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A Necessary and Sufficient Condition of a Mixed NE in Bimatrix Games

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Abstract. In the paper a mixed Nash equilibrium (NE) in bimatrix games is considered. Developing an algebraic approach, a necessary and sufficient condition for existence of the NE, the equilibrium profile, the value of its payoff are expressed with saddle point matrices of a special form, i.e. the block matrices with two vector blocks of ones.

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1. Introduction.

The aim of this paper is to apply in game theory some properties of a saddle point matrix, denoted by $(A, \mathbf{e})$, where the block matrix is defined as follows:

$$(A, \mathbf{e}) = \begin{bmatrix} A & \mathbf{e} \\ \mathbf{e}^{\mathrm{T}} & 0 \end{bmatrix}, \tag{1}$$

and the top-left block is a matrix $A \in \mathrm{R}^{n \times n}$, the top-right block **e** is the column vector with all entries of ones, the bottom-left block is **e** transposed, and the bottom-right block has the single entry 0. If A is symmetric, then (A, **e**) may be interpreted as the bordered Hessian of a standard quadratic program over the standard simplex, and it is called the Karush-Kuhn-Tucker matrix of the program, which is known to have a large spectrum of applications (for a review see, for instance, Bomze, 1998).

The matrix obtained from A by replacing the i-th row with $\mathbf{e}^{\mathrm{T}}$ is denoted by A^i, and A_j stands for the matrix of A with the j-th column replaced by **e**, and M/A denotes the Schur complement of A in M: if $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and A is nonsingular, then

$$M/A = D - CA^{-1}B, \tag{2}$$

The paper is mostly restricted to symmetric bimatrix games [A, A^{T}], where A is a real square payoff matrix for first player, often called Row, and A^{T} is the payoff matrix for second player, usually called Column. In concluding remarks our results are generalized for bimatrix games [A, B]. Row's strategy choice is denoted by **x**, and Column's strategy choice by **y**. The matrix A entries a_{ij} denote Row's and Column's payoffs when Row plays the i-th pure strategy in a contest against Column, that plays the j-th pure strategy. Notice that it is not here assumed that A is symmetric. A profile **p** is a NE if and only if

$$\mathrm{E}(\mathbf{p}, \mathbf{p}) \geq \mathrm{E}(\mathbf{x}, \mathbf{p}) \iff \mathbf{p}^{\mathrm{T}}A\mathbf{p} \geq \mathbf{x}^{\mathrm{T}}A\mathbf{p} \text{ for all strategies } \mathbf{x}. \tag{3}$$

In other words, the NE is a set of actions **p** which must be the best response to all strategies, especially to itself. A mixed strategy is a probability vector **x** specifying the probability with which each pure strategy is played. If these probabilities are all strictly positive, **x** is said to be completely mixed. Denote by E(A) the expected payoff of the given game

$$\mathrm{E}(A) = \mathbf{p}^{\mathrm{T}}A\mathbf{p} \tag{4}$$

2. Properties of (A, e)

In the paper we always assume that A, $E \in R^{n \times n}$ ($n > 1$), and $E = \mathbf{ee}^T$. The following properties, proved in Ostrowski (2007), of $(A, \mathbf{e})$ can be established:

Proposition 1. For any matrix A the following equalities hold:

$$\det(A, \mathbf{e}) = -\sum_{j=1}^{n} \det A^j = -\sum_{i=1}^{n} \det A_i \quad (5)$$

Proposition 2. For arbitrary real numbers α, β and vectors $\mathbf{v} \in R^n$ we have:

$$\det(\alpha A + \beta E, \mathbf{e}) = \alpha^{n-1} \cdot \det (A, \mathbf{e}) \quad (6)$$

Proposition 3. For any real numbers α, β the determinant of the matrix $\alpha A + \beta E$ can be expressed as follows:

$$\det(\alpha A + \beta E) = \alpha^{n-1}[\alpha \cdot \det A - \beta \cdot \det(A, \mathbf{e})] \quad (7)$$

Proposition 4. The determinant of $(A, \mathbf{e})$ is equal to the difference of the determinants of A and $A + E$:

$$\det(A, \mathbf{e}) = \det A - \det(A + E) \quad (8)$$

Proposition 5. If A is nonsingular, then the following equalities hold true:

$$\det(A, \mathbf{e}) = -\mathbf{e}^T A^{-1} \mathbf{e} \cdot \det A = (A, \mathbf{e})/A \cdot \det A \quad (9)$$

Note that nonsingularity of A does not say anything about singularity or nonsingularity of the saddle point matrix $(A, \mathbf{e})$. In the case above the condition of nonsingilarity of $(A, \mathbf{e})$ contains (17).

Proposition 6. If A is nonsingular and for some α, β the linear combination $\alpha A + \beta E$ is also nonsingular, then

$$(\alpha A + \beta E, \mathbf{e})/(\alpha A + \beta E) = \frac{(A, \mathbf{e})/A}{\alpha - \beta (A, \mathbf{e})/A} \quad (10)$$

Proposition 7. If A is symmetric and nonsingular, then (A, **e**) and $B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & -\mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e} \end{bmatrix}$ are congruent and, by (9), the determinants of both congruent matrices are equal. If both A and (A, **e**) are nonsingular, then

$$(A, \mathbf{e})^{-1} = \frac{1}{\mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e}} \begin{bmatrix} \mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e}A^{-1} - A^{-1}EA^{-1} & A^{-1}\mathbf{e} \\ \mathbf{e}^{\mathrm{T}}A^{-1} & -1 \end{bmatrix} \tag{11}$$

Proposition 8. If the saddle point matrix (A, **e**) is nonsingular, then the following holds:

$$\operatorname{rank}(A, \mathbf{e}) = n + 1 \;\Rightarrow\; \operatorname{rank}\begin{bmatrix} A \\ \mathbf{e}^{\mathrm{T}} \end{bmatrix} = n \tag{12}$$

Proposition 9. If $\det(A, \mathbf{e}) \neq 0$, then the rank of A is either n or $n - 1$:

$$(\operatorname{rank}(A, \mathbf{e}) = n + 1) \;\Rightarrow\; (n - 1 \leq \operatorname{rank} A \leq n) \tag{13}$$

The converse is not true, but if A is both symmetric and positive definite, then it immediately follows from (10) that $\det(A, \mathbf{e}) \neq 0$ (see also Benzi, Golub and Liesen, 2005). Moreover, positive definite (A, **e**) has one negative eigenvalue and n positive ones (Boyd and Vandenberghe, 2004).

Proposition 10. The following factorizations of (A, **e**) are equivalent (see also Higham and Cheng, 1998; Tian and Takane, 2005)

$$(A, \mathbf{e}) = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{e}^{\mathrm{T}}A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & -\mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} I & A^{-1}\mathbf{e} \\ \mathbf{0} & 1 \end{bmatrix} \tag{14}$$

$$(A, \mathbf{e}) = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{e} & -\mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e} \end{bmatrix} \begin{bmatrix} I & A^{-1}\mathbf{e} \\ \mathbf{0} & 1 \end{bmatrix} \tag{15}$$

$$(A, \mathbf{e}) = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{e}^{\mathrm{T}}A^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{e} \\ \mathbf{0} & -\mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e} \end{bmatrix} \tag{16}$$

The decompositions imply that if A is nonsingular then:

$$\det(A, \mathbf{e}) \neq 0 \iff \mathbf{e}^{\mathrm{T}}A^{-1}\mathbf{e} \neq 0 \tag{17}$$

If $A \in \mathrm{R}^{n \times n}$ is singular and rank $A = n-1$ then A can be replaced by nonsingular matrix $\alpha A + \beta E$ for some $\alpha, \beta \in \mathrm{R}$, and which does not change the optimizer (see also Bomze and de Klerk, 2002).

3. Bimatrix games.

It is well known that certain types of games like games with Minkowski-Leontief payoff matrices or generalized rock-scissors-paper games or anti-coordination game (for details see Kojima and Takahashi, 2004) posses completely mixed NE. Recall that Minkowski-Leontief matrix is a matrix of the form $I - V$, where V is a nonnegative matrix with zeroes on its diagonal, and the generalized rock-scissors-paper games have payoff matrices of the form

$$A = \begin{bmatrix} 1 & 2+a & 0 \\ 0 & 1 & 2+a \\ 2+a & 0 & 1 \end{bmatrix}, a \in \mathrm{R}.$$

It is established, Cheng et al. (2004), that any finite symmetric game has a symmetric equilibrium. The concept of the NE is of particular importance in evolutionary matrix games as it is applied in considering behavior adopted by various organisms and allows to make predictions about their behavior. Furthermore, the NE is the condition must be met if the proportion of individuals is to be stable. From this point of view, Holt and Roth (2004), the NE can be interpreted as a potential stable point of a dynamic adjustment process in which individuals adjust their behavior, searching for the choice that will give them better results.

Theorem. Let be given a symmetric bimatrix game $[A, A^{\mathrm{T}}]$, in which the square matrix A is the first player's payoff matrix, such that $(A, \mathbf{e})$ is nonsingular. A necessary and sufficient condition for the existence and uniqueness of a completely mixed NE is that

$$\det(A^i, \mathbf{e})\det(A^j, \mathbf{e}) > 0 \text{ for all pairs } i, j, \tag{18}$$

and in any mixed equilibrium in this game in which this player's strategy is completely mixed, the players payoff E(A) is given by

$$\mathrm{E}(A) = -\frac{\det A}{\det(A, \mathbf{e})}. \tag{19}$$

and coefficients of the equilibrium profile **p** have the form

$$\mathrm{p}^i = \frac{\det(A^i, \mathbf{e})}{\det(A, \mathbf{e})} (i = 1, \ldots, n). \tag{20}$$

Proof. In an equilibrium in which row player's strategy is completely mixed, the mixed strategy **y** used by the other player is such that $A\mathbf{y} = \mathrm{E}(A)\mathbf{e}$. Since the entries of the vector **y** sum up to one, $\mathrm{E}(A)\mathbf{e} = \mathrm{E}(A)E\mathbf{y}$. Therefore $[A - \mathrm{E}(A)E]\mathbf{y} = \mathbf{0}$, which implies that

$$\det[A - \mathrm{E}(A)E] = 0.$$

On the other hand, setting $\alpha = 1$ and $\beta = -\mathrm{E}(A)$ in (3) gives

$$\det[A - \mathrm{E}(A)E] = \det A + \mathrm{E}(A)\det(A, \mathbf{e}).$$

Since $\det(A, \mathbf{e}) \neq 0$ by assumption, we get (19). ■

The system of equations

$$\begin{bmatrix} A & \mathbf{e} \\ \mathbf{e}^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{E}(A)\mathbf{e} \\ 1 \end{bmatrix}. \tag{21}$$

expresses the double condition that each of the row player's n pure strategies give the same payoff when played against **p**, and the entries of **p** sum up to one. It follows immediately from Cramer's rule that (21) and (19) imply (20):

$$\mathrm{p}_i = \mathrm{E}(A)\frac{\det A^i}{\det A} = -\frac{\det A}{\det(A,\mathbf{e})}\left(-\frac{\det(A^i,\mathbf{e})}{\det A}\right) = \frac{\det(A^i,\mathbf{e})}{\det(A,\mathbf{e})}.$$

Similar arguments apply to the column player and A^{T}, because transposition of the payoff matrix makes the row player a column player, and note that

$$\det(A^{\mathrm{T}})_i = \det(A^i)^{\mathrm{T}} = \det A^i \text{ for all } i.$$

Suppose now that (18) holds. By (5), this implies that $\det(A, \mathbf{e}) \neq 0$, and that (20) unambiguously defines **p** as strictly positive probability vector. By Cramer's rule, the vector **p** and the scalar E(A) defined by (19) satisfy (21). This proves that **p** is a completely mixed equilibrium, with the payoff E(A).

It remains to prove the necessity of condition (18). Suppose that the game has a unique completely mixed equilibrium **p**, with the payoff E(A). As shown above, **p** and E(A) satisfy the double condition expressed by (21). Moreover, they constitute an isolated solution of (21). This is because replacing them with any other strictly positive vector **p**′ and scalar E′(A) satisfying (21) would give a different completely mixed equilibrium, which by assumption does not exist. The existence of an isolated solution of (21) implies that det(A, **e**) ≠ 0. Since the vector **p** is strictly positive, therefore (21) follows from (20). ∎

4. Concluding remarks.

The Theorem, using the same reasoning as above, can be easily generalized for bimatrix games [A, B], where A, B are n x n matrices. We obtain now that a necessary and sufficient condition for the existence and uniqueness of a completely mixed NE is that for all pairs i, j

$$\det(A^i, \mathbf{e})\det(A^j, \mathbf{e}) > 0 \text{ and } \det(B_i, \mathbf{e})\det(B_j, \mathbf{e}) > 0,$$

and in that case the row player's payoff E(A) and column player's payoff E(B) are equal

$$\mathrm{E}(A) = -\frac{\det A}{\det(A,\mathbf{e})},\ \mathrm{E}(B) = -\frac{\det B}{\det(B,\mathbf{e})},$$

and the row player's strategy **p** and column player's strategy **q** have coefficients

$$\mathrm{p}_i = \frac{\det(B_i,\mathbf{e})}{\det(B,\mathbf{e})},\ \mathrm{q}^i = \frac{\det(A^i,\mathbf{e})}{\det(A,\mathbf{e})},\ (i = 1, \ldots, n).$$

References

[1] M. Benzi, G. H. Golub and J. Liesen, Numerical solution of saddle point problems, *Acta Numerica* (2005), pp. 1–137

[2] I. M. Bomze, On standard quadratic optimization problems, *Journal of Global Optimization* **13** (1998), pp.369–387.

[3] I. M. Bomze and E. de Klerk, Solving standard quadratic optimization problems via linear, semidefinite and copositive programming, *Journal of Global Optimization* 24 (2002), pp. 163–185

[4] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, Cambridge, UK (2004)

[5] S. H. Cheng and D. M. Reeves and Y. Vorobeychik and M. P. Wellman, Notes on Equilibria in Symmetric Games, AAMAS-04 (2004), pp. 23–28

[6] N. J. Higham and S. H. Cheng, Modifying the Inertia of Matrices Arising in Optimization, *Linear Algebra and its Applications* 275/276, No 1–3 (1998), pp. 261–279

[7] C. A. Holt and A. E. Roth, The Nash Equilibrium: A perspective, *PNAS*, Vol. 101, no. 12 (2004), pp. 3999–4002

[8] F. Kojima and S. Takahashi, Satoru, Anti-Coordination Games and Dynamic Stability, *ISER Seminar Series*, 7 / 7 / 04 (2004)

[9] T. Ostrowski, On Some Properties of Saddle Point Matrices with Vector Blocks, *International Journal of Algebra*, Vol. 1, 2007, no.3, pp. 129–138

[10] Y. Tian and Y. Takane, Schur complements and Banachiewicz-Schur forms. *Electronic Journal of Linear Algebra*, 13 (2005), 405-418

[11] F. Zhang (Editor), *The Schur Complement And Its Applications*. Numerical Methods and Algorithms, Vol. 4, Springer (2005)

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