M \). The following characterization of filter regular sequences in included here for the reader’s convenience.

2.1 Proposition. [cf,1,2.2]. Let \( x_1, x_2, \ldots, x_n \) be a sequence of elements of \( \alpha \). Then the following statements are equivalent.

(i) \( x_1, \ldots, x_n \) is an \( \alpha \)-filter regular sequence on \( M \).

(ii) \( x_1, \ldots, x_n \) is a poor \( M \)-sequence for all \( p \in \text{supp } M - \text{Var} (\alpha) \).

(iii) \( x_1^{t_1}, \ldots, x_n^{t_n} \) is an \( \alpha \)-filter regular sequence on \( M \) for all positive integers \( t_1, \ldots, t_n \).

2.2 Proposition. [cf,6,1.2]. Suppose that \( \alpha \) is generated by \( n \) elements. Then \( \alpha \) has a sequence of generators of length \( n \) which forms an unconditioned \( \alpha \)-filter regular sequence on \( M \).

2.3 Remark. Let \( x_1, \ldots, x_n \) be a sequence of elements of \( R \) and let \( \alpha = < x_1, \ldots, x_n > \). Write the complex \( C(U(x), M) \) as

\[
0 \longrightarrow M \xrightarrow{e_0} U_1^{-1}M \xrightarrow{e_1} U_2^{-2}M \xrightarrow{e^2} \cdots \xrightarrow{e^{i-1}} U_i^{-i}M \xrightarrow{e_i} \cdots
\]

Note that, in view of [4,3.3] and [5,2.1], \( U(x)_i^{-i}M = 0 \) for all \( i > n \).

2.4 Proposition. [cf,2,3.1]. If \( U = (U_n)_{n \geq 1} \) is a chain of triangular subset on \( R \), then \( C(U, M) \) is exact if and only if for all \( n \in \mathbb{N} \), each elements of \( U_n \) is a poor \( M \)-sequence.
2.5 Lemma. [cf.5,2.2.8]. Let $w$ is a triangular subset of $R^n$. Suppose that, for any $(x_1, \ldots, x_n) \in w$, $x_n \in \alpha$, where $\alpha$ is an ideal of $R$. Then for an $R$-module $M$, $H^1_{\alpha}(w^nM) = 0$ for all $i \geq 0$.

2.6 Notation and definition. Let $x = x_1, \ldots, x_n$ be a sequence of elements of $R$ and let $F^\bullet : \cdots \to F_{t+1} \to F_t \to F_{t-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be a fix free resolution of $M$ such that $f_i = 0$ for all $i < 0$. We define $\theta_{p,q} : U_p^{-p}F_q \to U_p^{-p-1}F_q$ as $\theta_{p,q}(c) = \frac{c}{(t)}$ and $\lambda_{p,q} : U_p^{-p}F_q \to U_p^{-p}F_{q-1}$ as $\lambda_{p,q}(b) = (-1)^pU_p^{-p}f_q(b)$, $\text{Tot}(M)_n = U_1^{-1}F_{n+1} \bigoplus U_2^{-2}F_{n+2} \bigoplus \cdots$, $g_n : \text{Tot}(M)_n$ is given

$$g_n = ((g_n)_i)_{i \in \mathbb{N}} = (\theta i - 1, n + i - 1 + \lambda_{i,n+1}i)_{i \in \mathbb{N}}$$

2.7 Theorem. With the above notation, then

a) $g_ng_{n+1} = 0$ for all $n$.

b) $\frac{\ker g_n}{\text{im } g_{n+1}} = 0$ if $x_1, \ldots, x_n$ is a poor $M$-sequence

Proof.

a) $g_ng_{n+1} = ((g_ng_{n+1})_1, \ldots, (g_ng_{n+1})_t, \ldots)$ and

$$(g_ng_{n+1})_t = \theta_{t-1,n+t-1}(\theta_{t-2,n+t-1} + \lambda_{t-1,n+t}) + \lambda_{t,n+t} (\theta_{t-1,n+t} + \lambda_{t,n+t-1}) = 0$$

b) In view of [7,2.5] we may assume that

$$\frac{m}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})} = \frac{x_1^{\alpha_1 - \alpha_1} \cdots x_n^{\alpha_n - \alpha_n}}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})}.$$ By using (a), it is enough to show that there exists $b = (b_1, \ldots, b_t, \ldots) \in \text{Tot}(M)_{n+1}$ such that $g_{n+1}(b) = B$ if $B \in \ker g_n$.

Let $B = \left( \frac{r_{n+1}}{x_1^\gamma}, \frac{r_{n+2}}{x_1^\gamma x_2^\gamma}, \frac{r_{n+3}}{x_1^\gamma x_2^\gamma x_3^\gamma}, \cdots \right) \in \ker g_n$. Let $\text{ara}(\alpha) = k$. By 2.3, for every $R$-module $F_q$, we have $U_r^{-r}F_q = 0$ for all $r > k$. We now argue by descending induction on $t$. Suppose that $t \in \mathbb{N}$, and there exists $b_t, b_{t+1}, \ldots$ such that $g_{n+1}(b) = B$. So

$$\theta_{t-1,n+t}(b_{t-1}) + \lambda_{t,n+t+1}(b_t) = \frac{r_{n+t}}{(x_1^\gamma, \ldots, x_t^\gamma)}$$

We show there exist $b_{t-2} \in U_{t-2}^{-t+2}F_{n+t-1}$ such that

$$\theta_{t-2,n+t-1}(b_{t-2}) + \lambda_{t-1,n+t}(b_{t-1}) = \frac{r_{n+t-1}}{(x_1^\gamma, \ldots, x_{t-1}^\gamma)}$$
2.8 Lemma. Let $\theta_{t-1,n+t}(b_{t-1}) + \lambda_{t,n+t+1}(b_t) = \frac{r_{n+t}}{(x_1^\alpha, \ldots, x_t^\alpha)} \implies$

$$\lambda_{t,n+t}\theta_{t-1,n+t}(b_{t-1}) = \lambda_{t,n+t}\left(\frac{r_{n+t}}{(x_1^\alpha, \ldots, x_t^\alpha)}\right) \implies$$

$$\theta_{t-1,n+t+1}(f_{t+n}(b_{t-1})) = -\frac{f_{t+n}(r_{n+t})}{(x_1^\alpha, \ldots, x_t^\alpha)} \implies$$

$$\theta_{t-1,n+t+1}\left(\frac{r_{n+t+1}}{(x_1^\alpha, \ldots, x_t^\alpha)}\right) = \theta_{t-1,n+t+1}\lambda_{t-1,n+t}(b_{t-1}) \implies$$

$$\frac{r_{n+t+1}}{(x_1^\alpha, \ldots, x_t^\alpha)} - \lambda_{t-1,n+t}(b_{t-1}) \in \text{Ker}\theta_{t-1,n+t+1}.$$ 

By using (2.4) that $\text{im}\theta_{t-2,n+t-1} = \text{Ker}\theta_{t-1,n+t-1}$, there exists $b_{t-2}$ such that

$$\theta_{t-2,n+t-1}(b_{t-2}) = \frac{r_{n+t+1}}{(x_1^\alpha, \ldots, x_t^\alpha)} - \lambda_{t-1,n+t}(b_{t-1}),$$

the result follows.

2.9 Lemma. Let $x_1, x_2, \ldots, x_i$ be $\alpha$-filter regular sequence. Then $\Gamma_\alpha\left(\frac{\text{Ker}g_n}{\text{im}g_{n+1}}\right) = \frac{\text{Ker}g_n}{\text{im}g_{n+1}}$ for all $n$.

Proof. $\forall p \in \text{supp}(M) - \text{Var}(\alpha)$, then $(\frac{x_1}{1}, \frac{x_2}{1}, \ldots, \frac{x_i}{1}) \in \alpha_p^\parallel$ is $M_p$-sequence (2.1). By using (2.4) $\text{Ker}(g_n)_p = \text{im}(g_{n+1})_p$, then $H^i(\text{Tot}(M))_p = 0$ for $i \geq 0$. Therefore in the view of $\text{supp} H^i(\text{Tot}(M)) \subset \text{Var}(\alpha)$. It is now straightforward to see that the $\forall v \in H^i(\text{Tot}(M)) \exists t \in \mathbb{N}$ such that $\alpha^tv = 0$. It follows

$$\Gamma_\alpha\left(\frac{\text{Ker}g_n}{\text{im}g_{n+1}}\right) = \frac{\text{Ker}g_n}{\text{im}g_{n+1}}.$$ 

2.9 Lemma. With the notation (2.6). Then $H^i(\text{Tot}(M))_n = H^1_\alpha(\text{im}g_n)$.

Proof. We shall use the exact sequences

1. $0 \longrightarrow \text{im}g_{n+1} \longrightarrow \text{Ker}g_n \longrightarrow \frac{\text{Ker}g_n}{\text{im}g_{n+1}} \longrightarrow 0$ (1)

2. $0 \longrightarrow \text{im}g_{n+1} \longrightarrow \text{Tot}(M)_n \longrightarrow \frac{\text{Tot}(M)_n}{\text{im}g_{n+1}} \longrightarrow 0$ (2)

3. $0 \longrightarrow \text{Ker}g_n \longrightarrow \text{Tot}(M)_n \longrightarrow \text{im}g_n \longrightarrow 0$ (3)
By using (1), (2), (3) and (2.5) we have $H^i_{\mathfrak{A}}(\text{Ker} g_n) = H^i_{\mathfrak{A}}(\text{im} g_{n+1})$ for all $i \geq 2$ and $\Gamma_{\mathfrak{A}}(\text{im} g_{n+1}) = 0$ and $\Gamma_{\mathfrak{A}}(\text{Ker} g_n) = 0$, $H^i_{\mathfrak{A}}(\text{im} g_n) = H^{i+1}_{\mathfrak{A}}(\text{Ker} g_n)$ for all $i \geq 0$.

The above exact sequences induces an exact sequence

$$0 \to \frac{\text{Ker} g_n}{\text{im} g_{n+1}} \to H^1_{\mathfrak{A}}(\text{im} g_{n+1}) \to H^1_{\mathfrak{A}}(\text{Ker} g_n) \to 0.$$ 

It follows $\frac{\text{Ker} g_n}{\text{im} g_{n+1}} = H^1_{\mathfrak{A}}(\text{im} g_{n+1})$.

**2.10 Corollary.** Let, with the notation (2.6) and $\text{pd}(M) = l < \infty$. Then total complex $\text{Tot}(M)$ is exact sequence of length $< l + t$ where $\mathfrak{A} = \langle x_1, x_2, \ldots, x_t \rangle$.

**Proof.** Let $0 \to F_l \to F_{l-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ be a fix free resolution of $M$ such that $F_i = 0$ for all $i > l$. It is clear $\exists r, s$ such that $\text{Tot}(M)_n = 0$ for all $n > r$ or $n < s$.

We obtain the complex $0 \to \text{Tot}(M)_s \to \text{Tot}(M)_{s-1} \to \cdots \to \text{Tot}(M)_r \to 0$. By using (2.9) $\frac{\text{Ker} g_n}{\text{im} g_{n+1}} = H^1_{\mathfrak{A}}(\text{im} g_{n+1}) = H^2_{\mathfrak{A}}(\text{Ker} g_{n+1}) = \cdots = H^{s-n}_{\mathfrak{A}}(\text{Ker} g_{n+1}) = 0$, as required.

**References**


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