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σ-Lie Ideals with Derivations as Homomorphisms and Anti-homomorphisms

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Abstract

Let R be a 2-torsion free σ-prime ring, U a nonzero σ-square closed Lie ideal of R and d a derivation of R which commutes with σ. If d acts as a homomorphism or an anti-homomorphism on U, then either $d = 0$ or $U \subseteq Z(R)$.

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1 Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. R is said to be 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. We shall use basic commutator identities: $[x, yz] = y[x, z] + [x, y]z$, $[xy, z] = x[y, z] + [x, z]y$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = 0$ implies that $a = 0$ or $b = 0$. A ring R equipped with an involution σ is said to be a σ-prime ring if for any $a, b \in R$, $aRb = \sigma(a)Rb = 0$ implies that either $a = 0$ or $b = 0$. It is worthwhile to note that every prime ring having an involution σ is σ-prime but the converse is in general not true. In all that follows $Sa_\sigma(R)$ will denote the set of symmetric and skew symmetric elements of R i.e $Sa_\sigma(R) = \{x \in R / \sigma(x) = \pm\, x\}$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$ and $r \in R$. If U is a Lie ideal of R, then U is called a σ-square closed Lie ideal if $u^2 \in U$, for all

$u \in U$ and U is invariant under σ. Since $(u+v)^2 \in U$ and $[u,v] \in U$, we see that $2uv \in U$ for all $u, v \in U$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$.
A derivation d of R is called a derivation which acts as a homomorphism (resp. as an anti-homomorphism) on a subset S of R, if $d(xy) = d(x)d(y)$ (resp. $d(xy) = d(y)d(x)$), for all $x, y \in S$.

In [2], Bell and Kappe proved that if d is a derivation of a prime ring R which acts as a homomorphism or an anti-homomorphism on a nonzero right ideal I of R, then $d = 0$ on R. For 2-torsion free prime rings, A. Asma and all [1] extended this result on square closed Lie ideals. More precisely, they proved that if d is a derivation of a 2-torsion free prime ring R which acts as a homomorphism or an anti-homomorphism on a nonzero square closed Lie ideal U of R, then either $d = 0$ or $U \subseteq Z(R)$. In the present paper, our objective is to extend this result to σ-prime rings. More precisely, we shall prove the following theorem.

Theorem 1.1 *Let d be a derivation of a 2-torsion free σ-prime ring R which acts as a homomorphism or an anti-homomorphism on a nonzero σ-square closed Lie ideal U of R. If d commutes with σ, then either $d = 0$ or $U \subseteq Z(R)$.*

2 Proof of the main result

We begin with the following Lemmas which are essential in developing the proof of our main Theorem. The next Lemma shows that σ-primeness could also be defined by analogous property for σ-Lie ideals.

Lemma 2.1 ([5], Lemma 5) *If $U \not\subseteq Z(R)$ is a σ-Lie ideal of a 2-torsion free σ-prime ring R and $a, b \in R$ such that $aUb = \sigma(a)Ub = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 *Let R be a 2-torsion free σ-prime ring and U a nonzero σ-Lie ideal of R. If d is a derivation of R which commutes with σ and satisfying $d(U) = 0$, then either $d = 0$ or $U \subseteq Z(R)$.*

Proof. We have

$$d([u,r]) = [u, d(r)] = 0, \quad \text{for all } u \in U, \ r \in R. \tag{1}$$

Replacing r by rt in (1), where $t \in R$, we obtain

$$d(r)[t,u] + [r,u]d(t) = 0, \tag{2}$$

taking $r = t$ in (2) and applying (1) we find that $d(r)[r,u] + [r,u]d(r) = 0$. As $d(r)[r,u] = [r,u]d(r)$ by (1), then $2[r,u]d(r) = 0$ and so $[r,u]d(r) = 0$.

Substituting $2uv$ for u in this equality, with $v \in U$, we get $[r,u]vd(r) = 0$ and thus

$$[r,u]Ud(r) = 0, \text{ for all } u \in U,\ r \in R \tag{3}$$

Let $r \in Sa_\sigma(R)$, the fact that $\sigma(U) = U$ leads to

$$[r,u]Ud(r) = \sigma([r,u])Ud(r) = 0, \text{ for all } u \in U,\ r \in R \tag{4}$$

Applying Lemma 2.1, this yields that either $d(r) = 0$ or $[r,u] = 0$, for all $u \in U$. Let $r \in R$; since $r + \sigma(r) \in Sa_\sigma(R)$, then $d(r + \sigma(r)) = 0$ or $[r + \sigma(r), U] = 0$. If $d(r+\sigma(r)) = 0$, then $d(r) \in Sa_\sigma(R)$ and in view of (3) we conclude $d(r) = 0$ or $[r,U] = 0$. Now suppose that $[r + \sigma(r), U] = 0$; if $[r - \sigma(r), U] = 0$, then $2[r,U] = 0$ so that $[r,U] = 0$ because char$R \neq 2$. If $d(r - \sigma(r)) = 0$, then $d(r) \in Sa_\sigma(R)$ and once again using (3) we get $d(r) = 0$ or $[r,U] = 0$. In conclusion, for all $r \in R$ we have either $d(r) = 0$ or $[r,U] = 0$. Accordingly, R is a union of two additive subgroups G and H, where $G = \{r \in R \,/\, d(r) = 0\}$ and $H = \{r \in R \,/\, [r,u] = 0, \text{ for all } u \in U\}$. But a group can not be a union of two of its proper subgroups and thus $R = G$ or $R = H$. Consequently, either $d = 0$ or $U \subseteq Z(R)$. ∎

Proof of Theorem 1.1.

1) Suppose that $d(xy) = d(x)d(y)$, for all $x, y \in U$. Let $x, y, z \in U$, as $4xyz = 2(2xy)z$ it follows that $4xyz \in U$. Using char$R \neq 2$, we have

$$d(xyz) = d(xy)z + xyd(z) = d(x)d(y)z + xyd(z). \tag{5}$$

On the other hand

$$d(xyz) = d(x)d(yz) = d(x)d(y)z + d(x)yd(z). \tag{6}$$

Comparing (5) and (6), we conclude $(d(x) - x)yd(z) = 0$ for all $x, y, z \in U$. Therefore

$$(d(x) - x)Ud(z) = 0, \text{ for all } x, z \in U, \tag{7}$$

since $d \circ \sigma = \sigma \circ d$ and $\sigma(U) = U$, then $d(U) = 0$ or $d(x) = x$ for all $x \in U$. If $d(U) = 0$, by virtue of Lemma 2.2 it then follows that $d = 0$ or $U \subseteq Z(R)$. Now suppose that $d(x) = x$, for all $x \in U$. Let $r \in R$ and $u \in U$, using $d(u) = u$ and $d([r,u]) = [r,u]$, we find that

$$[d(r), u] = 0, \quad \text{for all } r \in R,\ u \in U.$$

Reasoning as in the proof of Lemma 2.2, we are forced to $d = 0$ or $U \subseteq Z(R)$.

2) Suppose that d acts as an anti-homomorphism on U. We then obtain

$$d(xy) = d(x)y + xd(y) = d(y)d(x), \quad \text{for all } x, y \in U. \tag{8}$$

Replace x by $2xy$ in (8) and using char$R \neq 2$, to get

$$xyd(y) = d(y)xd(y), \quad \text{for all } x, y \in U. \tag{9}$$

Writing $2zx$ instead of x in (9), where $z \in U$, and using $\mathrm{char}R \neq 2$, we find that

$$zxyd(y) = d(y)zxd(y), \quad \text{for all } \ x, y, z \in U. \tag{10}$$

Left multiplying (9) by z and comparing with (10), we obtain

$$[d(y), z]Ud(y) = 0, \quad \text{for all } \ y, z \in U. \tag{11}$$

For $y \in U \cap Sa_\sigma(R)$, this leads to $d(y) = 0$ or $[d(y), u] = 0$, for all $u \in U$. Since $d \circ \sigma = \sigma \circ d$ and $\sigma(U) = U$, using $u + \sigma(u), \ u - \sigma(u) \in U \cap Sa_\sigma(R)$, in view of (11) one can easily prove that

$$d(y) = 0 \quad \text{or} \quad [d(y), U] = 0, \quad \text{for all } \ y \in U.$$

Set $G = \{y \in U \,/\, d(y) = 0\}$ and $H = \{y \in U \,/\, [d(y), U] = 0\}$. Clearly, G and H are additive subgroups of U such that $U = G \cup H$ and so $U = H$ or $U = G$. If $U = G$, then $d(U) = 0$ and Lemma 2.2 yields $d = 0$ or $U \subseteq Z(R)$. Now, suppose that $U = H$; we then have

$$[d(x), u] = 0, \quad \text{for all } \ x, u \in U. \tag{12}$$

Replace x by x^2 in (12) to find that $d(x)[x, u] + [x, u]d(x) = 0$. As $d(x)[x, u] = [x, u]d(x)$, by (12), then $[x, u]d(x) = 0$. Write $2uv$ instead of u in this equality, to get $[x, u]vd(x) = 0$ so that

$$[x, u]Ud(x) = 0, \quad \text{for all } \ x, u \in U.$$

Let $x \in U \cap Sa_\sigma(R)$, as $\sigma(U) = U$, then either $d(x) = 0$ or $[x, u] = 0$ for all $u \in U$. Thus, reasoning as above, one can easily see that $d(x) = 0$ or $[x, U] = 0$, for all $x \in U$. Hence, U is a union of two additive subgroups L and K, where $L = \{x \in U \,/\, d(x) = 0\}, \ K = \{x \in U \,/\, [x, U] = 0\}$ and thus $U = L$ or $U = K$. If $U = L$, then $d(U) = 0$ and Lemma 2.2 assures that $d = 0$ or $U \subseteq Z(R)$. Suppose $U = K$, then $[x, y] = 0$ for all $x, y \in U$ in such a way that $[U, U] = 0$. Let $u \in U$, for $r, t \in R$ we then have $[u, [u, rt]] = 0$ and therefore

$$ur[u, t] + u[u, r]t = r[u, t]u + [u, r]tu.$$

As $u[u, r] = [u, r]u$ and $[u, t]u = u[u, t]$, then $[u, r][u, t] = 0$ because $\mathrm{char}R \neq 2$. Replacing r by rs in this equality, where $s \in R$, we have $[u, r]s[u, t] = 0$ and thus

$$[u, r]R[u, t] = 0, \quad \text{for all } \ r, t \in R, u \in U.$$

Since R is σ-prime, this gives $U \cap Sa_\sigma(R) \subseteq Z(R)$ and therefore $U \subseteq Z(R)$. Consequently, in all the cases we find that either $d = 0$ or $U \subseteq Z(R)$, which completes the proof. ∎

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