

Noncooperative-Equivalent Imputation Formula in Cooperative Differential Games

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Abstract

A benchmark payoff for a player is that his imputation under cooperation must at least be equal to his noncooperative payoff at each time instant along the cooperative path. A distribution formula that yields noncooperative-equivalent imputation in cooperative differential games is presented. An exegesis of the rationale of this formula is also given.

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1 Introduction

Cooperative differential games represent a complex form of optimization analysis. The recent work of Yeung and Petrosyan (2004 and 2006) developed a generalized method for the derivation of analytically tractable time-consistent solutions. A benchmark payoff for a player is that his imputation under cooperation must at least be equal to his noncooperative payoff at each time instant along the cooperative path. A distribution formula that yields noncooperative-equivalent imputation in cooperative differential games is presented. An exegesis of the rationale of this formula is also given.

2 Mathematical Preliminaries

Consider the n -person nonzero-sum differential game with initial state x_0 and duration $T - t_0$. The state space of the game is $X \in R^m$, with permissible state trajectories $\{x(s), t_0 \leq s \leq T\}$. The state dynamics of the game is characterized by the vector-valued differential equations:

$$\dot{x}(s) = f[s, x(s), u_1(s), u_2(s), \dots, u_n(s)], \quad x(t_0) = x_0, \quad (1)$$

Where $u_i(s) \in R^{m_i}$ is the control vector of player i .

The objective of player i is

$$\int_{t_0}^T g^i[s, x(s), u_1(s), u_2(s), \dots, u_n(s)] ds + q^i(x(T)), \text{ for } i \in \{1, 2, \dots, n\} \equiv N \quad (2)$$

and $g^i[s, x(s), u_1(s), u_2(s), \dots, u_n(s)]$ and $q^i(x(T))$ are non-negative. The players payoffs are transferable.

Invoking the work of Isaacs (1965) and Bellman (1957) a feedback Nash equilibrium of the game can be characterized the following well-known theorem:

Theorem 2.1 *An n -tuple of strategies $\{\phi_i(t, x), \text{ for } i \in N\}$, provides a feedback Nash equilibrium solution to the game (1)-(2) if there exist continuously differentiable functions $V^i(t, x) : [t_0, T] \times R^m \rightarrow R, i \in N$, satisfying the following set of partial differential equations:*

$$\begin{aligned} -V_t^i(t, x) = & \\ & \max_{u_i} \{g^i[t, x, \phi_1(t, x), \phi_2(t, x), \dots, \phi_{i-1}(t, x), u_i, \phi_{i+1}(t, x), \dots, \phi_n(t, x)] \\ & + V_x^i(t, x) f[t, x, \phi_1(t, x), \phi_2(t, x), \dots, \phi_{i-1}(t, x), u_i, \phi_{i+1}(t, x), \dots, \phi_n(t, x)]\} \\ V^i(T, x) = & q^i(x), \quad i \in N \end{aligned}$$

The noncooperative payoff of player i at time t given that $x(t) = t$ is given by the continuously differentiable function $V^i(t, x)$.

Under cooperation group rationality required the players to maximize their joint payoff

$$\int_{t_0}^T \sum_{j=1}^n g^j[s, x(s), u_1(s), u_2(s), \dots, u_n(s)] ds + \sum_{j=1}^n q^j(x(T)) \quad (3)$$

subject to (1).

Let $\{\psi_i(s, x), \text{ for } i \in N\}$ denote a set of strategies leading to an optimal control solution of the problem (1) and (3) the total cooperative payoff over the interval $[t, T]$ where $t \in [t_0, T]$ is:

$$\begin{aligned}
 W(t, x_t^*) = & \int_t^T \sum_{j=1}^n q^j [s, x^*(s), \psi_1(s, x^*(s)), \psi_2(s, x^*(s)), \dots, \psi_n(s, x^*(s))] ds \\
 & + \sum_{j=1}^n q^j (x^*(T)). \tag{4}
 \end{aligned}$$

In explicitly solvable cooperative differential games, continuously differentiable functions $W(t, x_t^*)$ and $V^i(\tau, x_\tau)$, for $i \in N$, could be obtained in closed-form.

The state dynamics under cooperation is:

$$\dot{x}^*(s) = f[s, x^*(s), \psi_1(s, x^*(s)), \psi_2(s, x^*(s)), \dots, \psi_n(s, x^*(s))], \quad x(t_0) = x_0. \tag{5}$$

The corresponding optimal trajectory under cooperation can be expressed as:

$$x^*(t) = x_0 + \int_{t_0}^t f[s, x^*(s), \psi_1^*(s, x^*(s)), \psi_2^*(s, x^*(s)), \dots, \psi_n^*(s, x^*(s))] ds. \tag{6}$$

For notational convenience, we use the terms $x^*(t)$ and x_t^* interchangeably.

3 Noncooperative-Equivalent Imputation Formula

Consider the situation when all players agree to act so that a group optimum could be obtained. A benchmark payoff for a player is that his imputation must at least be equal to his noncooperative payoff at each time instant along the cooperative path $\{x^*(t)\}$.

Let $\xi^i(\tau, x_\tau^*)$ denote the imputation to player i under cooperation over the time interval $[\tau, T]$ along the cooperative path $\{x_\tau^*\}_{\tau=t_0}^T$ for $\tau \in [t_0, T]$. An imputation distribution procedure as in Petrosyan and Danilov (1982) and Yeung and Petrosyan (2004 and 2006) has to be formulated so that the cooperative imputation $\xi^i(\tau, x_\tau^*) = V^i(\tau, x_\tau^*)$ can be realized along the cooperative path.

To do this we let $B_i(s, x^*(s))$ denote the instantaneous rate of cooperative payment received by player i at time s . In particular,

$$\xi^i(\tau, x_\tau^*) = V^i(\tau, x_\tau^*) = \int_\tau^T B_i(s, x^*(s))ds + q^i(x_T^*), \text{ for } \tau \in [t_0, T] \quad (7)$$

Theorem 3.1 *A payment scheme with a terminal payment $q^i(x_T^*)$ at time T and an instantaneous rate of payment at time $\tau \in [t_0, T]$ along the cooperative trajectory $\{x_\tau^*\}_{\tau=t_0}^T$ being*

$$B_i(\tau, x_\tau^*) = -V_\tau^i(\tau, x_\tau^*) - V_{x_\tau^*}^i(\tau, x_\tau^*) f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \dots, \psi_n(\tau, x_\tau^*)] \quad (8)$$

yield the noncooperative-equivalent imputation

$$\xi^i(\tau, x_\tau^*) = \int_\tau^T B_i(s, x^*(s))ds + q^i(x_T^*) = V^i(\tau, x_\tau^*), \text{ for } \tau \in [t_0, T]$$

Proof.

Using (7) one can obtain the identity:

$$\int_{t_0}^\tau B_i(s, x^*(s))ds + V^i(\tau, x_\tau^*) \equiv V^i(t_0, x_0). \text{ for } \tau \in [t_0, T] \quad (9)$$

Differentiating (8) with respect to τ yields

$$B_i(\tau, x_\tau^*) = -dV^i(\tau, x_\tau^*)/d\tau = -V_\tau^i(\tau, x_\tau^*) - V_{x_\tau^*}^i(\tau, x_\tau^*)\dot{x}^*(\tau) \quad (10)$$

Invoking (5) we obtain

$$B_i(\tau, x_\tau^*) = -V_\tau^i(\tau, x_\tau^*) - V_{x_\tau^*}^i(\tau, x_\tau^*) f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \dots, \psi_n(\tau, x_\tau^*)] \quad (11)$$

Hence Theorem 3.1 follows. \blacklozenge

Theorem 3.1 yields a distribution formula for noncooperative-equivalent imputation in a cooperative differential game with dynamics (1) and payoffs (2). Such a formula can be obtained in closed form for any explicitly solvable games.

4 An Exegesis of the formula

In this section an exegesis of the rationale for the noncooperative-equivalent imputation formula (11) is presented. Note that the Isaacs-Bellman equation in Theorem 2.1 for a feedback Nash equilibrium in the noncooperative game (1) and (2) leads to

$$\begin{aligned}
 -V_{\tau}^i(\tau, x_{\tau}^*) &= g^i[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)] \\
 &\quad + V_{x_{\tau}^*}^i(\tau, x_{\tau}^*) f[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)], \\
 &\quad \text{for } \tau \in [t_0, T]
 \end{aligned} \tag{12}$$

Using (12) the distribution formula in (10) can be expressed as:

$$\begin{aligned}
 B_i(\tau, x_{\tau}^*) &= g^i[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)] \\
 &\quad + V_{x_{\tau}^*}^i(\tau, x_{\tau}^*) \{f[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)] \\
 &\quad - f[\tau, x_{\tau}^*, \psi_1(\tau, x_{\tau}^*), \psi_2(\tau, x_{\tau}^*), \dots, \psi_n(\tau, x_{\tau}^*)]\}
 \end{aligned} \tag{13}$$

However, along the cooperative path $\{x_{\tau}^*\}_{\tau=t_0}^T$, the instantaneous rate of payoff to player i is :

$$g^i[\tau, x_{\tau}^*, \psi_1(\tau, x_{\tau}^*), \psi_2(\tau, x_{\tau}^*), \dots, \psi_n(\tau, x_{\tau}^*)] \text{ at time instant } \tau. \tag{14}$$

In order for player i to realize an instantaneous rate of payoff equaling $B_i(\tau, x_{\tau}^*)$ a *noncooperative-equivalent compensation formula* can be obtained as

$$\begin{aligned}
 \vartheta_i(\tau, x_{\tau}^*) &= B_i(\tau, x_{\tau}^*) - g^i[\tau, x_{\tau}^*, \psi_1(\tau, x_{\tau}^*), \psi_2(\tau, x_{\tau}^*), \dots, \psi_n(\tau, x_{\tau}^*)] \\
 &= g^i[\tau, x_{\tau}^*, \psi_1(\tau, x_{\tau}^*), \psi_2(\tau, x_{\tau}^*), \dots, \psi_n(\tau, x_{\tau}^*)] \\
 &\quad - g^i[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)] \\
 &\quad + V_{x_{\tau}^*}^i(\tau, x_{\tau}^*) \{f[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)] \\
 &\quad - f[\tau, x_{\tau}^*, \psi_1(\tau, x_{\tau}^*), \psi_2(\tau, x_{\tau}^*), \dots, \psi_n(\tau, x_{\tau}^*)]\}
 \end{aligned} \tag{15}$$

In formula (15) the term

$$\begin{aligned}
 &g^i[\tau, x_{\tau}^*, \phi_1(\tau, x_{\tau}^*), \phi_2(\tau, x_{\tau}^*), \dots, \phi_n(\tau, x_{\tau}^*)] \\
 &- g^i[\tau, x_{\tau}^*, \psi_1(\tau, x_{\tau}^*), \psi_2(\tau, x_{\tau}^*), \dots, \psi_n(\tau, x_{\tau}^*)]
 \end{aligned}$$

yields the difference between player i 's rate of instantaneous payoffs when he uses the noncooperative strategy and that when he adopts the cooperative

strategy. The term $V_{x_\tau}^i(\tau, x_\tau^*)$ reflects the marginal effects of a change in the state variables on agent i 's noncooperative payoff. The term $f[\tau, x_\tau^*, \phi_1(\tau, x_\tau^*), \phi_2(\tau, x_\tau^*), \dots, \phi_n(\tau, x_\tau^*)]$ yields the instantaneous change of the states over time if the agents act noncooperatively, while the term $f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \dots, \psi_n(\tau, x_\tau^*)]$ yields the instantaneous change of the states over time if the agents act cooperatively.

Hence, the expression

$$V_{x_\tau}^i(\tau, x_\tau^*) \{ f[\tau, x_\tau^*, \phi_1(\tau, x_\tau^*), \phi_2(\tau, x_\tau^*), \dots, \phi_n(\tau, x_\tau^*)] - f[\tau, x_\tau^*, \psi_1(\tau, x_\tau^*), \psi_2(\tau, x_\tau^*), \dots, \psi_n(\tau, x_\tau^*)] \}$$

represents the compensation to player i when the change in the state variable follows the cooperative trajectory instead of the noncooperative path. To sum up, at time instant τ the compensation to player i leading to the noncooperative-equivalent instantaneous rate of payoff $B_i(\tau, x_\tau^*)$ consists of (i) the compensation on the difference between player i 's rate of instantaneous payoffs when he uses the noncooperative strategy and that when he adopts the cooperative strategy, and (ii) the compensation to agent i for the difference in the change in the state variable on the cooperative trajectory and that on the noncooperative path.

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