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Jordan Generalized Derivations on σ-prime Rings

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Abstract

Let R be a 2-torsion free σ-prime ring with involution σ and let U be a nonzero σ-Lie ideal of R such that $u^2 \in U$ for all $u \in U$. The main goal of this work is to prove that if F is a Jordan generalized derivation on U, then F is a generalized derivation on U.

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1 Preliminaries

Throughout the present paper R will denote an associative ring with center $Z(R)$. A ring R with involution σ is said to be σ- prime if $aRb = aR\sigma(b) = (0)$ implies that $a = 0$ or $b = 0$. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. It is worthwhile to note that every prime ring having an involution σ is σ-prime but the converse is in general not true. An ideal I of R is called a σ-ideal if I is invariant under σ, i.e, $\sigma(I) = I$. In all that follows $Sa_\sigma(R)$ will denote the set of symmetric and skew symmetric elements of R i.e $Sa_\sigma(R) = \{x \in R/\sigma(x) = \pm x\}$. For any x, y in R we write $[x, y]$ for the commutator $xy-yx$ and we will use the following basic commutator identities:

$$[xy, z] = x[y, z] + [x, z]y \quad \text{and} \quad [x, yz] = y[x, z] + [x, y]z.$$

A Lie ideal of R is an additive subgroup of R satisfying $[R, U] \subset U$. Moreover if $\sigma(U) = U$, then U is called a σ-Lie ideal. An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$, (resp. $d(x^2) = d(x)x + xd(x)$) for all $x, y \in R$. Clearly, every derivation is a Jordan derivation but the converse need not be true in general.

An additive mapping $F : R \to R$ is called a generalized derivation (resp. Jordan generalized derivation) if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, (resp. $F(x^2) = F(x)x + xd(x)$) for all $x, y \in R$. Obviously, every generalized derivation is a Jordan generalized derivation. The converse statement does not hold in general.
In order to prove our main result we need the following Lemmas.

Lemma 1.1 ([6], Lemma 5) *Let R be a 2 torsion free σ-prime ring and let U be a nonzero σ-Lie ideal such that $U \nsubseteq Z(R)$. Then $aUb = aU\sigma(b) = 0$ implies that $a = 0$ or $b = 0$.*

Lemma 1.2 ([5], Lemma 2.4) *Let R be a 2 torsion free σ-prime ring and let U be a nonzero σ-Lie ideal of R. If d is a nonzero derivation on R such that $d(U) = 0$, then $U \subset Z(R)$.*

Lemma 1.3 ([2], Lemma 2.2) *Let R be a 2 torsion free ring and U be a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If $F : R \to R$ is an additive mapping satisfying $F(u^2) = F(u)u + ud(u)$ for all $u \in U$, then*

(i) $F(uv + vu) = F(u)v + F(v)u + ud(v) + vd(u)$, *for all* $u, v \in U$

(ii) $F(uvu) = F(u)vu + ud(v)u + uvd(u)$, *for all* $u, v \in U$

(iii) $F(uvw+wvu) = F(u)vw+F(w)vu+ud(v)w+uvd(w)+wd(v)u+wvd(u)$, *for all* $u, v, w \in U$

Lemma 1.4 ([2], Lemma 2.3) *Let Let R be a 2-torsion free ring and U be a nonzero Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If $F : R \to R$ is an additive mapping satisfying $F(u^2) = F(u)u + ud(u)$ for all $u \in U$, then $\delta(u, v)w[u, v] = 0$, for all $u, v, w \in U$ where $\delta(u, v) = F(uv) - F(u)v - ud(v)$.*

2 Main Result

In [4], Herstein showed that every Jordan derivation on a 2-torsion free prime ring is a derivation. In [3], Awtar extended this result on Lie ideals. In [1], M. Ashraf and N. Rehman showed that if R is a ring with a commutator which is not a divisor of zero, then every Jordan generalized derivation on R is a generalized derivation. The same authors with A. Shakir proved in [2] that if R is a 2-torsion free prime ring and U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$, then every Jordan generalized derivation on U is a generalized derivation on U. Our aim in this paper is to generalize this results for σ-prime rings as follows:

Theorem 2.1 *Let R be a 2-torsion free σ-prime ring and U be a nonzero σ-Lie ideal of R such that $u^2 \in U$, for all $u \in U$. If $F : R \to R$ is an additive mapping satisfying $F(u^2) = F(u)u + ud(u)$ for all $u \in U$, then $F(uv) = F(u)v + ud(v)$ for all $u, v \in U$.*

Proof. Let $\delta : R^2 \longrightarrow R$ the mapping defined by

$$\delta(u, v) = F(uv) - F(u)v - ud(v).$$

As we can see, δ is bi-additive. We begin by the case where $U \subset Z(R)$. According to Lemma 1.3 (iii), we have

$$F(uvw+wvu) = F(u)vw+F(w)vu+ud(v)w+uvd(w)+wd(v)u+wvd(u). \quad (1)$$

The fact that $u^2 \in U$ for all $u \in U$ yields $2uv \in U$ for all $u, v \in U$. It then follows, using Lemma 1.3 (i), that

$$\begin{aligned} 2F(uvw + wvu) &= F((2uv)w + w(2uv)) \\ &= F((2uv)w + 2uvd(w) + 2F(w)uv + wd(2uv)) \\ &= 2[F(uv)w + uvd(w) + F(w)uv + wd(uv)]. \end{aligned}$$

Since $char R \neq 2$, for all $u, v, w \in U$, we then get

$$F(uvw + wvu) = F(uv)w + uvd(w) + F(w)uv + wd(uv). \quad (2)$$

Combining (1) and (2), we obtain

$$\delta(u, v)w = 0, \text{ for all } u, v, w \in U. \quad (3)$$

Replace w by $[w, r]$ where $r \in R$ in (3), we get $\delta(u, v)rw = 0$, for all $u, v, w \in U$ and $r \in R$. Therefore, $\delta(u, v)RU = 0$ for all $u, v \in U$. Since $\sigma(U) = U \neq 0$ and R is σ-prime, then $\delta(u, v) = 0$, for all $u, v \in U$ and the Theorem is proved in this case.

Now, assume that $U \not\subseteq Z(R)$. By Lemma 1.4, we have $\delta(u, v)w[u, v] = 0$, for all $u, v, w \in U$. Thus

$$\delta(u, v)U[u, v] = 0, \text{ for all } u, v, \in U. \quad (4)$$

Let $u_0 \in Sa_\sigma(R) \cap U$. If $u_0 \in Z(R)$, replacing u by u_0 in (1) and (2) we find that $\delta(u_0, v) = 0$ for all $v \in U$. On the other hand if $u_0 \notin Z(R)$, then from (4) and Lemma 1.1, $\delta(u_0, v) = 0$ or $[u_0, v] = 0$ for all $v \in Sa_\sigma(R) \cap U$. Let $v \in U$; the fact that $v - \sigma(v) \in Sa_\sigma(R) \cap U$, yields $\delta(u_0, v - \sigma(v)) = 0$ or $[u_0, v - \sigma(v)] = 0$. Suppose that $[u_0, v - \sigma(v)] = 0$, then $[u_0, v] \in Sa_\sigma(R) \cap U$ so that $\delta(u_0, v) = 0$ or $[u_0, v] = 0$ by (4). Assume that $\delta(u_0, v - \sigma(v)) = 0$; the fact that $v+\sigma(v) \in Sa_\sigma(R) \cap U$ gives $\delta(u_0, v+\sigma(v)) = 0$ or $[u_0, v+\sigma(v)] = 0$. If

$\delta(u_0, v+\sigma(v)) = 0$, then $2\delta(u_0, v) = \delta(u_0, v+\sigma(v))+\delta(u_0, v-\sigma(v)) = 0$, so that $\delta(u_0, v) = 0$. If $[u_0, v+\sigma(v)] = 0$, then $[u_0, v] \in Sa_\sigma(R) \cap U$ and (4) implies that $\delta(u_0, v) = 0$ or $[u_0, v] = 0$. In conclusion, the additive group U is union of two its additive subgroups L and K such that $L = \{v \in U$ such that $\delta(u_0, v) = 0\}$ and $K = \{v \in U$ such that $[u_0, v] = 0\}$. By Brauer's trick, we have $U = L$ or $U = K$. If $U = K$, then $d_{u_0}(U) = 0$ where d_{u_0} is the inner-derivation defined by u_0. Since $u_0 \notin Z(R)$, then $0 \neq d_{u_0}$. Hence $U \subset Z(R)$ by Lemma 1.2, a contradiction. It follows that $U = L$ and consequently,

$$\delta(u, v) = 0, \text{ for all } u \in Sa_\sigma(R) \cap U, \ v \in U.$$

Let $u \in U$; then $2\delta(u, v) = \delta(u + \sigma(u), v) + \delta(u - \sigma(u), v) = 0$. As $char R \neq 2$, then $\delta(u, v) = 0$ for all $u, v \in U$, proving our Theorem.

Corollary 2.2 *Every Jordan generalized derivation on a 2-torsion free σ-prime ring is a generalized derivation.*

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