Cofinitely $\delta$-Supplemented and Cofinitely $\delta$-Semiperfect Modules

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Abstract

In this work, we prove that an $R$-module $M$ is cofinitely $\delta$-supplemented (i.e. each cofinite submodule of $M$ has a $\delta$-supplement) if and only if every maximal submodule of $M$ has a $\delta$-supplement. An $R$-module $M$ is called cofinitely $\delta$-semiperfect if each finitely generated factor module of $M$ has a projective $\delta$-cover, we prove that this is equivalent to the existence of a $\delta$-supplement, which is a direct summand of $M$, for each cofinite submodule of $M$. Cofinitely $\delta$-lifting modules are introduced and characterized. We also give new characterizations of $\delta$-semiperfect rings in terms of these concepts. Some examples are given at the end of this article.

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1 Introduction And Preliminary Notes

Throughout this paper $R$ is an associative ring with unity and all modules are unitary left $R$-modules. A submodule $K$ of a module $M$ is denoted by $K \subseteq M$. Let $M$ be a module, $K \subseteq M$ is called small in $M$ (denoted $K \ll M$) if for every $N \subseteq M$, the equality $N + K = M$ implies $N = M$. A submodule $U \subseteq M$ is called a supplement of $K \subseteq M$, if $M = K + U$ and $K \cap U \ll U$.

Zhou [12] introduced the concept of ”$\delta$-small submodule” as a generalization of small submodules. Let $K \subseteq M$, $K$ is called $\delta$-small in $M$, denoted by $K \ll_\delta M$, if whenever $M = N + K$ and $M/N$ is singular, we have $M = N$. The sum of all $\delta$-small submodules of a module $M$ is denoted by
δ(M), which defines a preradical on the category of \( R \)-modules. Zhou [12] proved that \( \delta(M) \) is the reject in \( M \) of the class of all singular simple modules, i.e. \( \delta(M) = \cap \{ N \subseteq M : M/N \text{ is singular} \} \).

We collect basic properties of \( \delta \)-small submodules in the following lemma which is taken from [12, Lemma 1.2 and 1.3]

**Lemma 1.1.** Let \( M \) be an \( R \)-module, \( N, L \subseteq M \)

1. If \( N \ll_{\delta} M \) and \( M = X + N \), then \( M = X \oplus Y \) for a projective semisimple submodule \( Y \) of \( M \) and \( Y \subseteq N \).

2. If \( N \ll_{\delta} M \) and \( f : M \rightarrow H \) is a homomorphism, then \( f(N) \ll_{\delta} H \); In particular, if \( K \ll_{\delta} N \subseteq M \), then \( K \ll_{\delta} M \).

3. If \( K_1 \ll_{\delta} X_1 \subseteq M \) and \( K_2 \ll_{\delta} X_2 \subseteq M \), then \( K_1 + K_2 \ll_{\delta} X_1 + X_2 \).

4. If \( K \subseteq N \subseteq M \), \( K \ll_{\delta} M \) and \( N \) is a direct summand of \( M \), then \( K \ll_{\delta} N \).

In [8] the concepts of \( \delta \)-supplemented and \( \delta \)-lifting modules were defined as generalizations of supplemented and lifting modules. Let \( K, N \subseteq M \), \( N \) is called a \( \delta \)-supplement of \( K \) in \( M \) if \( M = K + N \) and \( K \cap N \ll_{\delta} N \). \( M \) is called \( \delta \)-supplemented if every submodule of \( M \) has a \( \delta \)-supplement in \( M \), and it is called \( \delta \)-lifting if for every submodule \( K \subseteq M \), there exists a decomposition \( M = M_1 \oplus M_2 \) such that \( M_1 \subseteq K \) and \( K \cap M_2 \ll_{\delta} M_2 \). For a projective module \( P \), the author proved that the two concepts are equivalent.

An epimorphism \( f : P \rightarrow N \) is called a \( \delta \)-cover of \( N \) if \( \text{Ker}(f) \ll_{\delta} P \) and if moreover \( P \) is projective, then it is called a projective \( \delta \)-cover. In [12], a ring \( R \) is called \( \delta \)-semiperfect if every simple \( R \)-module has a projective \( \delta \)-cover. In [2] a module \( M \) is called \( \delta \)-semiperfect, if every factor module of \( M \) has a projective \( \delta \)-cover. For projective modules a similar result to that of Kasch and Mares [7] was obtained in [2], which states that a projective module is \( \delta \)-semiperfect if and only if it is \( \delta \)-supplemented.

A submodule \( N \) of a module \( M \) is called cofinite, if \( M/N \) is finitely generated. In [1] a module \( M \) is called cofinitely supplemented module if every cofinite submodule of \( M \) has a supplement. In [3], a module \( M \) is called \( \oplus \)-cofinitely supplemented, if every cofinite submodule of \( M \) has a supplement that is a direct summand of \( M \). Semiperfect rings were characterized in [1, 3] using these notions.

As a generalization of semiperfect modules, Çalişici and Pancar [4] introduced the notion of cofinitely semiperfect modules. A module \( M \) is called cofinitely semiperfect, if every factor module of \( M \) by a cofinite submodule has a projective cover. They proved that a projective module \( P \) is cofinitely semiperfect if and only if \( P \) is \( \oplus \)-cofinitely supplemented.
In this work, we introduce and study the notions of cofinitely $\delta$-supplemented and cofinitely $\delta$-semiperfect modules. A module $M$ is called cofinitely $\delta$-supplemented, briefly cof-$\delta$-supplemented, if each cofinite submodule of $M$ has a $\delta$-supplement. $M$ is called $\oplus$-cofinitely $\delta$-supplemented, briefly $\oplus$-cof-$\delta$-supplemented, if every cofinite submodule of $M$ has a $\delta$-supplement that is a direct summand of $M$. In section 2, we give some properties of these modules, among other things we show that an arbitrary sum of cof-$\delta$-supplemented modules is cof-$\delta$-supplemented. We also show that a module $M$ is cof-$\delta$-supplemented if and only if every maximal submodule of $M$ has a $\delta$-supplement.

A module $M$ is called cofinitely $\delta$-semiperfect, briefly cof-$\delta$-semiperfect, if every factor module of $M$ by a cofinite submodule has a projective $\delta$-cover. A module $M$ is called cofinitely $\delta$-lifting, if for every cofinite submodule $N \subseteq M$, there exists a decomposition $M = X \oplus Y$ such that $X \subseteq N$ and $Y \cap N \ll_\delta Y$.

We start section 3 by giving some properties of cof-$\delta$-semiperfect modules, after that we characterize these modules. We prove that a projective module is cof-$\delta$-semiperfect if and only if it is $\oplus$-cofinitely $\delta$-supplemented if and only if it is cofinitely $\delta$-lifting. We also characterize cof-$\delta$-semiperfect modules in the general case, by showing that a module is cofinitely $\delta$-semiperfect if and only if it is $\delta$-supplemented by $\delta$-supplements that have projective $\delta$-cover, this result is analogous to a result of Fieldhouse [5] for semiperfect modules, a result of Wang and Sun [10] for cofinitely semiperfect modules. We also characterize $\delta$-semiperfect rings using the notions mentioned above. We close this section by some examples to separate the above introduced notions.

2 Cofinitely $\delta$-Supplemented Modules

In this section we define and study cofinitely $\delta$-supplemented modules. Let $M$ be an $R$-module, a submodule $U \subseteq M$ is called cofinite in $M$ if the factor module $M/U$ is finitely generated. Recall that a submodule $N \subseteq M$ is called a $\delta$-supplement of $K \subseteq M$, if $M = N + K$ and $N \cap K \ll_\delta N$.

**Definition 2.1.** An $R$-module $M$ is called cofinitely $\delta$-supplemented, briefly cof-$\delta$-supplemented, if each cofinite submodule of $M$ has a $\delta$-supplement in $M$. $M$ is called $\oplus$-cofinitely $\delta$-supplemented, briefly $\oplus$-cof-$\delta$-supplemented, if every cofinite submodule of $M$ has a $\delta$-supplement, which is a direct summand of $M$.

It is clear from definitions, that a $\delta$-supplemented module $M$ is cof-$\delta$-supplemented, and if $M$ is finitely generated, then the converse also holds. Next we give some properties of cof-$\delta$-supplemented modules.

**Proposition 2.2.** Let $M$ be a cof-$\delta$-supplemented module, then any factor module of $M$ is cof-$\delta$-supplemented. Hence, homomorphic images and direct summands of $M$ are also cof-$\delta$-supplemented.
Proof. Assume \( M \) is cof-\( \delta \)-supplemented, and let \( N \subseteq M \). Any cofinite submodule of \( M/N \) is of the form \( U/N \), where \( U \) is cofinite in \( M \). So, there exists \( K \subseteq M \), such that \( M = K + U \) and \( K \cap U \ll_{\delta} K \). Thus \( M/N = U/N + (K + N)/N \), and \( U/N \cap (K + N)/N = U \cap (K + N)/N = (N + (K \cap U))/N \ll_{\delta} (K + N)/N \). Hence, \( M/N \) is cof-\( \delta \)-supplemented.

Lemma 2.3. Let \( M \) be an \( R \)-module, \( N, K, X \subseteq M \) such that \( X \ll_{\delta} M \). If \( K \) is a \( \delta \)-supplement of \( N \) in \( M \), then \( K \) is a \( \delta \)-supplement of \( N + X \) in \( M \). If \( N + X \) has a \( \delta \)-supplement \( L \) in \( M \), then there exists a projective semisimple direct summand \( Y \) of \( M \) such that \( L + Y \) is a \( \delta \)-supplement of \( N \) in \( M \).

Proof. Assume \( K \) is a \( \delta \)-supplement of \( N \) in \( M \). Thus, \( M = N + K \) and \( N \cap K \ll_{\delta} K \). We will show that \( K \) is a \( \delta \)-supplement of \( N + X \) in \( M \).

It is clear that, \( M = K + N + X \). To show that \( K \cap (N + X) \ll_{\delta} K \), assume \( K = S + K \cap (N + X) \), where \( K/S \) is singular. Now \( M = K + N = S + K \cap (N + X) + N = K \cap (S + N + X) + N = S + N + X \), and \( M/(S + N) \) is singular, since \( M/(S + N) \cong K/(S + (K \cap N)) \). But \( X \ll_{\delta} M \), thus \( M = S + N \). Since \( U/S \) is singular and \( M = S + N = K + N \), then \( K = S \).

Hence \( K \cap (N + X) \ll_{\delta} K \). Therefor, \( K \) is a \( \delta \)-supplement of \( N + X \) in \( M \).

Conversely, assume \( L \) is a \( \delta \)-supplement of \( N + X \) in \( M \). Then \( M = L + N + X \) and \( L \cap (N + X) \ll_{\delta} L \). Since \( X \ll_{\delta} M \), by lemma 1.1, there exists a projective, semisimple submodule \( Y \) of \( M \), such that \( Y \subseteq X \) and \( M = (L + N) \oplus Y \). We will show that \( L + Y \) is a \( \delta \)-supplement of \( N \) in \( M \).

First, we have \( M = L + Y + N \) and \( N \cap (L + Y) \subseteq L \cap (N + Y) + Y \cap (N + L) = L \cap (N + Y) \subseteq L \cap (N + X) \ll_{\delta} L \), hence \( N \cap (L + Y) \ll_{\delta} L + Y \) as required.

To show that arbitrary sum of cof-\( \delta \)-supplemented modules is cof-\( \delta \)-supplemented, we need the following standard lemma

Lemma 2.4. Let \( L, U \) be submodules of a module \( M \) such that \( L \) is cof-\( \delta \)-supplemented, \( U \) is cofinite in \( M \) and \( L + U \) has a \( \delta \)-supplement \( K \) in \( M \). Then \( L \cap (K + U) \) has a \( \delta \)-supplement \( K \) in \( L \). Moreover, \( K + X \) is \( \delta \)-supplement of \( U \) in \( M \).

Proof. Let \( K \) a \( \delta \)-supplement of \( L + U \) in \( M \). Thus \( M + K + L + U \) and \( K \cap (L + U) \ll_{\delta} K \). Now \( L/L \cap (K + U) \cong (M/U)/(K + U)/U \), which is finitely generated, hence \( L \cap (K + U) \) is cofinite in \( L \). Since \( L \) is cof-\( \delta \)-supplemented, there exists \( X \subseteq L \) a \( \delta \)-supplement of \( L \cap (K + U) \) in \( L \). Thus \( L = X + L \cap (K + U) \) and \( X \cap L \cap (K + U) = X \cap (K + U) \ll_{\delta} X \). To show that \( K + X \) is a \( \delta \)-supplement of \( U \) in \( M \), we have \( M = K + L + U = K + X + L \cap (K + U) + U = K + X + U \), and \( U \cap (K + X) \subseteq K \cap (U + X) + X \cap (K + U) \subseteq K \cap (U + L) + X \cap (K + U) \ll_{\delta} K + X \). Therefor, \( K + X \) is a \( \delta \)-supplement of \( U \) in \( M \).
Proposition 2.5. An arbitrary sum of cof-$\delta$-supplemented modules is cof-$\delta$-supplemented.

Proof. Suppose that $\{M_i\}_{i \in I}$ is a family of cofinitely $\delta$-supplemented modules, and $M = \sum_{i \in I} M_i$. Let $U$ be a cofinite submodule of $M$, so $M = U + M_{i_1} + \cdots + M_{i_n}$ for some $n \in \mathbb{N}, i_k \in I$. Applying lemma 2.4 we see by induction, that $U$ has a $\delta$-supplement in $M$.

If $M$ is a $\delta$-supplemented module, then $M/\delta(M)$ is semisimple (see [2]). Next we prove an analogue for this result

Proposition 2.6. Let $M$ be a cofinitely $\delta$-supplemented module, then every cofinite submodule of $M/\delta(M)$ is a direct summand.

Proof. Assume $M$ is cof-$\delta$-supplemented. Every cofinite submodule of $M/\delta(M)$ has the form $U/\delta(M)$, where $U$ is a cofinite submodule of $M$ and $\delta(M) \subseteq U$. By assumption, there exists $K \subseteq M$ such that $M = K + U$ and $K \cap U \ll_{\delta} K$, hence $K \cap U \subseteq \delta(M)$. Since $U \cap (K + \delta(M)) = \delta(M) + (U \cap K) = \delta(M)$, thus $M/\delta(M) = (U + K)/\delta(M) = U/\delta(M) \oplus (K + \delta(M))/\delta(M)$.

From this, the following is clear

Corollary 2.7. Let $M$ be a cofinitely $\delta$-supplemented module, then $M/\delta(M)$ is $\oplus$-cofinitely $\delta$-supplemented.

Now we are going to prove that a module $M$ is cofinitely $\delta$-supplemented if and only if every maximal submodule of $M$ has a $\delta$-supplement in $M$. To do this we need the following lemma

Lemma 2.8. Let $U, K$ be submodules of an $R$-module $M$. If $K$ is a $\delta$-supplement of a maximal submodule of $M$ and $K + U$ has a $\delta$-supplement in $M$, then $U$ has a $\delta$-supplement in $M$.

Proof. Let $K$ be a $\delta$-supplement of a maximal submodule $Q \subseteq M$, and $X$ a $\delta$-supplement of $K + U$ in $M$. Thus, $M = X + K + U = K + Q$, $X \cap (K + U) \ll_{\delta} X$ and $K \cap Q \ll_{\delta} K$. Next we consider $K \cap (X + U)$.

If $K \cap (X + U) \subseteq K \cap Q$. We will show that $X + K$ is a $\delta$-supplement of $U$ in $M$. It is clear that $M = X + K + U$, and $U \cap (X + K) \subseteq X \cap (U + K) + K \cap (U + X)$. Since $X \cap (U + K) \ll_{\delta} X$ and $K \cap (U + X) \subseteq K \cap Q \ll_{\delta} K$, we get, by lemma 1.1, that $U \cap (X + K) \ll_{\delta} X + K$, thus $X + K$ is a $\delta$-supplement of $X$ in $M$.

Now, assume that $K \cap (X + U) \nsubseteq K \cap Q$. Since $K/K \cap Q = (K + Q)/Q = M/Q$, then $K \cap Q$ is maximal in $K$, thus $K = K \cap Q + (K \cap (X + U))$. By lemma 1.1, and since $K \cap Q \ll_{\delta} K$, there exists a projective semisimple submodule $Y$ of $M$ such that $K = Y \oplus (K \cap (X + U))$, and $Y \subseteq K \cap Q$. We will show that $X + Y$ is a $\delta$-supplement of $U$ in $M$. We have $M = X + K \cap (X + U) + Y + U = X + Y + U.$
Since \( X \cap (U + Y) \subseteq X \cap (U + K) \ll_\delta X \), then \( X \cap (U + Y) \ll_\delta X \). Since \( Y \cap (U + X) \subseteq Y \subseteq K \cap Q \ll_\delta K \), and \( Y \) is a direct summand in \( K \), then, by lemma 1.1, \( X \cap (U + X) + Y \sim (U + X) \ll_\delta X + Y \). Now \( U \cap (X + Y) \subseteq X \cap (U + Y) + Y \cap (U + X) \), thus \( U \cap (X + Y) \ll_\delta X + Y \). Hence, \( X + Y \) is a \( \delta \)-supplement of \( U \) in \( M \). In both cases, \( U \) has a \( \delta \)-supplement in \( M \).

For a module \( M \), let \( \text{Cof}_\delta(M) \) be the sum of all submodules of \( M \) that are \( \delta \)-supplements of maximal submodules of \( M \), and \( \text{Cof}_\delta(M) = 0 \), if there is no such submodule.

**Theorem 2.9.** Let \( M \) be an \( R \)-module, then the following are equivalent:

1. \( M \) is \( \text{cof-}\delta \)-supplemented;
2. Every maximal submodule of \( M \) has a \( \delta \)-supplement in \( M \);
3. The module \( M/\text{Cof}_\delta(M) \) has no maximal submodules.

**Proof.** (1) \( \Rightarrow \) (2). Clear.

(2) \( \Rightarrow \) (3). Assume \( M/\text{Cof}_\delta(M) \) has a maximal submodule \( Q/\text{Cof}_\delta(M) \), then \( Q \) is maximal in \( M \) and \( \text{Cof}_\delta(M) \subseteq Q \). By assumption, \( Q \) has a \( \delta \)-supplement \( K \) in \( M \), then \( K \subseteq \text{Cof}_\delta(M) \subseteq Q \). Thus, \( M = K + Q = Q \), a contradiction. Hence \( M/\text{Cof}_\delta(M) \) has no maximal submodules.

(3) \( \Rightarrow \) (1). Let \( U \) be a cofinite submodule of \( M \), then \( U + \text{Cof}_\delta(M) \) is also cofinite in \( M \). Then \( U + \text{Cof}_\delta(M) = M \), otherwise \( M/\text{Cof}_\delta(M) \) would have a maximal submodule. Since \( M/U \) is finitely generated and \( U + \text{Cof}_\delta(M) = M \), there exist a finite number of submodules \( K_1, K_2, \ldots, K_n \) of \( M \) that are \( \delta \)-supplements of maximal submodules of \( M \), such that \( M = U + K_1 + K_2 + \cdots + K_n \). By lemma 2.8, since \( M = (U + K_1 + K_2 + \cdots + K_{n-1}) + K_n \) has 0 as a \( \delta \)-supplement in \( M \), \( U + K_1 + K_2 + \cdots + K_{n-1} \) has a \( \delta \)-supplement in \( M \). By repeated use of lemma 2.8, \( U \) has a \( \delta \)-supplement in \( M \). Hence \( M \) is \( \text{cof-}\delta \)-supplemented. \( \square \)

**Example 2.10.** The \( \mathbb{Z} \)-module \( \mathbb{Q} \) of rational integers has no maximal submodules, so \( \mathbb{Q} \) is \( \oplus \)-cofinitely \( \delta \)-supplemented, and hence cofinitely \( \delta \)-supplemented. But \( \mathbb{Q} \) is not \( \delta \)-supplemented.

### 3 Cofinitely \( \delta \)-semiperfect modules

As a proper generalization of semiperfect modules, Calisici and Pancar [4] introduced and studied the notion of cofinitely semiperfect modules. An \( R \)-module \( M \) is called cofinitely semiperfect if every finitely generated factor module of \( M \) has a projective cover. In this section we define cofinitely \( \delta \)-semiperfect modules and study their basic properties. We also characterize these modules using the notion of cof-\( \delta \)-supplemented modules for the projective and the not necessarily projective modules.
Let $M$ be an $R$-module. Recall that a module $N$ with an epimorphism $f : N \rightarrow M$ is called a $\delta$-cover of $M$, provided that $\text{Ker}(f) \ll_{\delta} N$. If $N$ is also projective, then it is called a projective $\delta$-cover of $M$.

**Definition 3.1.** An $R$-module $M$ is called cofinitely $\delta$-semiperfect, briefly cof-$\delta$-semiperfect, if every factor module of $M$ by a cofinite submodule has a projective $\delta$-cover.

It is clear from the definition, that $\delta$-semiperfect, and hence semiperfect, modules are cof-$\delta$-semiperfect (see [2]). It is also clear that finitely generated cof-$\delta$-semiperfect modules are $\delta$-semiperfect. In general cof-$\delta$-semiperfect modules need not be $\delta$-semiperfect (see the examples at the end of this work). The following lemma is of interest in this work, it is proved in [?], we include its proof here for the sake of completeness.

**Lemma 3.2.** Let $M, N, P$ be modules. Given homomorphisms $f : P \rightarrow M$, $g : P \rightarrow N$ and $h : N \rightarrow M$ such that $hg = f$, then

1. $f$ is an epimorphism if and only if $N = \text{Ker}(h) + g(P)$.
2. $f$ is a $\delta$-cover if and only if $g(P)$ is a $\delta$-supplement of $\text{Ker}(h)$ in $N$ and $\text{Ker}(g) \ll_{\delta} P$.

**Proof.** 1. This is well known.

2. Assume $f$ is a $\delta$-cover. By (1), it follows that $N = \text{Ker}(h) + \text{Im}(g)$. It is easy to show that $g(\text{Ker}(f)) = \text{Ker}(h) \cap \text{Im}(g)$. Since $\text{Ker}(f) \ll_{\delta} P$, it follows from lemma 1.1 that $g(\text{Ker}(f)) \ll_{\delta} \text{Im}(g)$, hence $\text{Im}(g)$ is a $\delta$-supplement of $\text{Ker}(h)$ in $N$. To show that $\text{Ker}(g) \ll_{\delta} P$, we know that $\text{Ker}(g) \subseteq \text{Ker}(f) \ll_{\delta} P$, which implies that $\text{Ker}(g) \ll_{\delta} P$. For the other direction assume that $\text{Im}(g)$ is a $\delta$-supplement of $\text{Ker}(h)$ in $N$ and $\text{Ker}(g) \ll_{\delta} P$, so $N = \text{Im}(g) + \text{Ker}(h)$ and $\text{Im}(g) \cap \text{Ker}(h) \ll_{\delta} \text{Im}(g)$. From (1) it follows that $f$ is an epimorphism. To show $\text{Ker}(f) \ll_{\delta} P$, assume $\text{Ker}(f) + S = P$, and $P/S$ is singular. So $g(P) = g(\text{Ker}(f)) + g(S)$, but $g(\text{Ker}(f)) = \text{Ker}(h) \cap g(P)$, hence $\text{Ker}(h) \cap g(P) + g(S) = g(P)$. Now since $g(P)/g(S)$ is singular, being a homomorphic image of a singular module, and $\text{Ker}(h) \cap g(P) \ll_{\delta} g(P)$, we get $g(P) = g(S)$, and so, $P = S + \text{Ker}(g)$, but by assumption $\text{Ker}(g) \ll_{\delta} P$ and $P/S$ is singular, so $P = S$, hence $\text{Ker}(f) \ll_{\delta} P$ as required.

**Theorem 3.3.** Let $M$ be a cof-$\delta$-semiperfect module, then

1. $M$ is cof-$\delta$-supplemented,
2. any factor module of $M$ is cof-$\delta$-semiperfect, hence any homomorphic image and any direct summand of $M$ is cof-$\delta$-semiperfect.
Proof. Assume $M$ is a cof-δ-semiperfect module.

1. Let $U \subseteq M$ be cofinite. By assumption, there exists a projective δ-cover $f : P \to M/U$. Let $\pi : M \to M/U$ be the canonical map, since $P$ is projective, there exists a homomorphism $g : P \to M$ such that $\pi g = f$. By lemma 3.2, $g(P)$ is a δ-supplement of $U$ in $M$.

2. Let $f : M \to N$ be an epimorphism, and $U \subseteq N$ be a cofinite submodule. Then we have, $M/f^{-1}(U) \cong (M/Ker(f))/(f^{-1}(U)/Ker(f)) \cong N/U$, hence $f^{-1}(U)$ is cofinite in $M$. By assumption and the above isomorphism, $N/U$ has a projective δ-cover. Therefore, $N$ is cofinitely δ-semiperfect.

Let $N$ be a submodule of an $R$-module $M$, $N$ is said to have ample δ-supplements in $M$ if every submodule $L$ of $M$ with $M = N + L$ contains a δ-supplement of $N$ in $M$. $M$ is called amply cofinitely δ-supplemented, if every cofinite submodule of $M$ has ample $δ$-supplements in $M$.

Next we characterize cof-δ-semiperfect modules using the concepts of (amply) cofinitely $δ$-supplemented modules.

**Theorem 3.4.** Let $M$ be an $R$-module, then the following are equivalent

(1) $M$ is cof-δ-semiperfect.

(2) $M$ is amply cofinitely δ-supplemented by δ-supplements, which have projective δ-covers.

(3) $M$ is cofinitely δ-supplemented by δ-supplements which have projective δ-covers.

Proof. (1) $\Rightarrow$ (2). Let $M = U + Y$, $U$ is a cofinite submodule of $M$. By assumption, there exists a projective δ-cover $f : P \to M/U$. Let $\pi : Y \to Y/U \cap Y = M/U$ be the canonical epimorphism. Since $P$ is projective, there exists a homomorphism $g : P \to Y$, such that $\pi g = f$. By lemma 3.2, $g(P)$ is a δ-supplement of Ker($\pi$) = $U \cap Y$ in $Y$, and Ker($\pi$) $\ll_δ P$. Thus, $Y = g(P) + U \cap Y$, and $g(P) \cap U \cap Y = g(P) \cap U \ll_δ g(P)$. Hence, $M = U + Y = U + g(P) + U \cap Y = U + g(P)$. From this it follows that $g(P)$ is a $δ$-supplement of $U$ in $M$ and $g(P) \subseteq Y$. It is clear that, $g : P \to g(P)$ is a projective δ-cover of $g(P)$.

(2) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). Let $U$ be a cofinite submodule of $M$. By assumption, $U$ has a δ-supplement $K$ in $M$ and $K$ has a projective δ-cover $f : P \to K$. Since $U \cap K \ll_δ K$, then the canonical map $\pi : K \to K/U \cap K = M/U$ is a δ-cover of $M/U$. The composition $\alpha = \pi f : P \to M/U$ is a projective δ-cover of $M/U$. Therefore, $M$ is cofinitely $δ$-supplemented. □
An $R$-module $M$ is called $\delta$-lifting (see [8]), if for every submodule $N \subseteq M$, there exists a decomposition $M = A \oplus B$, such that $A \subseteq N$ and $N \cap B \ll_\delta B$. We call an $R$-module $M$ cofinitely $\delta$-lifting, if for every cofinite submodule $U \subseteq M$, there exists a decomposition $M = X \oplus Y$ such that $X \subseteq U$ and $U \cap Y \ll_\delta Y$. Similar to the characterizations of $\delta$-lifting modules we give here some characterizations of cofinitely $\delta$-lifting modules, (see [8] and [11, 41.11])

**Theorem 3.5.** For an $R$-module $M$, the following are equivalent

1. $M$ is cofinitely $\delta$-lifting:
2. Every cofinite submodule $U \subseteq M$ has a $\delta$-supplement $V$ in $M$ such that $U \cap V$ is a direct summand of $U$;
3. Every cofinite submodule $U \subseteq M$ can be written as $U = A \oplus B$ such that $A$ is a direct summand of $M$ and $B \ll_\delta M$;
4. For every cofinite submodule $U \subseteq M$, there exists a direct summand $X \subseteq M$ such that $X \subseteq U$ and $U/X \ll_\delta M/X$;
5. For every cofinite submodule $U \subseteq M$, there exists a direct summand $X$ of $M$ and a submodule $Y$ of $M$ with $X \subseteq U$, $U = X + Y$ and $Y \ll_\delta M$;
6. For every cofinite submodule $U \subseteq M$, there exists an idempotent $e \in \text{End}(M)$ with $e(M) \subseteq U$ and $(1 - e)(U) \ll_\delta (1 - e)(M)$.

**Proof.** (1) \(\Rightarrow\) (2). Let $U \subseteq M$ be cofinite. By assumption, $M = X \oplus Y$, $X \subseteq U$ and $U \cap Y \ll_\delta Y$. Thus $Y$ is a $\delta$-supplement of $U$ in $M$ and $U = U \cap (X \oplus Y) = X \oplus (U \cap Y)$.

(2) \(\Rightarrow\) (3). Let $U$ be cofinite in $M$. By assumption, there exists $V \subseteq M$, $Y \subseteq U$, such that $M = U + V$, $U \cap V \ll_\delta V$ and $U \cap V + Y = U$. Thus

$M = U + V = U \cap V + V = Y + V$ and $Y \cap V = 0$, hence $M = Y \oplus V$.

(3) \(\Rightarrow\) (4). Let $U$ be cofinite in $M$. By assumption, $U = A \oplus B$, where $A$ is a direct summand of $M$ and $B \ll_\delta M$. Let $\pi : M \rightarrow M/A$ be the canonical map. Since $B \ll_\delta M$, then $\pi(B) \ll_\delta M/A$, so $U/A \ll_\delta M/A$.

(4) \(\Rightarrow\) (5). Let $U$ be cofinite in $M$. By assumption, $M = X \oplus X'$, $X \subseteq U$, $U/X \ll_\delta M/X$. Thus $U = U \cap (X \oplus X') = X + (U \cap X')$ and $U \cap X' \cong U/X \ll_\delta M/X \cong X'$, hence $U \cap X' \ll_\delta M$.

(5) \(\Rightarrow\) (1). Let $U$ be cofinite in $M$. By assumption, $M = X \oplus X'$, $X \subseteq U$, $U = X + Y$, $Y \ll_\delta M$. Thus $X'$ is a $\delta$-supplement of $X$ in $M$. By lemma 2.3, and since $Y \ll_\delta M$, we have $X'$ is a $\delta$-supplement of $X + Y$ in $M$. Hence $X' \cap U = X' \cap (X + Y) \ll_\delta X'$.

(1) \(\Rightarrow\) (6). Let $U$ be cofinite in $M$. By assumption, $M = X \oplus Y$, $X \subseteq U$ and $U \cap Y \ll_\delta Y$. For the decomposition, $M = X \oplus Y$, there exists an
idempotent \( e \in \text{End}(M) \), such that \( e(M) = X \) and \((1 - e)(M) = Y \). Since \( X \subseteq U \), we have \((1 - e)(U) = U \cap (1 - e)(M) = U \cap Y \ll_{\delta} (1 - e)(M) \).

(6) \(\Rightarrow\) (1). Let \( e(M) = X \) and \((1 - e)(M) = Y \).

Next we characterize projective cofinitely \(\delta\)-semiperfect modules

**Theorem 3.6.** Let \( P \) be a projective \( R \)-module. The following are equivalent

1. \( P \) is cofinitely \(\delta\)-semiperfect;
2. \( P \) is \(\oplus\)-cofinitely \(\delta\)-supplemented;
3. \( P \) is cofinitely \(\delta\)-lifting.

**Proof.** (1) \(\Rightarrow\) (2). Let \( U \) be a cofinite submodule of \( P \). By assumption, \( P/U \) has a projective \(\delta\)-cover. By [12, lemma 2.4], \( P = P_1 \oplus P_2 \), \( P_1 \subseteq U \) and \( P_2 \cap U \ll_{\delta} P \). Hence, \( P = U + P_2 \). By lemma 1.1, \( P_2 \cap U \ll_{\delta} P_2 \), i.e., \( P_2 \) is a \(\delta\)-supplement of \( U \) in \( P \). Therefore, \( P \) is \(\oplus\)-cofinitely \(\delta\)-supplemented module.

(2) \(\Rightarrow\) (3). Let \( U \) be a cofinite submodule of \( P \). By assumption, there exist \( K, K' \subseteq P \) such that, \( K + U = P \), \( K \cap U \ll_{\delta} K \) and \( K \oplus K' = P \). By [9, lemma 1.16], \( P = K \oplus U' \), where \( U' \subseteq U \). Hence \( P \) is cofinitely \(\delta\)-lifting.

(3) \(\Rightarrow\) (1). Let \( U \subseteq P \) be cofinite. By assumption, \( P = X \oplus Y \), \( X \subseteq U \), \( U \cap Y \ll_{\delta} Y \). Hence, the canonical map \( \pi : Y \to Y/(U \cap Y) = P/U \) is a projective \(\delta\)-cover of \( P/U \). Therefore, \( P \) is cofinitely \(\delta\)-semiperfect.

An \( R \)-module \( M \) is said to have the Summand Sum Property (SSP), if the sum of any two direct summands of \( M \) is a gain a direct summand of \( M \), (see [6]).

**Proposition 3.7.** Let \( f : Q \to M \) be a projective \(\delta\)-cover. If \( Q \) has the SSP, then the following are equivalent

1. \( M \) is cof-\(\delta\)-semiperfect;
2. \( Q \) is cof-\(\delta\)-semiperfect.

**Proof.** Let \( f : Q \to M \) be a projective \(\delta\)-cover, and \( Q \) has the SSP.

(1) \(\Rightarrow\) (2) Assume \( M \) is cof-\(\delta\)-semiperfect. Let \( U \) be a cofinite submodule of \( Q \). Then \( f(U) \) is cofinite in \( M \), hence \( M/f(U) \) has a projective \(\delta\)-cover \( \alpha : P \to M/f(U) \), where \( P \) is projective. If \( \pi : M \to M/f(U) \) denotes the canonical epimorphism, then \( \pi f : Q \to M/f(U) \) is an epimorphism. Thus, by [12, lemma 2.3], \( Q = X \oplus Y \), \( Y \subseteq \text{Ker}(\pi f) = U + \text{Ker}(f) \), and \( (\pi f)|_X : X \to M/f(U) \) is a projective \(\delta\)-cover. From this it follows that \( X \cap (U + \text{Ker}(f)) \ll_{\delta} X \), and \( Q = X + U + \text{Ker}(f) \). Hence \( X \) is a \(\delta\)-supplement of \( U + \text{Ker}(f) \) in \( Q \). Since \( \text{Ker}(f) \ll_{\delta} Q \), then by lemma 2.3, there exists \( Z \) a direct summand of \( Q \), such
that \( X + Z \) is a \( \delta \)-supplement of \( U \) in \( Q \). Since \( Q \) has the SSP, then \( X + Z \) is a direct summand of \( Q \). Hence \( Q \) is \( \oplus \)-cofinitely \( \delta \)-supplemented. By theorem 3.6, \( Q \) is cofinitely \( \delta \)-semiperfect.

(2) \( \Rightarrow \) (1). This follows from theorem 3.3.

\textbf{Corollary 3.8.} An arbitrary direct sum of projective modules \( P_i, i \in I \) is cofinitely \( \delta \)-semiperfect if and only if each \( P_i \) is cofinitely \( \delta \)-semiperfect.

\textit{Proof.} The first direction is clear, since a direct summand of a cofinitely \( \delta \)-semiperfect module is cofinitely \( \delta \)-semiperfect, see theorem 3.3. Now, assume that each \( P_i \) is cofinitely \( \delta \)-semiperfect, hence, by theorem 3.6, each \( P_i \) is \( \oplus \)-cofinitely \( \delta \)-supplemented. Let \( P = \bigoplus P_i \) and \( U \) a cofinite submodule of \( P \). Since \( P/U \) is finitely generated, \( P = \bigoplus_{i \in I} P_i + P_{i_2} + \cdots + P_{i_n} \), for some \( n \in \mathbb{N} \) and \( i_1, i_2, \ldots, i_n \in I \). Thus \( P = \bigoplus_{k=1}^n P_{i_k} + U \). Now, it is clear that, 0 is a \( \delta \)-supplement of \( P_{i_1} + (\bigoplus_{k=2}^n P_{i_k} + U) \).

Since \( P_{i_1} \) is \( \oplus \)-cofinitely \( \delta \)-supplemented, \( P_{i_1} \cap (\bigoplus_{k=2}^n P_{i_k} + U) \) has a \( \delta \)-supplement \( S_{i_1} \) in \( P_{i_1} \) and \( S_{i_1} \) is a direct summand of \( P_{i_1} \), hence \( S_{i_1} \) is also a direct summand of \( P \). By lemma 2.4, \( S_{i_1} \) is a \( \delta \)-supplement of \( \bigoplus_{k=2}^n P_{i_k} + U \) in \( P \). Continuing this way \( n \)-times, we get \( \sum_{k=1}^n S_{i_k} = \bigoplus_{k=1}^n S_{i_k} \), which is a direct summand of \( P \), is a \( \delta \)-supplement of \( U \) in \( P \). Hence, \( P \) is \( \oplus \)-cofinitely \( \delta \)-supplemented. Since \( P \) is projective, then by theorem 3.6 again, \( P \) is cofinitely \( \delta \)-semiperfect as required.

\textbf{Now we give some characterizations of \( \delta \)-semiperfect rings in terms of the notions introduced in this work}

\textbf{Theorem 3.9.} Let \( R \) be a ring, then the following are equivalent

1. \( R \) is \( \delta \)-semiperfect;
2. Every finitely generated free module is \( \delta \)-semiperfect;
3. Every free module is cofinitely \( \delta \)-semiperfect;
4. Every free module is \( \oplus \)-cofinitely \( \delta \)-supplemented;
5. Every free module is cofinitely \( \delta \)-lifting;
6. Every module is cofinitely \( \delta \)-semiperfect;
7. \( R \) is \( \oplus \)-cofinitely \( \delta \)-supplemented.
Proof. (1) $\Leftrightarrow$ (2). This follows from the fact that a finite direct sum of $\delta$-supplemented modules is $\delta$-supplemented.

(1) $\Rightarrow$ (3). Every free $R$-module is a direct sum of copies of $R$. The result follows now from corollary 3.8.

(3) $\Rightarrow$ (2). Since a finitely generated cofinitely $\delta$-semiperfect module is $\delta$-semiperfect.

(3) $\Rightarrow$ (6). Every module is a factor of a free module. The result follows now from theorem 3.3.

(6) $\Rightarrow$ (3) and (4) $\Rightarrow$ (7) are trivial.

(3) $\Leftrightarrow$ (4) and (4) $\Leftrightarrow$ (5) follows from theorem 3.6.

(7) $\Rightarrow$ (4). Follows from theorem 3.6 and corollary 3.8. $\square$

We close this section by some examples. We start with an example of Nicholson [9, Example 2.15]. This example was also considered by Zhou [12, Example 4.3].

Example 3.10. Let $F$ be a field, $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, and

$$R = \{(x_1, \cdots, x_n, x, x, \cdots) : n \in \mathbb{N}, x_i \in M_2(F)\},$$

with component-wise operations. By [9, example 2.15], the Jacobson radical $\text{Rad}(R) = 0$ and $R$ is not a regular ring, hence $R$ is not semiperfect. By [12, example 4.3], we have $\delta(R) = \{(x_1, \cdots, x_n, x, x, \cdots) : n \in \mathbb{N}, x_i \in M_2(F), x \in J\}$, where $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, and $R$ is $\delta$-semiperfect. From this it follows that $R$ as a left $R$-module is

1. $\delta$-supplemented but not supplemented,
2. cofinitely $\delta$-supplemented, but not cofinitely supplemented,
3. cofinitely $\delta$-semiperfect, but not cofinitely semiperfect, and
4. $\oplus$-cofinitely $\delta$-supplemented, but not $\oplus$-cofinitely supplemented.

Example 3.11. Let $R$ be a $\delta$-semiperfect ring, which is not $\delta$-perfect (see [12, Example 4.4]). Then every $R$-module is cofinitely $\delta$-semiperfect. If every $R$-module is $\delta$-semiperfect, then $R$ would be $\delta$-perfect (see [2]). So, there are $R$-modules, which are cofinitely $\delta$-semiperfect but not $\delta$-semiperfect. In fact, $R^{(N)}$ is such a module.

Also for this ring, every $R$-module is cofinitely $\delta$-supplemented. If every (projective) $R$-module is $\delta$-supplemented, then $R$ would be $\delta$-perfect (see [12]). So, there are (projective) $R$-modules, which are cofinitely $\delta$-supplemented, but not $\delta$-supplemented. $R^{(N)}$ satisfies this property.
Cofinitely δ-semiperfect modules

References


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