

## On Derivations in $\sigma$ -prime Rings

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### Abstract

Let  $(R, \sigma)$  be a 2-torsion free  $\sigma$ -prime ring with involution  $\sigma$  and  $d$  be a nonzero derivation of  $R$ . In the present paper it is shown that if  $d$  is centralizing, then  $R$  is commutative. Furthermore, if  $d$  commutes with  $\sigma$  and  $0 \neq I$  is a  $\sigma$ -ideal of  $R$  such that either  $[d(x), d(y)] = 0$  or  $d(xy) = d(yx)$  for all  $x, y \in I$ , then  $R$  is a commutative ring.

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## 1 Introduction

Throughout this paper,  $R$  will represent a unitary associative ring and  $Z(R)$  will be its center. Recall that  $R$  is prime if  $aRb = 0$  implies that  $a = 0$  or  $b = 0$ . If  $R$  has an involution  $\sigma$ , then  $R$  is said to be  $\sigma$ -prime if  $aRb = aR\sigma(b) = (0)$  implies that  $a = 0$  or  $b = 0$ . Obviously, every prime ring equipped with an involution  $\sigma$  is  $\sigma$ -prime. The converse need not be true in general. An ideal  $I$  of  $R$  is said to be a  $\sigma$ -ideal if  $\sigma(I) = I$ . An additive mapping  $d : R \rightarrow R$  is called a derivation if  $d(xy) = d(x)y + xd(y)$ , for all  $x, y \in R$ . A mapping  $F : R \rightarrow R$  is said to be centralizing if  $[F(x), x] \in Z(R)$ , for all  $x \in R$ .

Many authors have studied the relationship between the commutativity of a ring and the behavior of a special mapping on that ring. Especially, there has been considerable interest in centralizing automorphisms and derivations defined on rings. The first important result on centralizing maps is Posner's theorem [6] which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Considering this theorem from some distance it is not entirely clear what was Posner's motivation for

proving it and for which reasons he was able to conjecture that the theorem is true. Anyhow, it is a fact that the theorem has been extremely influential and it has been generalized by a number of authors in several ways.

In [3], I. N. Herstein proved that if  $R$  is a prime ring of characteristic not 2 which has a nonzero derivation  $d$  such that  $[d(x), d(y)] = 0$  for all  $x, y \in R$ , then  $R$  is commutative. Motivated by this result, H. E. Bell, in [1], studied derivation  $d$  satisfying  $d(xy) = d(yx)$ , for all  $x, y \in R$ . We here extend the above results to  $\sigma$ -prime rings. More precisely, we shall prove the following results.

**Theorem 1.1** *Let  $R$  be a 2-torsion free  $\sigma$ -prime ring and let  $d$  be a nonzero derivation of  $R$ . If  $[d(x), x] \in Z(R)$ , for all  $x \in R$ , then  $R$  is commutative.*

**Theorem 1.2** *Let  $R$  be a 2-torsion free  $\sigma$ -prime ring and let  $I$  be a nonzero  $\sigma$ -ideal of  $R$ . If  $R$  admits a nonzero derivation  $d$  such that  $[d(x), d(y)] = 0$ , for all  $x, y \in I$  and  $d$  commutes with  $\sigma$ , then  $R$  is commutative.*

**Theorem 1.3** *Let  $R$  be a 2-torsion free  $\sigma$ -prime ring and let  $I$  be a nonzero  $\sigma$ -ideal of  $R$ . If  $R$  admits a nonzero derivation  $d$  such that  $d(xy) = d(yx)$ , for all  $x, y \in I$  and  $d$  commutes with  $\sigma$ , then  $R$  is commutative.*

## 2 Proof of the main results

In all that follows, we assume that  $R$  is a 2-torsion free  $\sigma$ -prime ring, where  $\sigma$  is an involution of  $R$ . Let us define  $Sa_\sigma(R) = \{x \in R / \sigma(x) = \pm x\}$  to be the set of symmetric and skew symmetric elements of  $R$ . We will make frequent and important use of the following lemmas.

**Lemma 2.1** ([4], Lemma 3.1) *Let  $R$  be a  $\sigma$ -prime ring and let  $I$  be a nonzero  $\sigma$ -ideal of  $R$ . If  $a, b$  in  $R$  satisfy  $aIb = aI\sigma(b) = 0$ , then  $a = 0$  or  $b = 0$ .*

**Lemma 2.2** *Let  $I$  be a nonzero  $\sigma$ -ideal of  $R$  and  $0 \neq d$  be a derivation on  $R$  which commutes with  $\sigma$ . If  $[x, R]Id(x) = 0$  for all  $x \in I$ , then  $R$  is commutative.*

**Proof.** Let  $x \in I$ . Since  $t = x - \sigma(x) \in I$ , then  $[t, r]Id(t) = 0$  for all  $r \in R$ . As  $t \in Sa_\sigma(R)$ , we then get

$$[t, r]Id(t) = \sigma([t, r])Id(t) = 0, \text{ for all } r \in R,$$

which leads, in view of Lemma 2.1, to

$$d(t) = 0 \text{ or } [r, t] = 0, \text{ for all } r \in R.$$

If  $d(t) = 0$ , then  $d(x) = d(\sigma(x)) = \sigma(d(x))$ . Therefore

$$0 = [x, r]Id(x) = [x, r]I\sigma(d(x))$$

and thus  $d(x) = 0$  or  $[r, x] = 0$  for all  $r \in R$ , by Lemma 2.1. Consequently, either  $d(x) = 0$  or  $x \in Z(R)$ .

If  $[r, t] = 0$  for all  $r \in R$ , then  $t \in Z(R)$  and thus  $[x, r] = [\sigma(x), r]$  for all  $r \in R$ . Hence

$$[x, r]Id(x) = 0 = \sigma([x, r])Id(x).$$

Again using Lemma 2.1, we get  $d(x) = 0$  or  $x \in Z(R)$ .

In conclusion, for each  $x \in I$  either  $d(x) = 0$  or  $x$  in  $Z(R)$ . Let us consider  $G_1 = \{x \in I / d(x) = 0\}$  and  $G_2 = \{x \in I / x \text{ in } Z(R)\}$ . It is clear that  $G_1$  and  $G_2$  are additive subgroups of  $I$  such that  $I = G_1 \cup G_2$ . But a group can not be a union of two its proper subgroups and hence  $I = G_1$  or  $I = G_2$ .

If  $I = G_1$ , then  $d(x) = 0$  for all  $x \in I$ . For any  $s \in R$ , replace  $x$  by  $xs$  to get  $xd(s) = 0$ , for all  $x \in I$  so that  $Id(s) = 0$  for all  $s \in R$ . In particular

$$0 = 1Id(s) = \sigma(1)Id(s) \text{ for all } s \text{ in } R$$

and Lemma 2.1 gives  $d = 0$ , a contradiction. Hence,  $I = G_2$  so that  $I \subseteq Z(R)$ . Let  $r, s \in R$  and  $x \in I$ , from  $rsx = rxs = srx$  we conclude that  $[r, s]I = 0$  and then

$$[r, s]I1 = [r, s]I\sigma(1) = 0.$$

This yields, according to Lemma 2.1, that  $[r, s] = 0$  for all  $r, s \in R$  and consequently,  $R$  is a commutative ring. ■

**Lemma 2.3** *Let  $I$  be a nonzero  $\sigma$ -ideal of  $R$ . If  $R$  admits a derivation  $d$  such that  $d^2(I) = 0$  and  $d$  commutes with  $\sigma$  on  $R$  then  $d = 0$ .*

**Proof.** Suppose that  $d \neq 0$  and let  $r_0 \in R$  such that  $d(r_0) \neq 0$ . For any  $x \in I$ , we have  $d^2(x) = 0$ . Replacing  $x$  by  $xy$ , we obtain

$$d^2(x)y + 2d(x)d(y) + xd^2(y) = 0, \text{ for all } x, y \in I.$$

The fact that  $d^2(I) = (0)$  together with  $char R \neq 2$ , give  $d(x)d(y) = 0$ , for all  $x, y \in I$  so that  $d(x)d(I) = 0$ . In particular

$$d(x)d(yr_0) = d(x)y d(r_0) = 0, \text{ for all } y \in I$$

and therefore  $d(x)Id(r_0) = 0$ . As  $d$  commutes with  $\sigma$ , the fact that  $I$  is a  $\sigma$ -ideal gives  $\sigma(d(x))Id(r_0) = 0$ . Consequently

$$d(x)Id(r_0) = \sigma(d(x))Id(r_0) = 0$$

which yields, in view of Lemma 2.1, that

$$d(x) = 0, \text{ for all } x \in I. \quad (1)$$

If we replace  $x$  by  $xr_0$  in (1), then we get  $xd(r_0) = 0$  for all  $x \in I$  in such a way that  $Id(r_0) = 0$ . In particular,  $1Id(r_0) = \sigma(1)Id(r_0) = 0$  so that  $d(r_0) = 0$ , a contradiction. Consequently,  $d = 0$ . ■

**Proof of Theorem 1.1.** Linearizing  $[d(x), x] \in Z(R)$  gives

$$[d(x), y] + [d(y), x] \in Z(R), \text{ for all } x, y \in R. \quad (2)$$

Replacing  $y$  by  $x^2$  in (2) and using  $\text{char}R \neq 2$ , we have  $x[d(x), x] \in Z(R)$ . Henceforth,  $[r, x][d(x), x] = 0$ , for all  $x$  in  $R$ . Replacing  $r$  by  $d(x)$ , we then get  $[d(x), x]^2 = 0$ . Since  $[d(x), x] \in Z(R)$ , then

$$[d(x), x]R[d(x), x]\sigma([d(x), x]) = 0.$$

As  $[d(x), x]\sigma([d(x), x]) \in Sa_\sigma(R)$  and  $R$  is  $\sigma$ -prime, then

$$[d(x), x] = 0 \text{ or } [d(x), x]\sigma([d(x), x]) = 0.$$

Assume  $[d(x), x]\sigma([d(x), x]) = 0$ , the fact that  $[d(x), x] \in Z(R)$  gives

$$[d(x), x]R\sigma([d(x), x]) = [d(x), x]R[d(x), x] = 0,$$

and the  $\sigma$ -primeness of  $R$  yields  $[d(x), x] = 0$ . We then conclude that,

$$[d(x), x] = 0, \text{ for all } x \in R. \quad (3)$$

$$[d(x), y] + [d(y), x] = 0, \text{ for all } x, y \in R. \quad (4)$$

Let us write in (4)  $xy$  instead of  $y$  and using (4), we have

$$d(x)[x, y] = 0, \text{ for all } x, y \in R. \quad (5)$$

Replace  $y$  by  $yz$  in (5), to get  $d(x)y[x, z] = 0$ , for all  $x, y, z \in R$  and hence  $d(x)R[x, z] = 0$  for all  $x, z \in R$ . In particular,

$$0 = d(\sigma(x))R[\sigma(x), \sigma(z)] = \sigma(d(x))R\sigma([x, z])$$

because  $d$  commutes with  $\sigma$ . Applying  $\sigma$  to this last equality, we obtain

$$[x, z]Rd(x) = 0, \text{ for all } x, z \in R.$$

Hence we conclude from Lemma 2.2 that  $R$  is commutative. ■

**Proof of Theorem 1.2.** we have

$$[d(x), d(y)] = 0, \text{ for all } x, y \in I. \quad (6)$$

Let us write in (6)  $xy$  instead of  $y$ , thereby obtaining

$$d(x)[d(x), y] + [d(x), x]d(y) = 0, \text{ for all } x, y \in I.$$

Now for any  $r \in R$ , replace  $y$  by  $yr$  in the above expression to obtain

$$d(x)y[d(x), r] + [d(x), x]yd(r) = 0, \text{ for all } x, y \in I, r \in R. \tag{7}$$

If we replace  $r$  by  $d(z)$  in (7), then

$$[d(x), x]yd^2(z) = 0, \text{ for all } x, y, z \in I$$

so that  $[d(x), x]Id^2(z) = 0$ . As  $d$  commutes with  $\sigma$  and  $I$  is a  $\sigma$ -ideal, then

$$[d(x), x]Id^2(z) = \sigma([d(x), x])Id^2(z) = 0.$$

Applying Lemma 2.1, either  $d^2(z) = 0$  for all  $z \in I$  or  $[d(x), x] = 0$  for all  $x \in I$ . If  $d^2(z) = 0$  for all  $z \in I$ , then Lemma 2.3 assures  $d = 0$  which is impossible. Now suppose that

$$[d(x), x] = 0, \text{ for all } x \in I. \tag{8}$$

Linearizing (8), we get

$$[d(x), y] + [d(y), x] = 0, \text{ for all } x, y \in I. \tag{9}$$

If we replace  $y$  by  $yx$  in (9) and then employ (9), we have  $[y, x]d(x) = 0$ . Hence  $[x, y]d(x) = 0$ , for all  $x, y \in I$ . For any  $r \in R$ , again replace  $y$  by  $ry$  to get  $[x, r]yd(x) = 0$ , for all  $x, y \in I, r \in R$  and thus

$$[x, r]Id(x) = 0, \text{ for all } x \in I, r \in R.$$

Applying Lemma 2.2,  $R$  is then commutative. ■

**Proof of Theorem 1.3.** Let  $x, y, z$  in  $I$ . Since  $d([x, y]) = 0$ , the condition that  $d([x, y]z) = d(z[x, y])$  yields

$$[x, y]d(z) = d(z)[x, y], \text{ for all } x, y, z \in I. \tag{10}$$

From  $d(xy) = d(yx)$ , for all  $x, y \in I$  it follows that

$$[d(x), y] = [d(y), x], \text{ for all } x, y \in I.$$

In particular,  $[d(x^2), y] = [d(y), x^2]$  in such a way that

$$d(x)[x, y] + [x, y]d(x) = 0, \text{ for all } x, y \in I.$$

As  $\text{char}(R) \neq 2$ , in view of (10), the above expression yields that

$$[x, y]d(x) = 0, \text{ for all } x, y \in I. \tag{11}$$

For any  $r$  in  $R$ , replace  $y$  by  $ry$  in (11), we obtain  $[x, r]yd(x) = 0$ , for all  $x, y \in I$ . Hence  $[x, R]Id(x) = 0$ , for all  $x \in I$  and according to Lemma 2.2 we conclude that  $R$  is commutative. ■

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