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A Short Note on the Primary Submodules of Multiplication Modules

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Abstract. Let M be an R-module. An R-module M is called multiplication if for any submodule N of M we have $N = IM$, where I is an ideal of R. In this paper we characterize primary submodules of multiplication modules.

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In this paper, all rings are commutative with identity and all modules are unitary. For a submodule N of an R-module M, the set $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$, this is the ideal $Ann\,(M/N)$. A submodule N of an R-module M is said to be primary if $N \neq M$ and whenever $r \in R, m \in M$ and $rm \in N$, then $m \in N$ or $r^n \in (N : M)$ for some positive integer n. Let $PSpec\,(M)$ denote all primary submodules of M. If N is primary submodule of an R-module M, it is easily shown that $(N : M)$ is primary ideal of R.

An R-module M is called multiplication if for any submodule N of M we have $N = IM$, where I is an ideal of R. One can easily show that if M is a multiplication module, then $N = (N : M) M$ for every submodule N of M [see, 1].

If P is a maximal ideal of R, $T_P(M) = \{m \in M : (1-p)m = 0 \text{ for some } p \in P\}$. Clearly $T_P(M)$ is a submodule of M. We say that M is P-cyclic provided there exist $q \in P$ and $m \in M$ such that $(1-q)M \subseteq Rm$.

In Example 1 we show that $PSpec(M)$ may be empty.

Example 1. *Let p be a fixed prime integer and $N_0 = Z^+ \cup \{0\}$. Then $M = E(p) = \{\alpha \in Q/Z : \alpha = r/p^n + Z$ for some $r \in Z$ and $n \in N_0\}$ is a nonzero submodule of the Z-module Q/Z. For each $t \in N_0$, set $G_t = \{\alpha \in Q/Z : \alpha = r/p^t + Z$ for some $r \in Z\}$. G_t is a cyclic submodule of $E(p)$ generated by $1/p^t + Z$ for each $t \in N_0$. Each proper submodule of $E(p)$ is equal to G_i for some $i \in N_0$. $(G_t :_Z E(p)) = 0$ for every $t \in N_0$. However no G_t is primary submodule of $E(p)$, for if $p^k \notin (G_t :_Z E(p)) = 0$ for all k and $\beta = 1/p^{i+t} + Z \notin G_t\,(i > 0)$, but $p^i\beta = 1/p^t + Z \in G_t$. Consequently, $PSpec(M) = \varnothing$.*

Theorem 1. *Let R be a commutative ring with identity. Then, an R-module M is a multiplication module if and only if for every maximal ideal P of R either $M = T_P(M)$ or M is P-cyclic.*

Proof. See [2, Theorem 1.2]. ∎

For an ideal I, the intersection of all prime ideals containing I is called radical of I and denoted by $\sqrt{I}$. It is well known that $\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n\}$.

Let M be an R-module and N a submodule of M. A submodule N of M is called prime if $N \neq M$ and whenever $r \in R, m \in M$ and $rm \in N$, then $m \in N$ or $r \in (N : M)$ [see, for example, 3 and 5, 6]. In [2], Zeinab Abd El-Bast and Patrick F. Smith proved that if M is a faithful multiplication module and P a prime ideal of R such that $M \neq PM$ then PM is a prime submodule of M. Now, we prove that if M is a faithful multiplication module and P a primary ideal of R such that $M \neq PM$ then PM is a primary submodule of M.

Theorem 2. *Let P be a primary ideal of R and M a faithful multiplication R-module. Let $a \in R, x \in M$ satisfy $ax \in PM$. Then $a \in \sqrt{P}$ or $x \in PM$.*

Proof. Let $a \notin \sqrt{P}$. Let $K = \{r \in R : rx \in PM\}$. Suppose $K \neq R$. Then there exists a maximal ideal Q of R such that $K \subseteq Q$. Clearly $x \notin T_Q(M)$. For if $x \in T_Q(M)$, then $(1-q)x = 0$ for some $q \in Q$. Therefore, $0 = (1-q)x \in PM$ and so $1 - q \in K \subseteq Q, 1 \in Q$, a contradiciton.

By Theorem 1, M is Q-cyclic, that is there exists $m \in M, q \in Q$ such that $(1-q)M \subseteq Rm$. In particular, $(1-q)x = sm$ and $(1-q)ax = asm = pm$ for some $s \in R$ and $p \in P$. Thus $(as - p)m = 0$. Since $(1-q)M \subseteq Rm, (1-q)Ann(m)M \subseteq RAnn(m)m = 0$ and so $(1-q)Ann(m)M =$

0.Now $[(1-q)\,Ann\,(m)]\,M = 0$ implies $(1-q)\,Ann\,(m) = 0$, because M is faithful, and hence $(1-q)\,as = (1-q)\,p \in P$. Indeed, $as - p \in Ann\,(m)$ and so $(1-q)\,(as-p) = 0, (1-q)\,as = (1-q)\,p$.

But $P \subseteq K \subseteq Q$, so that $s \in P($ Since $(1-q)^n \notin P, a^m \notin P$ for all $m, n \in Z^+$ and P is primary) and $(1-q)\,x = sm \in PM \Rightarrow 1-q \in K \subseteq Q$, a contradiction. It follows that $K = R$ and $x \in PM$, as required. ∎

Corollary 1. *Let M be a faithful multiplication $R-$module and P a primary ideal of R such that $M \neq PM$. Then PM is a primary submodule of M.*

Proof. Let P be a primary ideal of R and M a faithful multiplication R-module. Then, $ax \in PM \Rightarrow x \in PM$ or $a \in \sqrt{P} \subseteq \sqrt{(PM : M)}$ where $a \in R$ and $x \in M$ by Theorem 2. Therefore, PM is a primary submodule of M. ∎

Remark 1. *Let M be an R-module and N a submodule of M such that $N \neq M$. Let I be an ideal of R such that $I \subseteq Ann\,(M) = (0 : M)$. Then N is a primary R-submodule of M if and only if N is a primary submodule of M as an R/I-module.*

Corollary 2. *The following statements are equivalent for a proper submodule N of a multiplication R-module M.*

(i) N is primary submodule of M.
(ii) $(N : M)$ is primary ideal of R.
(iii) $N = QM$ for some primary ideal Q of R with $Ann\,(M) = (0 : M) \subseteq Q$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$. Clear.

$(iii) \Rightarrow (i)$. Since $N = QM \neq M$ and as an $R/\,(0 : M)$-module, N is primary, so is primary as an R-submodule of M by Remark 1. ∎

Let R be a commutative ring with identity. R is called a Q-ring if and only if every ideal in R is a product of primary ideals[see , 4] .

Definition 1. *Let M be an R-module. M is called a Q-module if every submodule N of M such that $N \neq M$ either is primary or has a primary factorization $N = Q_1 Q_2 ... Q_n N^*$, where $Q_1, Q_2, ..., Q_n$ are primary ideals of R and N^* is a primary submodule in M.*

Theorem 3. *Let R be a Q-ring and M a faithful multiplication R-module. Then, M is a Q-module.*

Proof. Let N be a submodule of M such that $N \neq M$. Then $N = IM$ for some ideal I of R. Since R is a Q-ring, $I = Q_1 Q_2 ... Q_n$ and so $N = Q_1 Q_2 ... Q_n M$ where $Q_1, Q_2, ..., Q_n$ are primary ideals of R. Since $N \neq M$, there exists a primary submodule $Q_i M \neq M, 1 \leq i \leq n$. Therefore, N either is a primary submodule of M or has a primary factorization by Corollary 1. ∎

Corollary 3. *If R is a Dedekind domain and M a faithful multiplication R-module, then M is a Q-module.*

Theorem 4. *Let M be a finitely generated faithful multiplication R-module. If M is a Q-module, then R is a Q-ring.*

Proof. Let I be any ideal of R such that $I \neq R$.Then IM is a submodule of M such that $IM \neq M$ [see 2, Theorem 3.1] . Since M is a Q-module $IM = Q_1Q_2...Q_nN^*$ where $Q_1, Q_2, ..., Q_n$ are primary ideals of R and N^* is a primary submodule of M. Since M is a multiplication module, $N^* = (N^* : M) M$, where $(N^* : M)$ is a primary ideal of R. Then $IM = Q_1Q_2...Q_n (N^* : M) M$. Then $I = Q_1Q_2...Q_n (N^* : M)$ [see 2, Theorem 3.1] . Consequently, R is a Q-ring. ∎

Theorem 5. *Let R be a Q-ring and M a finitely generated faithful multiplication R-module. Let S be a multiplicative closed subset of R. Then R_S-module M_S is a Q-module.*

Proof. Let N_S be a submodule of M_S such that $N_S \neq M_S$. Since M is a multiplication R-module, M_S is a multiplication R_S-module[1, Lemma 2 (i)] . Then $N_S = I_SM_S$ for some ideal I_S of R_S. Since R is a Q-ring, it is clear that R_S is a Q-ring. Therefore, $I_S = (P_1)_S (P_2)_S ... (P_n)_S$ where $(P_1)_S , (P_2)_S , ..., (P_n)_S$ are primary ideals of R_S. It is clear that $Ann (M_S) = 0_S$. Therefore, N_S either is a primary submodule or has a primary factorization. Consequently, R_S-module M_S is a Q-module. ∎

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