A Short Note on the Primary Submodules of Multiplication Modules

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Abstract. Let $M$ be an $R$-module. An $R$-module $M$ is called multiplication if for any submodule $N$ of $M$ we have $N = IM$, where $I$ is an ideal of $R$. In this paper we characterize primary submodules of multiplication modules.

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In this paper, all rings are commutative with identity and all modules are unitary. For a submodule $N$ of an $R$-module $M$, the set $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$, this is the ideal $Ann (M/N)$. A submodule $N$ of an $R$-module $M$ is said to be primary if $N \neq M$ and whenever $r \in R, m \in M$ and $rm \in N$, then $m \in N$ or $r^n \in (N : M)$ for some positive integer $n$. Let $PSpec (M)$ denote all primary submodules of $M$. If $N$ is primary submodule of an $R$-module $M$, it is easily shown that $(N : M)$ is primary ideal of $R$. 
An $R$-module $M$ is called multiplication if for any submodule $N$ of $M$ we have $N = IM$, where $I$ is an ideal of $R$. One can easily show that if $M$ is a multiplication module, then $N = (N : M)M$ for every submodule $N$ of $M$ [see, 1].

If $P$ is a maximal ideal of $R$, $T_P(M) = \{m \in M : (1 - p)m = 0 \text{ for some } p \in P\}$. Clearly $T_P(M)$ is a submodule of $M$. We say that $M$ is $P$-cyclic provided there exist $q \in P$ and $m \in M$ such that $(1 - q)M \subseteq Rm$.

In Example 1 we show that $PSpec(M)$ may be empty.

**Example 1.** Let $p$ be a fixed prime integer and $N_0 = \mathbb{Z}^+ \cup \{0\}$. Then $M = E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = r/p^n + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \text{ and } n \in N_0\}$ is a nonzero submodule of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each $t \in N_0$, set $G_t = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = r/p^t + \mathbb{Z} \text{ for some } r \in \mathbb{Z}\}$. $G_t$ is a cyclic submodule of $E(p)$ generated by $1/p^t + \mathbb{Z}$ for each $t \in N_0$. Each proper submodule of $E(p)$ is equal to $G_i$ for some $i \in N_0$. $(G_t : \mathbb{Z}E(p)) = 0$ for every $t \in N_0$. However no $G_t$ is primary submodule of $E(p)$, for if $p^k \notin (G_t : \mathbb{Z}E(p)) = 0$ for all $k$ and $\beta = 1/p^{i+t} + \mathbb{Z} \notin G_t(i > 0)$, but $p^{i+\beta} = 1/p^t + \mathbb{Z} \in G_t$. Consequently, $PSpec(M) = \emptyset$.

**Theorem 1.** Let $R$ be a commutative ring with identity. Then, an $R$-module $M$ is a multiplication module if and only if for every maximal ideal $P$ of $R$ either $M = T_P(M)$ or $M$ is $P$-cyclic.

**Proof.** See [2, Theorem 1.2].

For an ideal $I$, the intersection of all prime ideals containing $I$ is called radical of $I$ and denoted by $\sqrt{I}$. It is well known that $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n\}$.

Let $M$ be an $R$-module and $N$ a submodule of $M$. A submodule $N$ of $M$ is called prime if $N \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in N$, then $m \in N$ or $r \in (N : M)$ [see, for example, 3 and 5, 6]. In [2], Zeinab Abd El-Bast and Patrick F. Smith proved that if $M$ is a faithful multiplication module and $P$ a prime ideal of $R$ such that $M \neq PM$ then $PM$ is a prime submodule of $M$. Now, we prove that if $M$ is a faithful multiplication module and $P$ a primary ideal of $R$ such that $M \neq PM$ then $PM$ is a primary submodule of $M$.

**Theorem 2.** Let $P$ be a primary ideal of $R$ and $M$ a faithful multiplication $R$-module. Let $a \in R, x \in M$ satisfy $ax \in PM$. Then $a \in \sqrt{P}$ or $x \in PM$.

**Proof.** Let $a \notin \sqrt{P}$. Let $K = \{r \in R : rx \in PM\}$. Suppose $K \neq R$. Then there exists a maximal ideal $Q$ of $R$ such that $K \subseteq Q$. Clearly $x \notin T_Q(M)$. For if $x \in T_Q(M)$, then $(1 - q)x = 0$ for some $q \in Q$. Therefore, $0 = (1 - q)x \in PM$ and so $1 - q \in K \subseteq Q, 1 \in Q$, a contradiction.

By Theorem 1, $M$ is $Q$-cyclic, that is there exists $m \in M, q \in Q$ such that $(1 - q)M \subseteq Rm$. In particular, $(1 - q)x = sm$ and $(1 - q)ax = asm = pm$ for some $s \in R$ and $p \in P$. Thus $(as - p)m = 0$. Since $(1 - q)M \subseteq Rm, (1 - q)\text{Ann}(m)M \subseteq R\text{Ann}(m)m = 0$ and so $(1 - q)\text{Ann}(m)M =$
0. Now \((1 - q) \text{Ann}(m)\) \(M = 0\) implies \((1 - q) \text{Ann}(m) = 0\), because \(M\) is faithful, and hence \((1 - q) as = (1 - q)p \in P\). Indeed, \(as - p \in \text{Ann}(m)\) and so \((1 - q)(as - p) = 0\) \((1 - q)as = (1 - q)p\).

But \(P \subseteq K \subseteq Q\), so that \(s \in P\) (Since \((1 - q)^n \not\in P, a^m \not\in P\) for all \(m, n \in \mathbb{Z}^+\) and \(P\) is primary) and \((1 - q)x = sm \in PM \Rightarrow 1 - q \in K \subseteq Q\), a contradiction. It follows that \(K = R\) and \(x \in PM\), as required.

**Corollary 1.** Let \(M\) be a faithful multiplication \(R\)-module and \(P\) a primary ideal of \(R\) such that \(M \neq PM\). Then \(PM\) is a primary submodule of \(M\).

**Proof.** Let \(P\) be a primary ideal of \(R\) and \(M\) a faithful multiplication \(R\)-module. Then, \(ax \in PM \Rightarrow x \in PM\) or \(a \in \sqrt{P} \subseteq \sqrt{(PM : M)}\) where \(a \in R\) and \(x \in M\) by Theorem 2. Therefore, \(PM\) is a primary submodule of \(M\).

**Remark 1.** Let \(M\) be an \(R\)-module and \(N\) a submodule of \(M\) such that \(N \neq M\). Let \(I\) be an ideal of \(R\) such that \(I \subseteq \text{Ann}(M) = (0 : M)\). Then \(N\) is a primary \(R/I\)-submodule of \(M\) if and only if \(N\) is a primary submodule of \(M\) as an \(R/I\)-module.

**Corollary 2.** The following statements are equivalent for a proper submodule \(N\) of a multiplication \(R\)-module \(M\).

1. \(N\) is primary submodule of \(M\).
2. \((N : M)\) is primary ideal of \(R\).
3. \(N = QM\) for some primary ideal \(Q\) of \(R\) with \(\text{Ann}(M) = (0 : M) \subseteq Q\).

**Proof.** (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii). Clear.

(iii) \(\Rightarrow\) (i). Since \(N = QM \neq M\) and as an \(R/(0 : M)\)-module, \(N\) is primary, so \(N\) is primary as an \(R\)-submodule of \(M\) by Remark 1.

Let \(R\) be a commutative ring with identity. \(R\) is called a \(Q\)-ring if and only if every ideal in \(R\) is a product of primary ideals[see , 4].

**Definition 1.** Let \(M\) be an \(R\)-module. \(M\) is called a \(Q\)-module if every submodule \(N\) of \(M\) such that \(N \neq M\) either is primary or has a primary factorization \(N = Q_1Q_2...Q_nN^*\), where \(Q_1, Q_2, ..., Q_n\) are primary ideals of \(R\) and \(N^*\) is a primary submodule in \(M\).

**Theorem 3.** Let \(R\) be a \(Q\)-ring and \(M\) a faithful multiplication \(R\)-module. Then, \(M\) is a \(Q\)-module.

**Proof.** Let \(N\) be a submodule of \(M\) such that \(N \neq M\). Then \(N = IM\) for some ideal \(I\) of \(R\). Since \(R\) is a \(Q\)-ring, \(I = Q_1Q_2...Q_n\) and so \(N = Q_1Q_2...Q_nM\) where \(Q_1, Q_2, ..., Q_n\) are primary ideals of \(R\). Since \(N \neq M\), there exists a primary submodule \(Q_iM \neq M, 1 \leq i \leq n\). Therefore, \(N\) either is a primary submodule of \(M\) or has a primary factorization by Corollary 1.

**Corollary 3.** If \(R\) is a Dedekind domain and \(M\) a faithful multiplication \(R\)-module, then \(M\) is a \(Q\)-module.
Theorem 4. Let $M$ be a finitely generated faithful multiplication $R$-module. If $M$ is a $Q$-module, then $R$ is a $Q$-ring.

Proof. Let $I$ be any ideal of $R$ such that $I \neq R$. Then $IM$ is a submodule of $M$ such that $IM \neq M$ [see 2, Theorem 3.1]. Since $M$ is a $Q$-module $IM = Q_1Q_2\ldots Q_nN^*$ where $Q_1, Q_2, \ldots, Q_n$ are primary ideals of $R$ and $N^*$ is a primary submodule of $M$. Since $M$ is a multiplication module, $N^* = (N^* : M) M$, where $(N^* : M)$ is a primary ideal of $R$. Then $IM = Q_1Q_2\ldots Q_n (N^* : M) M$. Then $I = Q_1Q_2\ldots Q_n (N^* : M)$ [see 2, Theorem 3.1]. Consequently, $R$ is a $Q$-ring.

Theorem 5. Let $R$ be a $Q$-ring and $M$ a finitely generated faithful multiplication $R$-module. Let $S$ be a multiplicative closed subset of $R$. Then $R_S$-module $M_S$ is a $Q$-module.

Proof. Let $N_S$ be a submodule of $M_S$ such that $N_S \neq M_S$. Since $M$ is a multiplication $R$-module, $M_S$ is a multiplication $R_S$-module[1, Lemma 2 (i)]. Then $N_S = I_SM_S$ for some ideal $I_S$ of $R_S$. Since $R$ is a $Q$-ring, it is clear that $R_S$ is a $Q$-ring. Therefore, $I_S = (P_1)_S(P_2)_S \ldots (P_n)_S$ where $(P_1)_S, (P_2)_S, \ldots, (P_n)_S$ are primary ideals of $R_S$. It is clear that $Ann(M_S) = 0_S$. Therefore, $N_S$ either is a primary submodule or has a primary factorization. Consequently, $R_S$-module $M_S$ is a $Q$-module.

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