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A Probabilistic Meaning of Certain Quasinormal Subgroups

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Abstract

The role of the cyclic quasinormal subgroups has been recently described in groups both finite and infinite by S.Stonehewer and G.Zacher. This role can be better analyzed in the class of compact groups, obtaining restrictions for the probability that two randomly chosen elements commute.

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1 Introduction

Following [1,2,5,6,7,10,12], if G is a finite group, the probability that two randomly chosen elements of G commute is defined to be $\#com(G)/|G|^2$, where $|G|^2$ is the order of the set $G \times G$ and $\#com(G)$ is the number of pairs $(x, y) \in G \times G$ such that $xy = yx$. For infinite groups this ratio is no longer meaningful, but in the class of the compact groups it is possible to proceed in analogy with the finite case.

We will denote with $cp(G)$ the probability that two randomly chosen elements of G commute. If G is a finite group, then $cp(G) = \#com(G)/|G|^2$ and literature on $cp(G)$ can be found in [1,2,3,5,6,7,10,12] when G is a finite group.

If G is a non-abelian finite group, then $cp(G) \leq 5/8$. Furthermore, this bound is achieved if and only if $G/Z(G)$ is isomorphic to the non-cyclic group of order 4, where $Z(G)$ denotes the center of the group G. This result has been known for a long time [6].

The present paper adopts the standard formulation for the notion of measure space in probability, for the notion of locally compact topological group and for the notion of Haar measure. They can be found in [8], which will be used as general reference of such subjects.

It is useful recall that every locally compact topological group G admits a left Haar measure μ, which is a positive Radon measure on a σ-algebra containing Borel sets with the property that $\mu(xE) = \mu(E)$ for each element x of the measure space X (see [8, Sections 18.1 and 18.2]). The support of μ is G and it is usually unbounded, but if G is compact, then μ is bounded. For this reason we may assume without ambiguity that a compact group G has a unique probability measure space $(G, \mathcal{M}, \mu)$ with normalized Haar measure μ (see [8, Proposition 18.2.1]).

Now, let us state the following definition which is adapted for the compact groups.

Let G be a compact group with the normalized Haar measure μ. On the product measure space $G \times G$, it is possible to consider the product measure $\mu \times \mu$ which is a probability measure. If

$$C_2 = \{(x, y) \in G \times G \mid xy = yx\},$$

then $C_2 = f^{-1}(1_G)$, where $f : G \times G \to G$ is defined via $f(x, y) = x^{-1}y^{-1}xy$ and 1_G denotes the neutral element of G. It is clear that f is continuous and so C_2 is a compact and measurable subset of $G \times G$. Therefore it is possible to define

$$cp(G) = (\mu \times \mu)(C_2).$$

Obviously if G is finite, then G is a compact group with the discrete topology and so the Haar measure of G is the counting measure.

Following [13], a cyclic subgroup A of an arbitrary group G is called *quasinormal* in G if $AH = HA$ for each subgroup H of G. Literature on quasinormal subgroups can be found in [9,11] and many results are known on their role with respect to subnormality, modularity [11] and similar conditions weaker than normality [11]. We recall that in a group G, the subgroup $[H, G]$ is defined to be $\langle [h, g] : h \in H, g \in G\rangle$ where H is a subgroup of G. Furthermore

$$[K, H, G] = [K, [H, G]] = \langle [k, [h, g]] : k \in K, h \in H, g \in G\rangle$$

where K is a subgroup of G. Finally the symbol H_G will denote the core of H in G. See [9] or [11] for details.

The main results of the present paper are listed below.

Theorem A. *Let G be a non-abelian compact group, n be a positive integer and A be a cyclic quasinormal 2-subgroup of G.*

(i) If the index $|G : [A, G, A]| \leq 4$, then $cp(G) = \frac{5}{8}$.

(ii) If $cp(G) = \frac{5}{8}$ and the index $|Z(G) : [A, G, A]| = 2^n$, then the index $|G : [A, G, A]| \leq 2^{n+2}$ and G contains a homomorphic image which is 2-elementary abelian of rank 2.

Theorem B. *Let G be a non-abelian compact group, p be an odd prime and n be a positive integer. If A is a cyclic quasinormal subgroup of order p^n of G and the quotient $G/[A, G]_G$ is a p-group, then*

$$cp(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Most of our notation is standard and can be found in [9] and [11].

2 Proof of Theorem A

In this Section G is assumed to be a non-abelian compact group (not necessarily finite even uncountable) with normalized Haar measure μ. The purpose of most of the results of this Section can be found in [3] and here they have been presented for convenience of the reader.

Lemma 2.1. *Let $C_G(x)$ be the centralizer of an element x in G. Then*

$$cp(G) = \int_G \mu(C_G(x))d\mu(x),$$

where $\mu(C_G(x)) = \int_G \chi_{C_2}(x, y)d\mu(y)$ and χ_{C_2} denotes the characteristic map of the set C_2.

Proof. Since $\mu(C_G(x)) = \int_G \chi_{C_2}(x, y)d\mu(y)$, we have by Fubini-Tonelli's Theorem:

$$\begin{aligned} cp(G) = (\mu \times \mu)(C_2) &= \int_{G\times G} \chi_{C_2} d(\mu \times \mu) \\ &= \int_G \int_G \chi_{C_2}(x, y)d\mu(x)d\mu(y) \\ &= \int_G \mu(C_G(x))d\mu(x). \end{aligned}$$

$\diamondsuit$

Lemma 2.2. *Let H be a closed subgroup of G and n be a positive integer.*

(i) If $|G : H| \geq n$, then $\mu(H) \leq \dfrac{1}{n}$.

(ii) If $|G : H| \leq n$, then $\mu(H) \geq \dfrac{1}{n}$.

Proof. Assume that $|G : H| = k$, where k is a positive integer. Then the proof follows from the equality

$$1 = \mu(G) = \mu(\bigcup_{i=1}^{k} \mu(x_i H)) = k\mu(H). \qquad \diamondsuit$$

Lemma 2.3. *Let $G/Z(G)$ be a p-group of order p^k, where k is a positive integer and p is a prime. An element x does not belong to $Z(G)$ if and only if $\mu(C_G(x)) \leq \frac{1}{p}$.*

Proof. It is clear that if $x \in Z(G)$ then $|C_G(x) : Z(G)| \leq p^{k-1}$ and therefore

$$p^k = |G : Z(G)| = |G : C_G(x)||C_G(x) : Z(G)| \geq p^{k-1}|G : C_G(x)|$$

which implies that $\mu(C_G(x)) \leq \frac{1}{p}$. Conversely, assume that $\mu(C_G(x)) \leq \frac{1}{p}$ and $x \in Z(G)$. Then $\mu(C_G(x)) = \mu(Z(G))) = 1$ which is a contradiction. Hence, $x \notin Z(G)$ and the proof is completed. $\diamondsuit$

Theorem 2.4. *If $G/Z(G)$ is a 2-elementary abelian group of rank 2, then $cp(G) = \frac{5}{8}$.*

Proof. Assume that $G/Z(G)$ is 2-elementary abelian of rank 2. Then we may write G as the union of 4 distinct cosets

$$G = Z(G) \cup x_1 Z(G) \cup x_2 Z(G) \cup x_3 Z(G)$$

and so $1 = \mu(G) = 4\mu(Z(G))$, since μ is a left Haar-measure.

If $a, b \in x_i Z(G)$, for $1 \leq i \leq 3$, then $a = x_i z_1$ and $b = x_i z_2$ for some $z_1, z_2 \in Z(G)$ so that

$$ab = x_i z_1 x_i z_2 = x_i x_i z_1 z_2 = x_i x_i z_2 z_1 = x_i z_2 x_i z_1 = ba.$$

Thus, if $a \in x_i Z(G)$, then $C_G(a) = Z(G) \cup aZ(G)$ and so

$$\mu(C_G(a)) \;\; = \;\; \mu(Z(G)) + \mu(aZ(G)) = 2\mu(Z(G)) = \frac{1}{2}.$$

Thus, we have

$$\begin{aligned} cp(G) &= \int_G \mu(C_G(x))d\mu(x) \\ &= \int_{Z(G)} \mu(C_G(x))d\mu(x) + \Sigma_{i=1}^{3} \int_{x_iZ(G)} \mu(C_G(x))d\mu(x) \\ &= \mu(Z(G)) + \Sigma_{i=1}^{3} \tfrac{1}{2}\mu(x_iZ(G)) \\ &= \mu(Z(G)) + \tfrac{3}{2}\mu(Z(G)) \\ &= \tfrac{5}{2}\mu(Z(G)) \\ &= \tfrac{5}{8}\,. \end{aligned} \qquad \diamondsuit$$

Now, we are able to prove Theorem A.

Proof of Theorem A. [13, Theorem 3.5] implies that $B = [A, G, A]$ is normal in G. [13, Theorem 3.5] implies also $B \leq Z(H) = \bigcap_{x\in H} C_H(x)$, where $H = [A, G]$. Put $A = \langle a\rangle$, [13, Corollary 3.2] implies $[A, G] = \langle [a, g] | g \in G\rangle$. Then $Z(H) = \bigcap_{g\in G} C_H([a, g])$, where $x = [a, g]$. But $[a, g]^b = [a^b, g^b] = [a, b]$ for each $b \in B$, so $a^b = a$ and $g^b = g$. Therefore $b \in C_G(g)$ for each $g \in G$ so $B \leq Z(G)$. If G/B is cyclic of order 4, then G is abelian and this cannot be. Assume that G/B is 2-elementary abelian of rank 2. Since the class of 2-elementary abelian groups of finite rank is closed with respect to subgroups, homomorphic images and extensions of its members, we are in the condition to apply Theorem 2.4 and (i) follows.

Now we proceed to prove (ii). Assume that $cp(G) = \frac{5}{8}$ and $G/Z(G)$ be not 2-elementary abelian of rank 2. If $|G : Z(G)| \in \{1, 2, 3\}$, then $G/Z(G)$ is cyclic and so G is abelian which is a contradiction. Thus we should have $|G : Z(G)| \geq 5$ and therefore $\mu(Z(G)) \leq \frac{1}{5}$ by Lemma 2.2. Moreover, if $x \notin Z(G)$ then $\mu(C_G(x)) \leq \frac{1}{2}$ by Lemma 2.3. Then we have

$$\begin{aligned} \tfrac{5}{8} = cp(G) &= \int_G \mu(C_G(x))d\mu(x) \\ &= \int_{Z(G)} \mu(C_G(x))d\mu(x) + \int_{G\setminus Z(G)} \mu(C_G(x))d\mu(x) \\ &\leq \mu(Z(G)) + \tfrac{1}{2}(1 - \mu(Z(G))) \\ &= \tfrac{1}{2}\mu(Z(G)) + \tfrac{1}{2} \\ &\leq \tfrac{1}{2}(\tfrac{1}{5}) + \tfrac{1}{2} \\ &= \tfrac{3}{5}\,. \end{aligned}$$

which is a contradiction and this implies that $G/Z(G)$ is 2-elementary abelian of rank 2. We have found a homomorphic image of G which is 2-elementary abelian of rank 2. Following the initial remark on B, we conclude that $B \leq$

$Z(G)$, so that$|G : B| \leq |G : Z(G)||Z(G) : B| = 2^2 \cdot 2^n = 2^{n+2}$. Our result is proved. ♢

3 Proof of Theorem B

Lemma 4.1. *If G is a compact non-abelian p-group, where p is a prime, then*

$$cp(G) \leq \frac{p^2 + p - 1}{p^3}.$$

Proof. Firstly we prove that the index $|G : Z(G)| \geq p^2$. The statement is obviously true when the index of $Z(G)$ in G is infinite. Assume that the index $|G : Z(G)|$ is finite. Since G is a p-group, $|G : Z(G)|$ is a p-power. If $|G : Z(G)| = 1$, then G is abelian and this can not be. Then $|G : Z(G)| > 1$. If $|G : Z(G)| \leq p^2$, the only possibility for $|G : Z(G)|$ is $|G : Z(G)| = p$ so again G is abelian. This can not be, then $|G : Z(G)| \geq p^2$.

From Lemma 2.2 and the previous remark, we conclude that $\mu(Z(G)) \leq \frac{1}{p^2}$.

On the other hand it is obvious that if $x \notin Z(G)$, then $|G : C_G(x)| \geq p$, so $\mu(C_G(x)) \leq \frac{1}{p}$ by Lemma 2.2.

In the following relations we will use Lemma 2.1, Lemma 2.3, the fact that $\mu(Z(G)) \leq \frac{1}{p^2}$, the fact that $\mu(G) = 1$, the fact that $\mu(C_G(x)) \leq \frac{1}{p}$ for each $x \notin Z(G)$.

$$\begin{aligned}
cp(G) &= \int_{Z(G)} \mu(C_G(x))d\mu(x) + \int_{G-Z(G)} \mu(C_G(x))d\mu(x) \\
&\leq \mu(Z(G)) + \frac{1}{p}\mu(G - Z(G)) \\
&= \mu(Z(G)) + \frac{1}{p}|\mu(G) - \mu(Z(G))| \\
&= \frac{p-1}{p}\mu(Z(G)) + \frac{1}{p} \\
&\leq \frac{p-1}{p}(\frac{1}{p^2}) + \frac{1}{p} \\
&= \tfrac{p^2+p-1}{p^3}.
\end{aligned}$$

♢

Proof of Theorem B. [13, Theorem 2.12] implies that [A,G] is a p-group, so $[A, G]_G$ is a p-group. The class of p-groups is closed with respect to extensions, subgroups and homomorphic images of its members, then G is a p-group. Now Lemma 4.1 finishes the proof. ♢

4 Examples

In final Section, we give some examples of finite groups which are satisfying the main theorems of the present paper.

Example 1. This Example modifies [12, Example 2]. Let H and X be cyclic groups of order 27 and 81, respectively, and let $H = \langle h \rangle$ and $X = \langle x \rangle$. We form the split extension $G = HX$ where H is normal in G and $h^x = h^{-1}$. Let $a = hx^3$ and $A = \langle a \rangle$, a cyclic group of order 27. Then $C_G(A) = HX^3$. Also $A \cap X = A \cap H = 1$, so $G = AX$ and

$$[A, G] = \langle [h, x] \rangle = H^3 \simeq C_9.$$

Thus $A \cap [A, G] = 1$. Also the only elements of form $[a, g]$ are 1 and $h^{-3} = [a, x]$. Therefore $[A, G] \neq \{[a, g] : g \in G\}$. However A is quasinormal in G. Each cyclic subgroup of G of form $\langle h^i x^{3j} \rangle$ commutes with A, where $i \in \{0, \ldots 26\}$ and $j \in \{0, \ldots 80\}$. On the other hand $\langle h^i x^j \rangle$ with j odd, has order 81, because $(h^i x^j)^3 = h^i x^j h^i x^{-j} x^{3j} = x^{3j}$, and intersects A trivially. So $|A\langle h^i x^j \rangle| = 3^7$ and hence $G = A\langle h^i x^j \rangle$. Actually A is quasinormal in G. Here we have that G is a finite 3-group which satisfies the conditions of Theorem B. In particular $cp(G) \leq \frac{3^2+3-1}{3^3} = \frac{11}{27}$.

Example 2. The dihedral group G of order 8 can be written as split extension of a cyclic group X of order 2 by a cyclic group Y of order 4. G has $|Z(G)| = |G'| = 2$ and a maximal 2-subgroup A of G is cyclic of order 4. Now A is quasinormal in G, $[A, [G, A]] \leq G' = Z(G)$ and $|G : [A, [G, A]]| \leq 4$. We are in the situation to apply Theorem A. It is well known that G has the homomorphic image $G/Z(G)$ which is 2-elementary abelian of rank 2.

Example 3. Each non-abelian compact group G which has infinite nontrivial center $Z(G)$ and $G/Z(G)$ which is 2-elementary abelian of rank 2 satisfies trivially Theorems A and B of the present paper.

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