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Tot(M) and Modules of Generalized Fractions

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1. Introduction

Throughout this paper, R is a commutative Notherian ring with non-zero identity and M is a finitely generated R-module. This paper is concerned with a certain complex $C(U(x), M)$ of R-modules and R-homomorphism which involves modules of generalized fractions derived from M and the sequence $x_1, x_2, \ldots, x_n$. The complex is described as follows:

for each $i \in \mathbb{N}$ (throughout we use $\mathbb{N}$ to denote the set of positive integers). Set $U(x)_i = \{(x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_i^{\alpha_i}) \mid \exists j,\ 0 \leq j \leq i$ s.t. $\alpha_1, \ldots, \alpha_j \in \mathbb{N}$, and $\alpha_{j+1} = \cdots = \alpha_i = 0\}$, where x_r is interpreted as x_n whenever $r > n$. Then, for each $i \in \mathbb{N}$, $U(x)_i$ is a triangular subset [4,2.1] of R^i. We can [2, page 420] from the complex $0 \to M \xrightarrow{e^0} U(x)_1^{-1}M \xrightarrow{e^1} \cdots \xrightarrow{e^i} U(x)_{i+1}^{-i-1}M \xrightarrow{e^{i+1}} \cdots$ of R-modouls and R-homomorphisms, which we denote by $C(U(x), M)$. Here $U(x)_i^{-i}M$ is modoule of generalized fractions of M with respect to the triangular subset $U(x)_i$ of R^i, and the homomorphisms e^i ($i \geq 0$) are given by the following formulas. $e^0(b) = \frac{b}{(1)}$ for all $b \in M$ and, for each $i \in \mathbb{N}$, $e^i\left(\frac{b}{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i})}\right) = \frac{(-1)^i b}{(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}, 1)}$ for all $b \in M$ and $(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}) \in U(x)_i$. let $F : \cdots \xrightarrow{f_{t+1}} F_t \xrightarrow{f_t} F_{t-1} \xrightarrow{f_{t-1}} \cdots F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$ be a fix free resolution of M such that $F_i = 0$ if $i < 0$.

We define a bigraded modoule $M = \{M_{p,q}\}$ by $M_{p,q} = U_p^{-p} F_q$. therefore we

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have the total complex $\mathrm{Tot}(M)$ where $\mathrm{Tot}(M)_n = \bigoplus_{q-p=n} M_{p,q}$ and $p > 0$, $q \geq 0$.

We show $\mathrm{Tot}(M)$ is exact sequence either $x_1, \ldots x_n$ is M-sequence or $\mathrm{pd}(M)$ is finite.

1. The result

Throughtout this paper $\underline{\alpha}$ is an ideal of R. For a submodoule N of M, we set $N{:}_M < \underline{\alpha} >= \{m \in M | \ \underline{\alpha}^r m \subset N \text{ for some } r \in \mathbb{N}\}$. We recall the definition of a filter regular sequence. A sequence $x_1, \ldots, x_n$ of elements of $\underline{\alpha}$ is called an $\underline{\alpha}$-filter regular on M, if $(x_1, \ldots, x_n)M{:}_M x_{i+1} \subset (x_1, \ldots, x_i)M{:}_M < \underline{\alpha} >$ for all $0 \leq i < n$.

When such property for $x_1, \ldots, x_n$ holds in any order, we will say that the sequence $x_1, \ldots, x_n$ forms an unconditioned $\underline{\alpha}$-filter sequence on M. Note that $x_1, \ldots, x_n$ is a poor M-sequence if and only if it is an R-filter regular sequence on M. The following characterization of filter regular sequences in included here for the reader's convenience.

2.1 Proposition. [cf,1,2.2]. Let $x_1, x_2, \ldots, x_n$ be a sequence of elements of $\underline{\alpha}$. Then the following statements are equivalent.

(i) $x_1, \ldots, x_n$ is an $\underline{\alpha}$-filter regular sequence on M.

(ii) $x_1, \ldots, x_n$ is a poor M-sequence for all $p \in \mathrm{supp}\, M - \mathrm{Var}(\underline{\alpha})$.

(iii) $x_1^{t_1}, \ldots, x_n^{t_n}$ is an $\underline{\alpha}$-filter regular sequence on M for all positive integers $t_1, \ldots, t_n$.

2.2 Proposition. [cf,6,1.2]. Suppose that $\underline{\alpha}$ is generated by n elements. Then $\underline{\alpha}$ has a sequence of generators of length n which forms an unconditioned $\underline{\alpha}$-filter regular sequence on M.

2.3 Remark. Let $x_1, \ldots, x_n$ be a sequence of elements of R and let $\underline{\alpha} =< x_1, \ldots, x_n >$. Write the complex $C(U(x), M)$ as

$$0 \longrightarrow M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} U_2^{-2}M \xrightarrow{e^2} \cdots \xrightarrow{e^{i-1}} U_i^{-i}M \xrightarrow{e^i} \cdots$$

Note that, in view of [4,3.3] and [5,2.1], $U(x)_i^{-i}M = 0$ for all $i > n$.

2.4 Proposition. [cf,2,3.1]. If $U = (U_n)_{n \geq 1}$ is a chain of triangular subset on R, then $C(U, M)$ is exact if and only if for all $n \in \mathbb{N}$, each elements of U_n is a poor M-sequence.

2.5 Lemma. [cf,5,2.2.8]. Let w is a triangular subset of R^n. suppose that, for any $(x_1, \ldots, x_n) \in w$, $x_n \in \underline{\alpha}$, where $\underline{\alpha}$ is an ideal of R. Then for an R-modoule M, $H^i_{\underline{\alpha}}(w^{-n}M) = 0$ for all $i \geq 0$.

2.6 Notation and definition. Let $x = x_1, \ldots, x_s$ be a sequence of elements of R and let $F^{\cdot} : \cdots \xrightarrow{f_{t+1}} F_t \xrightarrow{f_t} F_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$ be a fix free resoloution of M such that $f_i = 0$ for all $i < 0$. We define $\theta_{p,q} : U_p^{-p} F_q \longrightarrow U_{p+1}^{-p-1} F_q$ as $\theta_{p,q}(c) = \frac{c}{(1)}$ and $\lambda_{p,q} : U_p^{-p} F_q \longrightarrow U_p^{-p} F_{q-1}$ as $\lambda_{p,q}(b) = (-1)^p U_p^{-p} f_q(b)$, $\mathrm{Tot}(M)_n = U_1^{-1} F_{n+1} \bigoplus U_2^{-2} F_{n+2} \bigoplus \cdots$, $g_n : \mathrm{Tot}(M)_n$ is given

$$g_n = ((g_n)_i)_{i \in \mathbb{N}} = (\theta i - 1, n + i - 1 + \lambda_{i,n+i})_{i \in \mathbb{N}}$$

2.7 Theorem. With the above notation, then

a) $g_n g_{n+1} = 0$ for all n.

b) $\dfrac{\ker g_n}{\mathrm{im}\, g_{n+1}} = 0$ if $x_1, \ldots, x_n$ is a poor M-sequence

Proof.

a) $g_n g_{n+1} = ((g_n g_{n+1})_1, \ldots, (g_n g_{n+1})_t, \ldots)$ and

$$(g_n g_{n+1})_t = \theta_{t-1,n+t-1}(\theta_{t-2,n+t-1} + \lambda_{t-1,n+t}) + \lambda_{t,n+t}(\theta_{t-1,n+t} + \lambda_{t,n+t-1}) = 0$$

b) In view of [7,2.5] we may assume that $\dfrac{m}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})} = \dfrac{x_1^{\alpha - \alpha_1} \ldots x_n^{\alpha - \alpha_n}}{(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})}$. By using (a), it is enough to show that there exists $b = (b_1, \ldots, b_t, \ldots) \in \mathrm{Tot}(M)_{n+1}$ such that $g_{n+1}(b) = B$ if $B \in \mathrm{Ker}\, g_n$.

Let $B = \left(\frac{r_{n+1}}{(x_1^{\alpha})}, \frac{r_{n+2}}{(x_1^{\alpha}, x_2^{\alpha})}, \frac{r_{n+3}}{(x_1^{\alpha}, x_2^{\alpha}, x_3^{\alpha})}, \cdots \right) \in \mathrm{Ker}\, g_n$. Let $\mathrm{ara}(\underline{\alpha}) = k$. By 2.3, for every R-module F_q, we have $U_r^{-r} F_q = 0$ for all $r > k$. We now argue by descending induction on t. Suppose that $t \in \mathbb{N}$, and there exists $b_t, b_{t+1}, \ldots$ such that $g_{n+1}(b) = B$. So

$$\theta_{t-1,n+t}(b_{t-1}) + \lambda_{t,n+t+1}(b_t) = \frac{r_{n+t}}{(x_1^{\alpha}, \ldots, x_t^{\alpha})}$$

We show there exist $b_{t-2} \in U_{t-2}^{-t+2} F_{n+t-1}$ such that

$$\theta_{t-2,n+t-1}(b_{t-2}) + \lambda_{t-1,n+t}(b_{t-1}) = \frac{r_{n+t-1}}{(x_1^{\alpha}, \ldots, x_{t-1}^{\alpha})}$$

Since that

$$\theta_{t-1,n+t}\left(b_{t-1}\right)+\lambda_{t,n+t+1}\left(b_{t}\right)=\frac{r_{n+t}}{\left(x_1^{\alpha},\ldots,x_t^{\alpha}\right)}\Longrightarrow$$

$$\lambda_{t,n+t}\theta_{t-1,n+t}\left(b_{t-1}\right)=\lambda_{t,n+t}\left(\frac{r_{n+t}}{\left(x_1^{\alpha},\ldots,x_t^{\alpha}\right)}\right)\Longrightarrow$$

$$\theta_{t-1,n+t-1}\left(f_{t+n}(b_{t-1})\right)=-\frac{f_{t+n}(r_{n+t})}{\left(x_1^{\alpha},\ldots,x_t^{\alpha}\right)}\Longrightarrow$$

$$\theta_{t-1,n+t-1}\left(\frac{r_{n+t-1}}{\left(x_1^{\alpha},\ldots,x_{t-1}^{\alpha}\right)}\right)=\theta_{t-1,n+t-1}\lambda_{t-1,n+t}\left(b_{t-1}\right)\Longrightarrow$$

$$\frac{r_{n+t-1}}{\left(x_1^{\alpha},\ldots,x_{t-1}^{\alpha}\right)}-\lambda_{t-1,n+t}\left(b_{t-1}\right)\in\operatorname{Ker}\theta_{t-1,n+t-1}.$$

By using (2.4) that $\operatorname{im}\theta_{t-2,n+t-1}=\operatorname{Ker}\theta_{t-1,n+t-1}$, there exists b_{t-2} such that

$$\theta_{t-2,n+t-1}(b_{t-2})=\frac{r_{n+t-1}}{\left(x_1^{\alpha},\ldots,x_{t-1}^{\alpha}\right)}-\lambda_{t-1,n+t}\left(b_{t-1}\right),$$

the result follows.

2.8 Lemma. Let $x_1,x_2,\ldots,x_i$ be $\underline{\alpha}$-filter regular sequence. Then $\Gamma_{\underline{\alpha}}\left(\frac{\operatorname{Ker}g_n}{\operatorname{im}g_{n+1}}\right)=\frac{\operatorname{Ker}g_n}{\operatorname{im}g_{n+1}}$ for all n.

Proof. $\forall p\in\operatorname{supp}(M)-\operatorname{Var}(\underline{\alpha})$, then $\left(\frac{x_1}{1},\frac{x_2}{1},\ldots,\frac{x_i}{1}\right)\in\underline{\alpha}_p$ is M_p-sequence (2.1). By using (2.4) $\operatorname{Ker}(g_n)_p=\operatorname{im}(g_{n+1})_p$, then $H^i(\operatorname{Tot}(M))_p=0$ for $i\geq 0$. Therefore in the view of $\operatorname{supp}H^i(\operatorname{Tot}(M))\subset\operatorname{Var}(\underline{\alpha})$. It is now straightforward to see that the $\forall v\in H^i(\operatorname{Tot}(M))$ $\exists t\in\mathbb{N}$ such that $\underline{\alpha}^t v=0$. It follows

$$\Gamma_{\underline{\alpha}}\left(\frac{\operatorname{Ker}g_n}{\operatorname{im}g_{n+1}}\right)=\frac{\operatorname{Ker}g_n}{\operatorname{im}g_{n+1}}.$$

2.9 Lemma. With the notation (2.6). Then $H^i\left(\operatorname{Tot}(M)\right)_n=H^1_{\underline{\alpha}}(\operatorname{im}g_n)$.

Proof. We shall use the exact sequences

$$0\longrightarrow\operatorname{im}g_{n+1}\longrightarrow\operatorname{Ker}g_n\longrightarrow\frac{\operatorname{Ker}g_n}{\operatorname{im}g_{n+1}}\longrightarrow 0\tag{1}$$

$$0\longrightarrow\operatorname{im}g_{n+1}\longrightarrow\operatorname{Tot}(M)_n\longrightarrow\frac{\operatorname{Tot}(M)_n}{\operatorname{im}g_{n+1}}\longrightarrow 0\tag{2}$$

$$0\longrightarrow\operatorname{Ker}g_n\longrightarrow\operatorname{Tot}(M)_n\longrightarrow\operatorname{im}g_n\longrightarrow 0\tag{3}$$

By using (1), (2), (3) and (2.5) we have $H^i_{\underline{\alpha}}(\mathrm{Ker} g_n) = H^i_{\underline{\alpha}}(\mathrm{im}\, g_{n+1})$ for all $i \geq 2$ and $\Gamma_{\underline{\alpha}}(\mathrm{im}\, g_{n+1}) = 0$ and $\Gamma_{\underline{\alpha}}(\mathrm{Ker}\, g_n) = 0$, $H^i_{\underline{\alpha}}(\mathrm{im}\, g_n) = H^{i+1}_{\underline{\alpha}}(\mathrm{Ker}\, g_n)$ for all $i \geq 0$.
The above exact sequences induces an exact sequence

$$0 \longrightarrow \frac{\mathrm{Ker}\, g_n}{\mathrm{im}\, g_{n+1}} \longrightarrow H^1_{\underline{\alpha}}(\mathrm{im}\, g_{n+1}) \longrightarrow H^1_{\underline{\alpha}}(\mathrm{Ker} g_n) \longrightarrow 0.$$

It follows $\frac{\mathrm{Ker}\, g_n}{\mathrm{im}\, g_{n+1}} = H^1_{\underline{\alpha}}(\mathrm{im}\, g_{n+1})$.

2.10 Corollary. Let, with the notation (2.6) and $\mathrm{pd}(M) = l < \infty$. Then total complex $\mathrm{Tot}(M)$ is exact sequence of length$< l + t$ where $\underline{\alpha} =< x_1, x_2, \dots, x_t >$.

Proof. Let $0 \longrightarrow\longrightarrow F_l \longrightarrow F_{l-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be a fix free resolution of M such that $F_i = 0$ for all $i > l$. It is clear $\exists\, r$, s such that $\mathrm{Tot}(M)_n = 0$ for all $n > r$ or $n < s$.
We obtain the complex $0 \longrightarrow \mathrm{Tot}(M)_s \longrightarrow \mathrm{Tot}(M)_{s-1} \longrightarrow \cdots \longrightarrow \mathrm{Tot}(M)_r \longrightarrow 0$. By using (2.9) $\frac{\mathrm{Ker}\, g_n}{\mathrm{im}\, g_{n+1}} = H^1_{\underline{\alpha}}(\mathrm{im}\, g_{n+1}) = H^2_{\underline{\alpha}}(\mathrm{Ker}\, g_{n+1}) = \cdots = H^{s-n}_{\underline{\alpha}}(\mathrm{Ker}\, g_{n+1} = 0$, as required.

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