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# On Generalized Derivations and Commutativity in $\sigma$-prime Rings

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**Abstract**

Let $R$ be a 2-torsion free $\sigma$-prime ring, $0 \neq I$ a $\sigma$-ideal of $R$ and $F$ a nonzero generalized derivation associated with a derivation $d$ of $R$ which commutes with $\sigma$. In this paper it is shown that if either $F[x, y] = [x, y]$ or $F(x \circ y) = x \circ y$ for all $x, y \in I$, then $R$ is a commutative ring.

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## 1 Preliminaries

Throughout this paper, $R$ will represent an associative ring with center $Z(R)$. $R$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anticommutator $xy + yx$. If $R$ has an involution $\sigma$, we set $Sa_\sigma(R) := \{r \text{ in } R \text{ such that } \sigma(r) = \pm\, r\}$. Recall that $R$ is said to be $\sigma$-prime if $aRb = \sigma(a)Rb = 0$ implies that $a = 0$ or $b = 0$. It is straightforward to see that every prime ring having an involution $\sigma$ is a $\sigma$-prime ring but the converse is in general not true. An ideal $I$ of $R$ is called a $\sigma$-ideal if $I$ is invariant under $\sigma$. An additive map $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the map $d_a : R \longrightarrow R$ defined by $d_a(x) = [a, x]$ for all $x \in R$ is a derivation which is said to be an inner derivation. An additive map $g_{a,b} : R \longrightarrow R$ is called a generalized inner derivation if $g_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. It is easy to see that if $g_{a,b}$ is a generalized inner derivation, then

$$g_{a,b}(xy) = g_{a,b}(x)y + xd_{-b}(y) \text{ for all } x, y \in R,$$

where $d_{-b}$ is an inner derivation. Following this observation and M. Brešar [1], an additive map $F : R \longrightarrow R$ is called a generalized derivation associated with $d$ if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. We can easily check that the notion of generalized derivation covers the notions of a derivation and a left multiplier (i.e. $F(xy) = F(x)y$ for all $x, y \in R$).

Throughout the present paper we shall make extensive use of the following basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z, \ [xy, z] = x[y, z] + [x, z]y$$

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

To prove our results we will need the following two lemmas.

**Lemma 1.1** ([4], Lemma 3.1) *Let $R$ be a $\sigma$-prime ring and let $I$ be a nonzero $\sigma-$ideal of $R$. If $a$, $b$ in $R$ satisfy $aIb = aI\sigma(b) = 0$, then $a = 0$ or $b = 0$.*

**Lemma 1.2** ([5], Lemma 2.2) *Let $R$ be a 2-torsion free $\sigma$-prime ring, let $F$ be an additive mapping on $R$, let $d$ be a derivation which commutes with $\sigma$ and let $I$ be a nonzero $\sigma$-ideal of $R$. If $[F(y), z]Id(y) = 0$ for all $y, z \in I$, then either $d = 0$ or $[F(y), z] = 0$ for all $y, z \in I$.*

# 2 Main Results

There exist various results concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivation of $R$. In [2], M. N. Daif and H. E. Bell established that a prime ring $R$ must be commutative if it admits a derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y$ in a nonzero ideal of $R$. Recently, M. A. Quadri et al. [6] proved that the Daif and Bell's result obtained by considering a generalized derivation instead of the derivation in a prime ring, is still true. The purpose of the following theorem is to prove the corresponding result for $\sigma$-prime rings as follows.

**Theorem 2.1** *Let $R$ be a 2-torsion free $\sigma$-prime ring and let $I$ be a nonzero $\sigma$-ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ commuting with $\sigma$ such that $F[x, y] = [x, y]$ for all $x, y \in I$, then $R$ is commutative.*

**Proof.** Suppose that $F[x, y] = [x, y]$ for all $x, y \in I$. Then

$$F(x)y + xd(y) - F(y)x - yd(x) - [x, y] = 0, \text{ for all } x, y \in I. \qquad (1)$$

Replace $y$ by $yz$ in (1), where $z \in I$, to obtain

$$F(x)yz+xd(y)z+xyd(z)-F(y)zx-yd(z)x-yzd(x)-y[x,z]-[x,y]z=0. \quad (2)$$

By employing (1), we see that the relation (2) is reduced to

$$F(y)[x,z]+y[d(x),z]+[x,y]d(z)+y[x,d(z)]-y[x,z]=0, \text{ for all } x,y,z \in I. \quad (3)$$

Substituting $zx$ for $z$ in (3), in view of (3) we find that

$$[x,yz]d(x)=0 \text{ for all } x,y,z \in I. \quad (4)$$

Write $zt$ instead of $z$ in (4) to get $[x,y]ztd(x)=0$ and therefore

$$[x,y]zId(x)=0 \text{ for all } x,y,z \in I. \quad (5)$$

Let $x \in I \cap Sa_\sigma(R)$; since $d$ commutes with $\sigma$, for the relation (5) yields

$$[x,y]zId(x)=0=[x,y]zI\sigma(d(x)) \text{ for all } y,z \in I.$$

Applying Lemma 1.1, either $d(x)=0$ or $[x,y]z=0$. If $[x,y]z=0$ for all $y,z \in I$, then $[x,y]I=0$ so that $[x,y]=0$ by Lemma 1.1. Therefore, for each $x \in I \cap Sa_\sigma(R)$ we have $d(x)=0$ or $[x,y]=0$ for all $y \in I$.
Let $u \in I$, the fact that $u-\sigma(u) \in I \cap Sa_\sigma(R)$ assures that $d(u-\sigma(u))=0$ or $[u-\sigma(u),y]=0$ for all $y \in I$.
If $d(u-\sigma(u))=0$, then $d(u) \in Sa_\sigma(R)$ and in view of (5) this yields $d(u)=0$ or $[u,y]=0$ for all $y \in I$. Now, suppose that $[u-\sigma(u),y]=0$ for all $y \in I$. Since $u+\sigma(u) \in Sa_\sigma(R)$, then $d(u+\sigma(u))=0$ or $[u+\sigma(u),y]=0$ for all $y \in I$.
If $[u+\sigma(u),y]=0$, then $2[u,y]=0$ so that $[u,y]=0$, for all $y \in I$, because $\text{char}R \neq 2$. If $d(u+\sigma(u))=0$, then $d(u) \in Sa_\sigma(R)$ and (5) leads to $d(u)=0$ or $[u,y]=0$ for all $y \in I$. In conclusion, for all $u \in I$, we have either $d(u)=0$ or $[u,v]=0$ for all $v \in I$. This means that $I$ is the union of two its additive subgroups $L=\{u \in I \,/\, [u,v]=0, \text{ for all } v \in I\}$ and $K=\{u \in I \,/\, d(u)=0\}$. Since a group cannot be the union of two proper subgroups, then $I=K$ or $I=L$. The fact that $d \neq 0$ forces $I=L$ and thus $[u,v]=0$ for all $u,v \in I$. A similar reasoning as in ([5], proof of Theorem 1.1) assures that $R$ is commutative. ∎

In [6] it is proved that a prime ring $R$ must be commutative if $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ satisfying $F(x \circ y)=x \circ y$ for all $x,y$ in a nonzero ideal $I$ of $R$. Our next goal in this paper is to prove an analogous result for $\sigma$-prime rings as follows.

**Theorem 2.2** *Let $R$ be a 2-torsion free $\sigma$-prime ring and let $0 \neq I$ be a $\sigma$-ideal of $R$. If $R$ has a generalized derivation $F$ associated with a nonzero derivation $d$ commuting with $\sigma$ such that $F(x \circ y)=x \circ y$ for all $x,y \in I$, then $R$ is commutative.*

**Proof.** Assume that $x, y \in I$ for all $x, y \in I$. Hence $F(x \circ y) = (x \circ y)$, which can be written as

$$F(x)y + xd(y) + F(y)x + yd(x) - x \circ y = 0, \text{ for all } x, y \in I. \tag{6}$$

Write $yx$ instead of $y$ in (6), where $x \in I$, to obtain

$$F(x)yx + xd(y)x + xyd(x) + F(y)x^2 + yd(x)x + yxd(x) - (x \circ y)x = 0.$$

According to (6), this yields that $(x \circ y)d(x) = 0$, for all $x, y \in I$.
Replacing $y$ by $yz$ in this equation we conclude

$$[x, y]zd(x) = 0 \text{ for all } x, y, z \in I$$

and therefore $[x, y]Id(x) = 0$, for all $x, y \in I$. Applying Lemma 1.2 for the particular case $F = 1$, we deduce that $[x, y] = 0$ for all $x, y \in I$, because $d \neq 0$. From ([5], proof of Theorem 1.1) this yields that $R$ is commutative. ∎

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