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The Rank Conditions such that AB and BA are Similar and Applications[1]

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Abstract

Suppose that A and B are two complex $n \times n$ matrices. What is the sufficient or necessary condition such that AB and BA are similar? In this note, we give an equivalent rank condition to answer the question. We also show several sufficient rank conditions of the same problem. As applications, some problems about matrix similar and generalized inverses are solved.

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1 Introduction and examples

Let $\mathbf{C}$ be the field of complex numbers. Denote by $M_n(\mathbf{C})$ the set of all $n \times n$ matrices over $\mathbf{C}$, and by $GL_n(\mathbf{C})$ the general linear group which consists of $n \times n$ invertible matrices over $\mathbf{C}$. Let $A, B \in M_n(\mathbf{C})$. As well known, AB and BA have the same characteristic polynomial, and if $A \in GL_n(\mathbf{C})$ or $B \in GL_n(\mathbf{C})$, then AB and BA are similar (see [4]). There is a natural question here. What is the sufficient or necessary condition such that AB and BA are similar? In this note, we will show some rank conditions which answer this problem. As applications, we solve some problems about matrix similar and generalized inverses.

We denote by I_k the $k \times k$ identity matrix. Let $X, Y \in M_n(\mathbf{C})$. If X and Y are similar, then we write $X \sim Y$. For a complex matrix A, denote

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by $A^t, A^*, A^{(1)}, A^{(1,2)}, A^{\#}, A^D$ and A^+ the transpose, transpose conjugate, (1)-inverse, (1,2)-inverse, group inverse, Drazin inverse and *M-P* inverse of A, respectively (see [2]). Also, we denote the rank, index and range of A by rank$A$, ind$A$ and R(A), respectively. Let $A\{1,2\}$ denote the set of all $\{1,2\}$-inverses of A. If A is an Hermitian semi-positive matrix, then we write as $A \geqslant 0$. Recall that the index t of A, denoted by ind$A$, is the smallest nonnegative integer for which rank$A^t = $ rankA^{t+1}.

Firstly, let us consider the following interesting examples.

Example 1.1 *Suppose $A \in M_n(\mathbf{C})$. Then $AA^* \sim A^*A$ and $AA^{(1)} \sim A^{(1)}A$.*

Example 1.2 *Suppose that $A, B \in M_n(\mathbf{C})$ with $A \geqslant 0$ and $B \geqslant 0$. Then $AB \sim BA$.*

Proof. Since $A \geqslant 0$, we can assume that $A = P\text{diag}(I_r, 0)P^*$, where $P \in GL_n(\mathbf{C})$. Thus,

$$AB = P\text{diag}(I_r, 0)P^*B \sim \text{diag}(I_r, 0)P^*BP.$$

Let $P^*BP = \begin{bmatrix} B_1 & B_2 \\ B_2^* & B_3 \end{bmatrix}$, where $B_1 \in M_r(\mathbf{C})$. By [1], there is an $r \times (n-r)$ matrix X such that $B_2 = B_1X$. Hence,

$$AB \sim \begin{bmatrix} B_1 & B_1X \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_r & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & X \\ 0 & I \end{bmatrix}.$$

It is easy to see that $B_1 \geqslant 0$. Hence AB is similar to a diagonal matrix. Similarly, we see that BA is similar to a diagonal matrix. Note that AB and BA have the same eigenvalue. We get $AB \sim BA$. ■

Example 1.3 *Suppose that A, B are two $n \times n$ orthogonal project matrices. Then by Example 1.2, we have $AB \sim BA$.*

Example 1.4 *Suppose that A, B are two $m \times n$ matrices. Then by Example 1.3 we have $AA^+BB^+ \sim BB^+AA^+$.*

Example 1.5 *Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then we have $AB = B$ and $BA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Clearly, we see that AB is not similar to BA.*

2 Main result

Lemma 2.1 *Suppose A and B are two nilpotent matrices. Then $A \sim B$ if and only if* $\mathrm{ind}A = \mathrm{ind}B$ *and* $\mathrm{rank}A^i = \mathrm{rank}B^i$ *for all* $i = 1, 2, \cdots, k-1$, *where* $k = \mathrm{ind}A$.

Proof. It is not different to prove the conclusion by Theory on Jordan forms of matrices. ■

Lemma 2.2 *Suppose that A is an $m \times n$ matrix, and B is an $n \times m$ matrix. Then there are $P \in GL_m(C)$ and $Q \in GL_n(C)$ such that*

$$A = P \begin{bmatrix} I_r & o \\ o & o \end{bmatrix} Q, \quad B = Q^{-1} \begin{bmatrix} D & 0 & 0 \\ 0 & N & T \\ 0 & S & Y \end{bmatrix} P^{-1}, \tag{1}$$

$$AB = P \begin{bmatrix} D & 0 & 0 \\ 0 & N & T \\ 0 & 0 & 0 \end{bmatrix} P^{-1}, \quad BA = Q^{-1} \begin{bmatrix} D & 0 & 0 \\ 0 & N & 0 \\ 0 & S & 0 \end{bmatrix} Q, \tag{2}$$

$$AB \sim \mathrm{diag}(D, N_1), \quad BA \sim \mathrm{diag}(D, N_2) \tag{3}$$

where $D \in GL_{r_1}(C)$ and N is an $r_2 \times\, r_2$ nilpotent matrix with $r_1 + r_2 = r$, and N_1 and N_2 are nilpotent.

Proof. By direct computation, one can from (1) conclude that (2), and from (2) conclude that (3). So, we only prove (1).

Firstly, let $A = P_1 \begin{bmatrix} I_r & o \\ o & o \end{bmatrix} Q_1$, where $P_1 \in GL_m(C)$ and $Q_1 \in GL_n(C)$. We write B as $Q_1^{-1} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} Q_1^{-1}$, where $B_1 \in M_r(C)$. By the Jordan form of B, we may assume that

$$B_1 = P_2 \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P_2^{-1},$$

where $P_2 \in GL_r(C)$, $D \in GL_{r_1}(C)$ and N is an $r_2 \times r_2$ nilpotent matrix such that $r_1 + r_2 = r$. Therefore, we see that

$$B = Q_1^{-1} \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} D & 0 & X_1 \\ 0 & N & X_2 \\ R_1 & R_2 & Y \end{bmatrix} \begin{bmatrix} P_2 & 0 \\ 0 & I_{m-r} \end{bmatrix} P_1^{-1},$$

and

$$A = P_1 \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I_{m-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_2 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q_1.$$

Next, let

$$P_3 = \begin{bmatrix} I_{r_1} & 0 & D^{-1}X_1 \\ 0 & I_{r_2} & 0 \\ 0 & 0 & I_{m-r} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{r_2} & 0 \\ R_1 D^{-1} & 0 & I_{n-r} \end{bmatrix}.$$

Then we have

$$B = Q_1^{-1} \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I_{n-r} \end{bmatrix} Q_2 \begin{bmatrix} D & 0 & 0 \\ 0 & N & X_2 \\ 0 & R_2 & Y_1 \end{bmatrix} P_3 \begin{bmatrix} P_2 & 0 \\ 0 & I_{m-r} \end{bmatrix} P_1^{-1}.$$

Finally, let $Q = Q_2^{-1} \begin{bmatrix} P_2 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q_1$, $P = P_1 \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I_{m-r} \end{bmatrix} P_3^{-1}$ and $T = S_2$, $S = R_2$. It is not difficult to see that (1) holds. This completes the proof. ■

Theorem 2.3 *Suppose that $A, B \in M_n(C)$. Then $AB \sim BA$ if and only if* $\mathrm{ind}(AB) = \mathrm{ind}(BA)$ *and* $\mathrm{rank}(AB)^i = \mathrm{rank}(BA)^i$ *for all $i = 1, 2, \cdots, k-1$, where $k = \mathrm{ind}(AB)$.*

Proof. By (3) of Lemma 2.2, we see that $AB \sim BA$ if and only if $\mathrm{diag}(D, N_1) \sim \mathrm{diag}(D, N_2)$. By the theory of Jordan form, one can conclude that $AB \sim BA$ if and only if $N_1 \sim N_2$. Noting that Lemma 2.1, the proof is completed. ■

By Lemma 2.2 and Theorem 2.3, we can prove the following corollary.

Corollary 2.4 *Suppose that $A, B \in M_n(C)$. Then we have the following conclusions.*

(a) $AB \sim BA$ if and only if $\mathrm{rank}(AB)^i = \mathrm{rank}(BA)^i$ *for all $i = 1, 2, 3, \cdots$;*
(b) $(AB)^D \sim (BA)^D$;
(c) If $\mathrm{ind}(AB) = \mathrm{ind}(BA) = 1$, *then $AB \sim BA$;*
(d) If $\mathrm{rank}(AB) = \mathrm{rank}(BA)$ *and* $\mathrm{ind}(AB) = 1$, *then $AB \sim BA$;*
(e) If $\mathrm{rank}(AB) = \mathrm{rank}(BA) = \mathrm{rank}(ABA)$, *then $AB \sim BA$;*
(f) If $\mathrm{rank}(A) = \mathrm{rank}(AB) = \mathrm{rank}(BA)$, *then $AB \sim BA$;*
(g) If $\mathrm{rank}(A) = \mathrm{rank}(BA)$ *and* $\mathrm{rank}(B) = \mathrm{rank}(AB)$, *then $AB \sim BA$;*
(h) If $\mathrm{rank}(A) = \mathrm{rank}(AB)$ *and* $\mathrm{rank}(B) = \mathrm{rank}(BA)$, *then $AB \sim BA$;*
(i) If $\mathrm{rank}(A) = \mathrm{rank}(AB)$ *and* $\mathrm{ind}(AB) = 1$, *then $AB \sim BA$;*
(j) If $\mathrm{rank}(B) = \mathrm{rank}(AB)$ *and* $\mathrm{ind}(AB) = 1$, *then $AB \sim BA$;*
(k) If $\mathrm{rank}(A) = \mathrm{rank}(AB) = \mathrm{rank}(ABA)$, *then $AB \sim BA$;*
(l) If $\mathrm{rank}(B) = \mathrm{rank}(AB) = \mathrm{rank}(BAB)$, *then $AB \sim BA$.*

Proof. (a) It follows from Theorem 2.3.

(b) Recall that if $X = \text{diag}(S, N)$ where S is invertible and N is nilpotent, then $X^D = \text{diag}(S^{-1}, 0)$. This, together with (3), implies that $(AB)^D \sim (BA)^D$.

(c) By Lemma 2.2 and (b) we see that $(AB)^{\#} \sim (BA)^{\#} \sim \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Hence $AB \sim BA \sim \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$.

(d) Noting that $\text{ind}(AB) = 1$, by Lemma 2.2 we have $AB \sim \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. Again by $\text{rank}(AB) = \text{rank}(BA)$ and Lemma 2.2, one can conclude that $BA \sim \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$. This shows that $AB \sim BA$.

(e) Firstly, by Lemma 2.2, we have

$$ABA = P\text{diag}(D, N, 0)Q.$$

Next, from (2) and $\text{rank}(AB) = \text{rank}(BA) = \text{rank}(ABA)$, we have

$$\text{rank}N = \text{rank}\begin{bmatrix} N & T \end{bmatrix} = \text{rank}\begin{bmatrix} N \\ S \end{bmatrix}.$$

Hence, there are an $r_2 \times (n-r)$ matrix T_0 and an $(n-r) \times r_2$ matrix S_0 such that $T = NT_0$ and $S = S_0 N$. Thus, we see that

$$\begin{bmatrix} I & T_0 \\ 0 & I \end{bmatrix} \begin{bmatrix} N & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -T_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} I & 0 \\ -S_0 & I \end{bmatrix} \begin{bmatrix} N & 0 \\ S & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ S_0 & I \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $AB \sim BA$.

(f) Since $\text{rank}(A) = \text{rank}(AB)$, we have $\text{R}(A) = \text{R}(AB)$. Furthermore, $\text{R}(BA) = \text{R}(BAB)$. This means that $\text{rank}(AB) = \text{rank}(BA) = \text{rank}(BAB)$. It follows from the conclusion (e).

(g) By Theorem 4.3 of [3], one can obtain that $\text{ind}(AB) = 1$ and $\text{ind}(BA) = 1$. Then using the conclusion of (c), we have $AB \sim BA$.

(h) It is easy to see that A^t and B^t satisfying the condition of (g). Thus, $A^t B^t \sim B^t A^t$. Furthermore, we have $AB \sim BA$.

(i) Noting that Theorem 4.3 of [3], we see that $\text{ind}(BA) = 1$. Next, form (c), we complete the proof.

(j) By Theorem 4.1 of [3], one can see that $\text{ind}(BA) = 1$. This, together with (c), gives that $AB \sim BA$.

(k) It follows form known conditions that

$$\text{R}(A) = \text{R}(AB) = \text{R}(ABA) = \text{R}(ABAB).$$

Hence, we have $\text{ind}(AB) = 1$. Moreover, we get $AB \sim BA$ by (i).

(l) It is clear that A^t and B^t satisfying the condition of (k). Then $A^tB^t \sim B^tA^t$, and we see that $AB \sim BA$. ■

Remark 2.5 *Let $A = diag(1, 1, 1, 1, 0, 0)$ and*

$$B = \begin{bmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 0 & 0 & & & \\ & & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix}.$$

From Theorem 2.3, one can get that $AB \sim BA$. But the pair of matrices A, B is not satisfying any conditions of (c)–(k). This tells us that the conditions of (c)–(k) only are sufficient but not are necessary.

Remark 2.6 *Let $A = diag(1, 1, 0)$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It is easy to see that the pair of matrices A, B satisfies the conditions of (e), but it is not satisfying the conditions of (c). Clearly, the any one of the conditions of (d), (g)–(k), satisfies the conditions of (c). On the other hand, one can conclude that the conditions of (c) implies the conditions of (e). In fact, by $\text{rank}(AB)^2 \leqslant \text{rank}(ABA) \leqslant \text{rank}(AB)$ and $\text{ind}(AB) = 1$, we see that $\text{rank}(AB) = \text{rank}(ABA)$. Similarly, we can know that $\text{rank}(BA) = \text{rank}(ABA)$.*

3 Applications

In this section, we shall give some applications of obtained results to problems of matrix similar and generalized inverses.

Theorem 3.1 *Suppose that $A, B \in M_n(C)$. Then the following conditions are equivalent.*

(i) AB and BA are similar to the same diagonal matrix;

(ii) AB is similar to a diagonal matrix and $\text{rank}(AB) = \text{rank}(BA)$;

(iii) BA is similar to a diagonal matrix and $\text{rank}(AB) = \text{rank}(BA)$.

Proof. It is clear that " $(i) \Longrightarrow (ii)$" and" $(i) \Longrightarrow (iii)$" hold true. We only prove that "$(ii) \Longrightarrow (i)$", since"$(iii) \Longrightarrow (i)$" can be proved in a similar way.

It follows form AB is similar to a diagonal matrix that ind(AB)=1. Thus, by the conclusion of (d) of Corollary 2.4, we have (i) holds. ■

Theorem 3.2 *Let* $A, B, C \in M_n(C)$ *such that* $\text{rank}A = \text{rank}B = \text{rank}C = r$ *and*

$$A = \begin{bmatrix} M & N \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} M_1 & N_1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} M_2 & 0 \\ N_2 & 0 \end{bmatrix},$$

where $M, M_1, M_2 \in M_r(C)$ *satisfying that they are similar each to other. Then* $A \sim B \sim C$.

Proof. It follows from $M \sim M_1$ that $M^t \sim M_1$. Hence there is an invertible matrix P such that $M_1 = PM^tP^{-1}$. Thus,

$$B = \begin{bmatrix} M_1 & N_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} M^t & P^{-1}N_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

This, together with $B \sim B^t$, implies that

$$B \sim \begin{bmatrix} M & 0 \\ (P^{-1}N_1)^t & 0 \end{bmatrix} \text{ and rank} \begin{bmatrix} M \\ (P^{-1}N_1)^t \end{bmatrix} = r.$$

Now, let

$$R = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \; S = \begin{bmatrix} M & N \\ (P^{-1}N_1)^t & 0 \end{bmatrix}.$$

It is easy to verify that $\text{rank}RS = \text{rank}SR = \text{rank}R = r$. So, by the conclusion of (f) of Corollary 2.4, one can see that $RS \sim SR$. Furthermore, we see that $A \sim \begin{bmatrix} M & 0 \\ (P^{-1}N_1)^t & 0 \end{bmatrix}$. This tells us that $A \sim B$. In a similar way, we can show that $B \sim C$. This completes the proof. ■

Theorem 3.3 *Suppose that* $A, B \in M_n(C)$ *such that* $A^2 = A$ *and* $I_n - A - B \in GL_n(C)$. *Then* $AB \sim BA$.

Proof. Without loss of generality, one can assume that $A = \text{diag}(I_r, 0)$, meanwhile, assume that $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, where $B_1 \in M_r(C)$. Noting that $I_n - A - B$ is invertible, we know that $\begin{bmatrix} -B_1 & -B_2 \\ -B_3 & I - B_4 \end{bmatrix}$ is invertible. Thus,

$$\text{rank} \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \text{rank} \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = r.$$

Finally, by Theorem 3.2, we have $AB = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix} = BA$. ■

Theorem 3.4 *Suppose that A is an $m \times n$ matrix and B is an $n \times m$ matrix. Then both $(AB)^{\#}$ and $(BA)^{\#}$ exist if and only if there are $P \in GL_m(C)$ and $Q \in GL_n(C)$ such that*

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q, \quad B = Q^{-1} \begin{bmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y \end{bmatrix} P^{-1},$$

where D is an $r_1 \times r_1$ invertible matrix and Y is any $(n-r) \times (m-r)$ matrix.

Proof. Form (2) in Lemma 2.2, the theorem can be easily verified. ■

Theorem 3.5 *Suppose that $A, B \in M_n(C)$. Then both $(AB)^{\#}$ and $(BA)^{\#}$ exist if and only if there is $P \in GL_n(C)$ such that*

$$A = P \begin{bmatrix} X \\ 0 \end{bmatrix} P^{-1}, \quad B = P \begin{bmatrix} X_1 D & 0 & X_2 \end{bmatrix} P^{-1}, \tag{4}$$

where X is the first r rows of some invertible matrix M, and X_1 is the first r_1 columns of M^{-1} $(r_1 \leq r)$; D is an $r_1 \times r_1$ invertible matrix and X_2 is any $n \times (n-r)$ matrix.

Proof. Firstly, we assume that both $(AB)^{\#}$ and $(BA)^{\#}$ exist. Noting that Theorem 3.4, we take $M = QP$. Thus,

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} QP \cdot P^{-1} = P \begin{bmatrix} X \\ 0 \end{bmatrix} P^{-1}$$

and

$$B = P \cdot P^{-1} Q^{-1} \begin{bmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y \end{bmatrix} P^{-1} = P \begin{bmatrix} X_1 D & 0 & X_3 Y \end{bmatrix} P^{-1},$$

where X_3 is the last $n-r$ columns of M^{-1}. Let $X_2 = X_3 Y$. Since Y is arbitrary, and X_2 is so. This has proved the necessity. Next we prove the sufficiently. Suppose that (4) holds. Let $M = NP$ for some $n \times n$ invertible matrix N. Let X_3 be the last $n-r$ columns of M^{-1}. Then there is Y such that $X_3 Y = X_2$. Therefore, we get

$$A = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} NP \cdot P^{-1} = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} N$$

and

$$B = P\cdot(NP)^{-1}\begin{bmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Y \end{bmatrix}P^{-1} = N^{-1}\begin{bmatrix} D & & \\ & 0 & \\ & & Y \end{bmatrix}P^{-1}.$$

Hence, one can obtain that ind(AB) = ind(AB) =1, and both $(AB)^{\#}$ and $(BA)^{\#}$ exist. The proof is completed. ■

Remark 3.6 *Similar to the proof of Theorem 3.5, we can prove that AB and BA are similar to the same diagonal matrix if and only if (4) holds true, where D is an diagonal matrix. Furthermore, both AB and BA are idempotent if and only if (4) holds true, where $D = I_{r_1}$.*

Theorem 3.7 *Suppose that A is an $m \times n$ matrix and B is an $n \times m$ matrix. Then there are $A^{(1,2)}$ and $B^{(1,2)}$ such that*

$$(AB)^D = B^{(1,2)}A^{(1,2)} \quad \textit{and} \quad (BA)^D = A^{(1,2)}B^{(1,2)}.$$

Proof. It follows from (2) that

$$(AB)^D = P\begin{bmatrix} D^{-1} & \\ & 0 \end{bmatrix}P^{-1} \quad \text{and} \quad (BA)^D = Q^{-1}\begin{bmatrix} D^{-1} & \\ & 0 \end{bmatrix}Q.$$

Noting that (1), let $B^{(1,2)} = P\begin{bmatrix} D^{-1} & 0 \\ 0 & \begin{bmatrix} N & T \\ S & Y \end{bmatrix}^{(1,2)} \end{bmatrix}Q$ and $A^{(1,2)} = Q^{-1}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$. It is easy to see that the theorem follows. ■

Remark 3.8 *Above Theorem 3.7 is a generalization of the results of Section 4 in [3] about group inverse.*

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