A New Bartholdi Zeta Function of a Graph

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Abstract

We define a new type of the Bartholdi zeta function of a graph G , and give a determinant expression of it. Furthermore, we define a new type of the Bartholdi L -function of G , and present a determinant expression for a new type of the Bartholdi L -function of G . As a corollary, we show that a new type of the Bartholdi zeta function of a regular covering of G is a product of new Bartholdi L -functions of G .

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1 Introduction

Graphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v . For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v . Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path* P of length n in G is a sequence $P = (e_1, \dots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ ($1 \leq i \leq n-1$), where indices are treated *mod* n . Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$ -*path*. We say that a path $P = (e_1, \dots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some i ($1 \leq i \leq n-1$). A (v, w) -path is called a v -*cycle* (or v -*closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \dots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \dots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \dots, e_m)$ and $C_2 = (f_1, \dots, f_m)$ are called *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j . The inverse cycle of C is in general not equivalent to C .

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Let $[C]$ be the equivalence class which contains a cycle C . Let B^r be the cycle obtained by going r times around a cycle B . Such a cycle is called a *power* of B . A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G .

Let G be a connected graph. For each $u, v \in V(G)$, let $[u, v]$ be the set of all (u, v) -paths in G . We say that a path $P = (e_1, \dots, e_n)$ has a *bump* at $t(e_i)$ if $e_{i+1} = e_i^{-1}$ ($1 \leq i \leq n$). The *bump count* $bc(P)$ of a path P is the number of bumps in P . Furthermore, the *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \dots, \pi_n)$ is

$$cbc(\pi) = |\{i = 1, \dots, n \mid \pi_i = \pi_{i+1}^{-1}\}|,$$

where $\pi_{n+1} = \pi_1$. Then the *Bartholdi zeta function* of G is defined to be a function of $u, t \in \mathbf{C}$ with $|u|, |t|$ sufficiently small, by

$$\zeta_G(u, t) = \zeta(G, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G (see [1]).

Bartholdi [1] gave a determinant expression of the Bartholdi zeta function of a graph.

Theorem 1 (Bartholdi) *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\zeta(G, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1 - u)(\mathbf{D} - (1 - u)\mathbf{I})t^2),$$

where $\mathbf{A}(G)$ is the adjacency matrix of G , and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ where $V(G) = \{v_1, \dots, v_n\}$.

If $u = 0$, then the Bartholdi zeta function of G is the (*Ihara*) *zeta function* of a graph G :

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G (see [6]).

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [6]. In [6], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of

the fundamental group of G was developed by Sunada [12,13]. Hashimoto [5] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial:

$$\mathbf{Z}(G, u)^{-1} = (1 - u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})).$$

Stark and Terras [12] gave an elementary proof of Bass' Theorem, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass' Theorem were given by Foata and Zeilberger [3], Kotani and Sunada [7].

Sato [9] defined a new zeta function of a graph by using not an infinite product but a determinant.

Let G be a connected graph and $V(G) = \{v_1, \dots, v_n\}$. Then we consider an $n \times n$ matrix $\mathbf{W} = (w_{ij})_{1 \leq i, j \leq n}$ with ij entry the complex variable w_{ij} if $(v_i, v_j) \in D(G)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(G)$ is called the *weighted matrix* of G . For each path $P = (v_{i_1}, \dots, v_{i_r})$ of G , the *norm* $w(P)$ of P is defined as follows: $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(G)$ and $w(e) = w_{ij}$, $e = (v_i, v_j) \in D(G)$.

Let G be a connected graph with n vertices and m unoriented edges, and $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G . Two $2m \times 2m$ matrices $\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in D(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \quad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *zeta function* of G is defined by

$$\mathbf{Z}_1(G, w, u) = \det(\mathbf{I}_n - u(\mathbf{B} - \mathbf{J}_0))^{-1}.$$

If $w(e) = 1$ for any $e \in D(G)$, then the zeta function of G is the Ihara zeta function of G .

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \dots, \mathbf{M}_s$.

Theorem 2 *Let G be a connected graph, and let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G . Then the reciprocal of the zeta function of G is given by*

$$\mathbf{Z}_1(G, w, u)^{-1} = (1 - u^2)^{m-n} \det(\mathbf{I}_n - u\mathbf{W}(G) + u^2(\mathbf{D} - \mathbf{I}_n)),$$

where $n = |V(G)|$, $m = |E(G)|$ and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(e)=v_i} w(e)$, $V(G) = \{v_1, \dots, v_n\}$.

In Section 2, we define a new type of the Bartholdi zeta function of a graph G , and give a determinant expression of it. In Section 3, we give a decomposition formula of a new type of the Bartholdi zeta function of a regular

covering of G . In Section 4, we define a new type of the Bartholdi L -function of G , and present a determinant expression for a new type of the Bartholdi L -function of G . Furthermore, we show that a new type of the Bartholdi zeta function of a regular covering of G is a product of new Bartholdi L -functions of G .

A general theory of the representation of groups and graph coverings, the reader is referred to [10] and [4], respectively.

2 A New Bartholdi zeta functions of graphs

We consider a new zeta function of a graph under Theorems 1,2, and define a new zeta function of a graph by using not an infinite product but a determinant.

Let G be a connected graph and $V(G) = \{v_1, \dots, v_n\}$. Then we consider an $n \times n$ matrix $\mathbf{W} = (w_{ij})_{1 \leq i, j \leq n}$ with ij entry the complex variable w_{ij} if $(v_i, v_j) \in D(G)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(G)$ is called the *weighted matrix* of G . For each path $P = (v_{i_1}, \dots, v_{i_r})$ of G , the *norm* $w(P)$ of P is defined as follows: $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(G)$ and $w(e) = w_{ij}$, $e = (v_i, v_j) \in D(G)$.

Let G be a connected graph with n vertices and m unoriented edges, and $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G . Two $2m \times 2m$ matrices $\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in D(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \quad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *zeta function* of G is defined by

$$\zeta_1(G, w, u, t) = \det(\mathbf{I}_n - t(\mathbf{B} - (1-u)\mathbf{J}_0))^{-1}.$$

If $w(e) = 1$ for any $e \in D(G)$, then the zeta function of G is the Bartholdi zeta function of G (see [1]). In the case of $u = 0$, then the zeta function of G is the new zeta function of G (see [9]).

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \dots, \mathbf{M}_s$.

Theorem 3 *Let G be a connected graph, and let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G . Then the reciprocal of the zeta function of G is given by*

$$\zeta_1(G, w, u, t)^{-1} = (1 - (1-u)^2 t^2)^{m-n} \det(\mathbf{I}_n - t\mathbf{W}(G) + (1-u)t^2(\mathbf{D} - (1-u)\mathbf{I}_n)),$$

where $n = |V(G)|$, $m = |E(G)|$ and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(e)=v_i} w(e)$, $V(G) = \{v_1, \dots, v_n\}$.

Proof. Similar to the proof of Theorem 6. $\square$

If $u = 0$, then Theorem 3 implies Theorem 2.

3 Zeta functions of regular coverings of graphs

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v, w) \in D(G)\}$ denote the neighbourhood of a vertex v in G . A graph H is called a *covering* of G with projection $\pi : H \rightarrow G$ if there is a surjection $\pi : V(H) \rightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \rightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G , the *quotient graph* G/Π is a graph whose vertices are the Π -orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G . A covering $\pi : H \rightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $\text{Aut}H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G .

Let G be a graph and Γ a finite group. Then a mapping $\alpha : D(G) \rightarrow \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called an *ordinary voltage graph*. The *derived graph* G^α of the ordinary voltage graph (G, α) is defined as follows: $V(G^\alpha) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $k = h\alpha(u, v)$. The *natural projection* $\pi : G^\alpha \rightarrow G$ is defined by $\pi(u, h) = u$. The graph G^α is called a *derived graph covering* of G with voltages in Γ or a Γ -*covering* of G . The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in D(G)$ and the left action of Γ on the fibers: $g(u, h) = (u, gh), g \in \Gamma$, which is free and transitive. Thus, the Γ -covering G^α is a $|\Gamma|$ -fold regular covering of G with covering transformation group Γ . Furthermore, every regular covering of a graph G is a Γ -covering of G for some group Γ (see [4]).

Let G be a connected graph, Γ a finite group and $\alpha : D(G) \rightarrow \Gamma$ an ordinary voltage assignment. In the Γ -covering G^α , set $v_g = (v, g)$ and $e_g = (e, g)$, where $v \in V(G), e \in D(G), g \in \Gamma$. For $e = (u, v) \in D(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1} = (e^{-1})_{g\alpha(e)}$.

Let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G . Then we define the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(G^\alpha) = (\tilde{w}(u_g, v_h))$ of G^α derived from $\mathbf{W}$ as follows:

$$\tilde{w}(u_g, v_h) := \begin{cases} w(u, v) & \text{if } (u, v) \in D(G) \text{ and } h = g\alpha(u, v), \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, let the matrix $\mathbf{W}_g = (w_{uv}^{(g)})$ be defined by

$$w_{uv}^{(g)} := \begin{cases} w(u, v) & \text{if } \alpha(u, v) = g \text{ and } (u, v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{M}_1 = \mathbf{M}_2 = \dots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \dots \oplus \mathbf{M}_s$. The *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 4 *Let G be a connected graph with ν vertices and ϵ unoriented edges, $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G , Γ a finite group and $\alpha : D(G) \rightarrow \Gamma$*

an ordinary voltage assignment. Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_k$ be the irreducible representations of Γ , and f_i the degree of ρ_i for each i , where $f_1 = 1$. Suppose that the Γ -covering G^α of G is connected. Then the reciprocal of the zeta function of G^α is

$$\begin{aligned} \zeta_1(G^\alpha, \tilde{w}, u, t)^{-1} &= \zeta_1(G, w, u, t)^{-1} \\ &\cdot \prod_{i=2}^k \{ (1 - (1-u)^2 t^2)^{(\epsilon-\nu)f_i} \det(\mathbf{I}_{\nu f_i} - t \sum_{h \in \Gamma} \rho_i(h) \otimes \mathbf{W}_h \\ &\quad + (1-u)t^2(\mathbf{I}_{f_i} \otimes (\mathbf{D} - (1-u)\mathbf{I}_\nu))) \}^{f_i}. \end{aligned}$$

Proof. Let $V(G) = \{v_1, \dots, v_\nu\}$ and $\Gamma = \{1 = g_1, g_2, \dots, g_m\}$. Arrange arcs of G^α in m blocks: $(v_1, 1), \dots, (v_\nu, 1); (v_1, g_2), \dots, (v_\nu, g_2); \dots; (v_1, g_m), \dots, (v_\nu, g_m)$. We consider the adjacency matrix $\mathbf{W}(G^\alpha)$ under this order. For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $((u, g_i), (v, g_j)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $g_j = g_i \alpha(u, v)$, i.e., $\alpha(u, v) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus we have

$$\mathbf{W}(G^\alpha) = \sum_{h \in \Gamma} \mathbf{P}_h \otimes \mathbf{W}_h.$$

Let ρ be the right regular representation of Γ . Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_k$ be the irreducible representations of Γ , and f_i the degree of ρ_i for each i , where $f_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1} \rho(h) \mathbf{P} = (1) \oplus f_2 \circ \rho_2(h) \oplus \dots \oplus f_k \circ \rho_k(h)$ for each $h \in \Gamma$ (see [10]). Putting $\mathbf{F} = (\mathbf{P}^{-1} \otimes \mathbf{I}_\nu) \mathbf{W}(G^\alpha) (\mathbf{P} \otimes \mathbf{I}_\nu)$, we have

$$\mathbf{F} = \sum_{h \in \Gamma} \{ (1) \oplus f_2 \circ \rho_2(h) \oplus \dots \oplus f_k \circ \rho_k(h) \} \otimes \mathbf{W}_h.$$

Note that $\mathbf{W}(G) = \sum_{h \in \Gamma} \mathbf{W}_h$ and $1 + f_2^2 + \dots + f_k^2 = m$. Therefore it follows that

$$\begin{aligned} \zeta_1(G^\alpha, \tilde{w}, u, t)^{-1} &= (1 - (1-u)^2 t^2)^{(\epsilon-\nu)m} \det(\mathbf{I}_{\nu m} - t \mathbf{W}(G^\alpha) + (1-u)(\mathbf{I}_m \otimes (\mathbf{D} - (1-u)\mathbf{I}_\nu)) t^2) \\ &= (1 - (1-u)^2 t^2)^{\epsilon-\nu} \det(\mathbf{I}_\nu - t \mathbf{W}(G) + (1-u)(\mathbf{D} - (1-u)\mathbf{I}_\nu) t^2) \\ &\quad \times \prod_{i=2}^k \{ (1 - (1-u)^2 t^2)^{(\epsilon-\nu)f_i} \det(\mathbf{I}_{\nu f_i} - t \sum_{h \in \Gamma} \rho_i(h) \otimes \mathbf{W}_h + (1-u)t^2(\mathbf{I}_{f_i} \otimes (\mathbf{D} - (1-u)\mathbf{I}_\nu))) \}^{f_i}. \end{aligned}$$

□

We give another decomposition formula for the zeta function of a regular covering G^α of a graph G .

Let G be a connected graph, Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Let $w : D(G) \longrightarrow \mathbf{C}$ be a weight of G . Then two matrices $\tilde{\mathbf{B}} = \mathbf{B}(G^\alpha) = (\tilde{b}(e_g, f_h))$ and $\tilde{\mathbf{J}}_0 = \mathbf{J}_0(G^\alpha) = (\tilde{c}(e_g, f_h))$ of G^α are given by

$$\tilde{b}(e_g, f_h) := \begin{cases} w(f) & \text{if } t(e_g) = o(f_h), \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{c}(e_g, f_h) := \begin{cases} 1 & \text{if } e_g^{-1} = f_h, \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, let the matrix $\mathbf{B}_g = (b_{ef}^{(g)})$ be defined by

$$b_{ef}^{(g)} := \begin{cases} w(f) & \text{if } \alpha(e) = g \text{ and } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let the matrix $\mathbf{J}_g = (c_{ef}^{(g)})$ be defined by

$$c_{ef}^{(g)} := \begin{cases} 1 & \text{if } \alpha(e) = g \text{ and } e^{-1} = f, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5 *Let G be a connected graph with l unoriented edges, $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G , Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_k$ be all inequivalent irreducible representations of Γ , and f_i the degree of ρ_i for each i , where $f_1 = 1$. Suppose that the Γ -covering G^α of G is connected. Then the reciprocal of the zeta function of G^α is*

$$\zeta_1(G^\alpha, \tilde{w}, u, t)^{-1} = \zeta_1(G, w, u, t)^{-1} \cdot \prod_{i=2}^k \det(\mathbf{I}_{2lf_i} - t \sum_{h \in \Gamma} \rho_i(h) \otimes (\mathbf{B}_h - (1-u)\mathbf{J}_h))^{f_i}.$$

Proof. Let $D(G) = \{e_1, \dots, e_l, e_{l+1}, \dots, e_{2l}\}$ and $\Gamma = \{1 = g_1, g_2, \dots, g_m\}$. Arrange arcs of G^α in m blocks: $(e_1, 1), \dots, (e_{2l}, 1); (e_1, g_2), \dots, (e_{2l}, g_2); \dots; (e_1, g_m), \dots, (e_{2l}, g_m)$. We consider the matrix $\tilde{\mathbf{B}} - \tilde{\mathbf{J}}_0$ under this order. For $h \in \Gamma$, let $\mathbf{P}_h = (p_{ij}^{(h)})$. Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $t(e, g_i) = o(f, g_j)$ if and only if $t(e) = o(f)$ and $(o(f), g_j) = o(f, g_j) = t(e, g_i) = t(e), g_i \alpha(e)$, i.e., $\alpha(e) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus we have

$$\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}}_0 = \sum_{h \in \Gamma} \mathbf{P}_h \otimes (\mathbf{B}_h - (1-u)\mathbf{J}_h).$$

Let ρ be the right regular representation of Γ . Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_t$ be all inequivalent irreducible representations of Γ , and f_i the degree of ρ_i for each i , where $f_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$.

Furthermore, there exists a regular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_t \circ \rho_t(h)$ for each $h \in \Gamma$. Putting $\mathbf{F} = (\mathbf{P}^{-1} \otimes \mathbf{I}_{2l})(\tilde{\mathbf{B}} - \tilde{\mathbf{J}}_0)(\mathbf{P} \otimes \mathbf{I}_{2l})$, we have

$$\mathbf{F} = \sum_{h \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_t \circ \rho_t(h)\} \otimes (\mathbf{B}_h - (1-u)\mathbf{J}_h).$$

Note that $\mathbf{B} - \mathbf{J}_0 = \sum_{h \in \Gamma} (\mathbf{B}_h - \mathbf{J}_h)$ and $1 + f_2^2 + \cdots + f_t^2 = m$. Therefore it follows that

$$\begin{aligned} \zeta_1(G^\alpha, \tilde{w}, u, t)^{-1} &= \det(\mathbf{I}_{2lm} - t(\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}}_0)) \\ &= \det(\mathbf{I}_{2l} - t(\mathbf{B} - (1-u)\mathbf{J}_0)) \prod_{i=2}^k \det(\mathbf{I}_{2lf_i} - t \sum_h \rho_i(h) \otimes (\mathbf{B}_h - (1-u)\mathbf{J}_h))^{f_i}. \end{aligned}$$

□

4 L -functions of graphs

Let G be a connected graph with n vertices and l unoriented edges, $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G , Γ a finite group and $\alpha : D(G) \rightarrow \Gamma$ an ordinary voltage assignment. For each path $P = (e_1, \dots, e_r)$ of G , set $\alpha(P) = \alpha(e_1) \cdots \alpha(e_r)$. This is called the *net voltage* of P . Furthermore, let ρ be a unitary representation of Γ and d its degree. The L -function of G associated with ρ and α is defined by

$$\zeta_1(G, w, u, t, \rho, \alpha) = \det(\mathbf{I}_{2ld} - t \sum_{h \in \Gamma} \rho(h) \otimes (\mathbf{B}_h - (1-u)\mathbf{J}_h))^{-1}.$$

If $w(e) = 1$ for each $e \in D(G)$, then the L -function of G is the Bartholdi L -function of G :

$$\zeta_1(G, w, u, t, \rho, \alpha) = \zeta_G(u, t, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - \rho(\alpha(C)) u^{cbc(C)} t^{|C|})^{-1},$$

where $cbc(C)$ is the cyclic bump count of V , and $[C]$ runs over all equivalence classes of prime, reduced cycles of G (see [8]). In the case of $u = 0$, then the L -function of G is the new L -function of G (see [9]).

A determinant expression for the L -function of G associated with ρ and α is given as follows. Let $1 \leq i, j \leq n$. Then, the (i, j) -block $\mathbf{F}_{i,j}$ of a $dn \times dn$ matrix $\mathbf{F}$ is the submatrix of $\mathbf{K}$ consisting of $d(i-1) + 1, \dots, di$ rows and $d(j-1) + 1, \dots, dj$ columns.

Theorem 6 *Let G be a connected graph with n vertices and l unoriented edges, $\mathbf{W}(G)$ a weighted matrix of G , Γ a finite group and $\alpha : D(G) \rightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ a representation of Γ and d the degree of ρ . Then the reciprocal of the L -function of G associated with ρ and α is*

$$\zeta_1(G, w, u, t, \rho, \alpha)^{-1} = (1 - (1-u)^2 t^2)^{(l-n)d} \det(\mathbf{I}_{nd} - t \sum_{h \in \Gamma} \rho(h) \otimes \mathbf{W}_h + (1-u)t^2(\mathbf{I}_d \otimes (\mathbf{D} - (1-u)\mathbf{I}_n))).$$

Proof. The argument is an analogue of Bass' method [2].

Let ρ be a unitary representation of Γ , and d the degree of ρ . Furthermore, let $D(G) = \{e_1, \dots, e_l, e_{l+1}, \dots, e_{2l}\}$ such that $e_{l+i} = e_i^{-1}$ ($1 \leq i \leq l$). Let $\mathbf{K} = (\mathbf{K}_{i,j})_{1 \leq i \leq 2l; 1 \leq j \leq n}$ be the $2ld \times nd$ matrix defined as follows:

$$\mathbf{K}_{i,j} := \begin{cases} w(e_i)^t \mathbf{I}_d & \text{if } o(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Furthermore, we define two $2ld \times nd$ matrices $\mathbf{L} = (\mathbf{L}_{i,j})_{1 \leq i \leq 2l; 1 \leq j \leq n}$ and $\mathbf{H} = (\mathbf{H}_{i,j})_{1 \leq i \leq 2l; 1 \leq j \leq n}$ as follows:

$$\mathbf{L}_{i,j} := \begin{cases} \rho(\alpha(e_i)) & \text{if } t(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}, \mathbf{H}_{i,j} := \begin{cases} \mathbf{I}_d & \text{if } o(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{L}^t \mathbf{K} = \sum_{h \in \Gamma} \mathbf{B}_h \otimes \rho(h). \quad (1)$$

and

$${}^t \mathbf{K} \mathbf{L} = \sum_{g \in \Gamma} \mathbf{W}_g \otimes \rho(g). \quad (2)$$

Furthermore,

$${}^t \mathbf{H} \mathbf{K} = \mathbf{D}_d = \mathbf{D} \otimes \mathbf{I}_d, \quad (3)$$

where ${}^t \mathbf{F}$ is the transpose of $\mathbf{F}$.

Now, let

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & w(e_1^{-1})\rho(\alpha(e_1)) \oplus \dots \oplus w(e_l^{-1})\rho(\alpha(e_l)) \\ w(e_1)\rho(\alpha(e_1))^{-1} \oplus \dots \oplus w(e_l)\rho(\alpha(e_l))^{-1} & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{N} = \mathbf{B}_\rho - \mathbf{M},$$

where $\mathbf{B}_\rho = \sum_{h \in \Gamma} \mathbf{B}_h \otimes \rho(h)$. Then we have

$$\mathbf{K}^t \mathbf{H} = {}^t \mathbf{N}^t \mathbf{J}_\rho + (w(e_1)\mathbf{I}_d \oplus \dots \oplus w(e_{2l})\mathbf{I}_d), \quad (4)$$

where $\mathbf{J}_\rho = \sum_{h \in \Gamma} \mathbf{J}_h \otimes \rho(h)$.

We introduce two $(2l+n)d \times (2l+n)d$ matrices as follows:

$$\mathbf{P} = \begin{bmatrix} (1 - (1-u)^2 t^2) \mathbf{I}_{nd} & -{}^t \mathbf{L} + (1-u)t {}^t \mathbf{H} \\ \mathbf{0} & \mathbf{I}_{2ld} \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} \mathbf{I}_{nd} & {}^t \mathbf{L} - (1-u)t {}^t \mathbf{H} \\ {}^t \mathbf{K} & (1 - (1-u)^2 t^2) \mathbf{I}_{2ld} \end{bmatrix}$$

By (2) and (3), we have

$$\begin{aligned} \mathbf{P} \mathbf{Q} &= \begin{bmatrix} (1 - (1-u)^2 t^2) \mathbf{I}_{nd} - t {}^t \mathbf{L} \mathbf{K} + (1-u)t^2 {}^t \mathbf{H} \mathbf{K} & \mathbf{0} \\ {}^t \mathbf{K} & (1 - (1-u)^2 t^2) \mathbf{I}_{2ld} \end{bmatrix} \\ &= \begin{bmatrix} (1 - (1-u)^2 t^2) \mathbf{I}_{nd} - t \sum_{g \in \Gamma} {}^t \mathbf{W}_g \otimes {}^t \rho(g) + (1-u)t^2 \mathbf{D}_d & \mathbf{0} \\ {}^t \mathbf{K} & (1 - (1-u)^2 t^2) \mathbf{I}_{2ld} \end{bmatrix}. \end{aligned}$$

By (1) and (4),

$$\mathbf{QP} = \begin{bmatrix} (1 - (1 - u)^2 t^2) \mathbf{I}_{nd} & \mathbf{0} \\ t(1 - (1 - u)^2 t^2) \mathbf{K} & -t \mathbf{K}^t \mathbf{L} + (1 - u) t^2 \mathbf{K}^t \mathbf{H} + (1 - (1 - u)^2 t^2) \mathbf{I}_{2ld} \end{bmatrix}.$$

Since

$$w(e_1) \mathbf{I}_d \oplus \cdots \oplus w(e_{2l}) \mathbf{I}_d = {}^t \mathbf{M}^t \mathbf{J}_\rho$$

and $({}^t \mathbf{J}_\rho)^2 = \mathbf{I}_{2ld}$, we have

$$\begin{aligned} & -t \mathbf{K}^t \mathbf{L} + (1 - u) t^2 \mathbf{K}^t \mathbf{H} + (1 - (1 - u)^2 t^2) \mathbf{I}_{2ld} \\ &= \mathbf{I}_{2ld} - t({}^t \mathbf{N} + {}^t \mathbf{M}) + (1 - u) t^2 ({}^t \mathbf{N}^t \mathbf{J}_\rho + {}^t \mathbf{M}^t \mathbf{J}_\rho - (1 - u) {}^t \mathbf{J}_\rho {}^t \mathbf{J}_\rho) \\ &= (\mathbf{I}_{2ld} - t({}^t \mathbf{N} + {}^t \mathbf{M} - (1 - u) {}^t \mathbf{J}_\rho)) (\mathbf{I}_{2ld} - (1 - u) {}^t \mathbf{J}_\rho). \end{aligned}$$

Thus,

$$\mathbf{QP} = \begin{bmatrix} (1 - (1 - u)^2 t^2) \mathbf{I}_{nd} & \mathbf{0} \\ t(1 - (1 - u)^2 t^2) \mathbf{K} & (\mathbf{I}_{2ld} - t({}^t \mathbf{N} + {}^t \mathbf{M} - (1 - u) {}^t \mathbf{J}_\rho)) (\mathbf{I}_{2ld} - (1 - u) {}^t \mathbf{J}_\rho) \end{bmatrix}.$$

Since $\det(\mathbf{PQ}) = \det(\mathbf{QP})$, we have

$$\begin{aligned} & (1 - (1 - u)^2 t^2)^{2ld} \det(\mathbf{I}_{nd} - t \sum_{g \in \Gamma} {}^t \mathbf{W}_g \bigotimes {}^t \rho(g) + (1 - u) (\mathbf{D}_d - (1 - u) \mathbf{I}_{nd}) t^2) \\ &= (1 - (1 - u)^2 t^2)^{nd} \det(\mathbf{I}_{2ld} - t({}^t \mathbf{N} + {}^t \mathbf{M} - (1 - u) {}^t \mathbf{J}_\rho)) \det(\mathbf{I}_{2ld} - (1 - u) {}^t \mathbf{J}_\rho). \end{aligned}$$

But,

$$\begin{aligned} \det(\mathbf{I}_{2ld} - (1 - u) {}^t \mathbf{J}_\rho) &= \det \left(\begin{bmatrix} \mathbf{I}_{ld} & (1 - u) t \mathbf{S} \\ \mathbf{0} & \mathbf{I}_{ld} \end{bmatrix} \right) \det \left(\begin{bmatrix} \mathbf{I}_{ld} & -(1 - u) t \mathbf{S} \\ -(1 - u) t \mathbf{S}^{-1} & \mathbf{I}_{ld} \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} (1 - (1 - u)^2 t^2) \mathbf{I}_{ld} & \mathbf{0} \\ * & \mathbf{I}_{ld} \end{bmatrix} \right) = (1 - (1 - u)^2 t^2)^{ld}, \end{aligned}$$

where $\mathbf{S} = \rho(\alpha(e_1)) \oplus \cdots \oplus \rho(\alpha(e_l))$. Therefore it follows that

$$\begin{aligned} & (1 - (1 - u)^2 t^2)^{2ld} \det(\mathbf{I}_{nd} - t \sum_{g \in \Gamma} {}^t \mathbf{W}_g \bigotimes {}^t \rho(g) + (1 - u) (\mathbf{D}_d - (1 - u) \mathbf{I}_{nd}) t^2) \\ &= (1 - (1 - u)^2 t^2)^{(l+n)d} \det(\mathbf{I}_{2ld} - t({}^t \mathbf{N} + {}^t \mathbf{M} - (1 - u) {}^t \mathbf{J}_\rho)). \end{aligned}$$

Hence

$$\begin{aligned} & \det(\mathbf{I}_{2ld} - t \sum_h \rho(h) \bigotimes (\mathbf{B}_h - (1 - u) \mathbf{J}_h)) \\ &= (1 - (1 - u)^2 t^2)^{(l-n)d} \det(\mathbf{I}_{nd} - t \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{W}_g + (1 - u) (\mathbf{I}_d \bigotimes (\mathbf{D} - (1 - u) \mathbf{I}_n)) t^2). \end{aligned}$$

□

By Theorems 4,5,6, the following result holds.

Corollary 1 *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G , Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\zeta_1(G^\alpha, \tilde{w}, u, t) = \prod_{\rho} \zeta_1(G, w, u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ .

In the case that $w(v_i, v_j) = 1$ for each $(v_i, v_j) \in D(G)$, we obtain a decomposition formula for the Bartholdi zeta function of a regular covering of a graph by Mizuno and Sato [8].

Corollary 2 (Mizuno and Sato) *Let G be a connected graph, Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Suppose that the Γ -covering G^α of G is connected. Then we have*

$$\zeta(G^\alpha, u, t) = \prod_{\rho} \zeta_G(u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ .

If $u = 0$, we obtain a decomposition formula for the new zeta function of a regular covering of a graph by Sato [9].

Corollary 3 (Sato) *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G , Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\mathbf{Z}_1(G^\alpha, \tilde{w}, t) = \prod_{\rho} \mathbf{Z}_1(G, w, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ .

5 Example

Finally, we give an example. Let $G = K_3$ be the complete graph with three vertices v_1, v_2, v_3 , and let the following matrix $\mathbf{W}$ be a weighted matrix of K_3 :

$$\mathbf{W} = \begin{bmatrix} 0 & a & b \\ p & 0 & c \\ q & r & 0 \end{bmatrix}.$$

Then, by Theorem 3, we have

$$\begin{aligned} \zeta_1(K_3, w, u, t)^{-1} &= \det(\mathbf{I}_3 - t\mathbf{W} + (1-u)t^2(\mathbf{D} - (1-u)\mathbf{I}_3)) \\ &= \det \left(\begin{bmatrix} 1 + (1-u)(a+b-1+u)t^2 & -ta & -tb \\ -tp & 1 + (1-u)(p+c-1+u)t^2 & -tc \\ -tq & -tr & 1 + (1-u)(q+r-1+u)t^2 \end{bmatrix} \right) \end{aligned}$$

$$= \alpha\beta\gamma - (bq\beta + ap\gamma + cr\alpha)t^2 - (acq + bpr)t^3,$$

where $\alpha = 1 + (1 - u)(a + b - 1 + u)t^2$, $\beta = 1 + (1 - u)(p + c - 1 + u)t^2$ and $\gamma = 1 + (1 - u)(q + r - 1 + u)t^2$.

Let $\Gamma = Z_3 = \{1, \tau, \tau^2\} (\tau^3 = 1)$ be the cyclic group of order 3, and let $\alpha : D(K_3) \rightarrow Z_3$ be the ordinary voltage assignment such that $\alpha(v_1, v_2) = \tau$ and $\alpha(v_1, v_3) = \alpha(v_2, v_3) = 1$. Then, the $\mathbf{Z}_3$ -covering K_3^α is the cycle graph C_9 with nine vertices. The characters of $\mathbf{Z}_3$ are given as follows: $\chi_i(\tau^j) = (\xi^i)^j$, $0 \leq i, j \leq 2$, where $\xi = \frac{-1+\sqrt{-3}}{2}$.

Now, we present the L -function $\mathbf{Z}_1(K_3, w, u, \chi_1, \alpha)$ of K_3 associated with χ_1 and α . Theorem 6 implies that

$$\begin{aligned} \zeta_1(K_3, w, u, t, \chi_1, \alpha)^{-1} &= \det(\mathbf{I}_3 - t \sum_{i=0}^2 \chi_1(\tau^i) \mathbf{W}_{\tau^i} + (1 - u)t^2(\mathbf{D} - (1 - u)\mathbf{I}_3)) \\ &= \det \left(\begin{bmatrix} \alpha & -\xi t a & -t b \\ -\xi^2 t p & \beta & -t c \\ -t q & -t r & \gamma \end{bmatrix} \right) \\ &= \alpha\beta\gamma - (bq\beta + ap\gamma + cr\alpha)t^2 - (acq\xi + bpr\xi^2)t^3. \end{aligned}$$

Similarly, we have

$$\zeta_1(K_3, w, u, t, \chi_2, \alpha)^{-1} = \alpha\beta\gamma - (bq\beta + ap\gamma + cr\alpha)t^2 - (acq\xi^2 + bpr\xi)t^3.$$

By Corollary 1, it follows that

$$\begin{aligned} \zeta_1(C_9, w, u, t)^{-1} &= \zeta_1(K_3^\alpha, w, u, t)^{-1} = \zeta_1(K_3, w, u, t)^{-1} \zeta_1(K_3, w, u, t, \chi_1, \alpha)^{-1} \zeta_1(K_3, w, u, t, \chi_2, \alpha)^{-1} \\ &= \{\alpha\beta\gamma - (bq\beta + ap\gamma + cr\alpha)t^2 - (acq + bpr)t^3\} \{(\alpha\beta\gamma - (bq\beta + ap\gamma + cr\alpha)t^2)^2 \\ &\quad + (acq + bpr)(\alpha\beta\gamma - (bq\beta + ap\gamma + cr\alpha)t^2)t^3 + (a^2c^2q^2 - abcpqr + b^2p^2r^2)t^6\}. \end{aligned}$$

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