Compactly Generated Lattices
with a Unique Essential Element

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Abstract
Lattices with a unique essential element are investigated. It is shown that, a compactly generated lattice with a unique essential element different from the top element, and with finite Goldie dimension has finite length.

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1 Introduction
In the literature an essential element of a lattice is not assumed to be different from the top element. Therefore compactly generated lattices with a unique essential element are nothing but the semiatomic ones, see [3] section 1.8. Modules with unique proper essential submodule, first introduced and studied in [2]. Our aim in this note is to find basic properties of lattices with a unique essential element.(ue-lattices). It is clear that every maximal element in a compactly generated lattice is either essential or is a complement. We also study lattices which has a unique essential maximal element.(uem-lattices). In the preliminary section definitions and results which are essential in the article are given. In the main section uem and ue-lattices will be investigated. Examples are given at the end of this section.

2 Preliminary Notes

Definition 2.1 A lattices $L$ is said to be upper-continuous if it is complete and such that for each subset $A$ of $L$, we have: $a \land \left( \bigvee_{x \in A} x \right) = \bigvee_{x \in A} (a \land x)$. 

Definition 2.2 An element $a$ in an upper-continuous lattice $L$ is said to be compact if for each directed subset $A$ of $L$ for which $a \leq \bigvee_{x \in A} x$, there exists an element $x_0 \in A$ such that $a \leq x_0$. $L$ is compact if the top element is compact in $L$, and $L$ in compactly generated if each element of $L$ is compact. Throughout this note $L$ will always denotes an upper-continuous and modular lattice with the top element $1$ and bottom element $0$. For all $a \leq b$

$$[a, b] = \{x \in L : a \leq x \leq b\}$$

is called the factor of $b$ by $a$. Obviously $[a, b]$ is a sublattice of $L$ and $\phi_b : x \to x \lor b$ is an isomorphism of $[a, a \lor b]$ onto $[a \land b, b]$. The inverse isomorphism is $\Psi_a : x \to x \lor a$.

A subset $I$ of $L \setminus \{0\}$ is called join-independent if for any finite subset $X$ of $I$ and $x \in I \setminus X$, we have $\bigvee X \land x = (0)$, where $\bigvee X$ denotes the join of all elements of $X$.

Definition 2.3 The Goldie dimension of a lattice $L$, denoted by $G\dim L$ is defined to be the supremum of all cardinal numbers $\aleph$ such that there exists a set $X$ of join-independent set of cardinality $\aleph$.

Definition 2.4 A nonzero element $a$ in a lattice $L$ is essential in $L$, if for any nonzero element $x \in L$, $a \land x \neq 0$.

Definition 2.5 An element $b$ in a lattice $L$ is a complement of $a \in L$, if $a \lor b = 1$ and $a \land b = 0$. Any $b$ such that $a \land b = 0$ and $b$ is maximal with this property is called a pseudo-complement of $a$ in $L$. It is known that if $b$ is a pseudo-complement of $a$, then $a \lor b$ is essential in $L$.

Lemma 2.6 Suppose $L$ is a compact lattice, given $a \neq 0$ in $L$, then there exists a maximal element $m$ such that $a \leq m$.


An element $a \in L$ is called an atom if $b \leq a$ implies $b = 0$ or $b = a$. The join of all atom of $L$ which in denoted by $s(L)$ is called the socle of $L$ and whenever $1$ is a join of atoms, we call $L$ semiatomic.

For the proof of the following lemma see [3] section 1.9.

Lemma 2.7 If $L$ is compactly generated, then $s(L)$ is the meet of all essential elements of $L$. 
Definition 2.8 The Loewy chain of a lattice $L$ is defined inductively. If $\alpha = 0$, $s_\alpha(L) = 0$, if $\alpha = 1$ $s_1(L) = L$. If $\alpha = \beta + 1$, put $s_\alpha = s[s_\beta(L), 1]$ and if $\alpha$ in a limit ordinal, we define $s_\alpha(L) = \bigvee_{\beta<\alpha} s_\beta(L)$. so we obtain

$$0 = s_\alpha(L) \leq s_1(L) \leq \cdots \leq s_\alpha(L) < \cdots$$

since $L$ is a set, there exists a least ordinal $\alpha$ such that $s_\alpha(L) = s_{\alpha+1}(L) = \cdots$.

This ordinal is called the Loewy length of $L$ and is denoted by $l(L)$. If $s_\alpha(L) = 1$, $L$ is called a Loewy lattice or a semiartinian lattice.

3 Uem-lattices and ue-lattices

It is well-known and easy to prove that if every maximal right ideal of a ring $R$ is a direct summand, then $R$ is Artin semisimple. The next proposition is the lattice theoretical formulation of this result. First we recall that a complemented lattice is a one, such that every element has a complement and a semiatomic lattice is complemented see [3] section 1.8.2. The converse is true if $L$ is compactly generated, see [3] section 1.8.7. Next we observe that for a compactly generated lattice to be complemented (semiatomic) it is enough that all its maximal elements have a complement. First we need the following easy lemma.

Lemma 3.1 Let $L$ be a compactly generated lattice, if $m \in L$ has a complement $n$, then $m$ is maximal if and only if $n$ is an atom.

Proof. Since $L$ is modular $[m, m \lor n]$ is lattice isomorphic to $[m \land n, n]$ i.e $[m.1] \cong [0, n]$ and we are through.

Proposition 3.2 A compactly generated lattice $L$ is complemented if and only if every maximal element has a complement.

Proof. It is sufficient to show that $s(L) = 1$, If $s(L) \neq 1$ then $s(L) \leq m$ for some maximal element $m$ of $L \setminus \{1\}$ If $n$ is a complement of $m$, then by the above lemma $n$ is an atom and in view of lemma 2.7, $n \leq s(L) \leq m$, which is absurd. Therefore $s(L) = 1$, and $L$ is complemented.

Next we investigate lattices in which all maximal elements, except one has a complement.
**Definition 3.3** A lattice $L$ is a uem-lattice if it has a unique essential maximal element different from 1.

**Proposition 3.4** A compactly generated lattice is a uem-lattice if and only if $[s(L), 1]$ has only one maximal element in $L \setminus \{1\}$.

**Proof.** If $R$ has only one maximal element, then the statement is clear. Hence we assume that $L$ has more than one maximal element and that $L$ is uem-lattice. Let $m$ be the non-essential maximal element of $L$, then clearly $s(L) \leq m$, and $m$ is the only element with this property. Therefore $[s(L), 1]$ has only one maximal element. Conversely if $[s(L), 1]$ has only one maximal element, namely $m$, clearly $m$ is essential in $L$. Now we claim that each maximal element of $L$ different from $m$, has a complement. So let $n$ be such a maximal element, then $n$ is not essential as otherwise $n \geq s(L)$ which implies $n = m$, a contradiction. Therefore $n$ has a complement.

**Definition 3.5** A lattice $L$ is said to be a ue-lattice if it contains only one essential element different from 1.

**Proposition 3.6** Let $L$ be a compactly generated lattice, then the following are equivalent:

1. $L$ is a ue-lattice,
2. $s(L)$ is a maximal element of $L \setminus \{1\}$,
3. For any $x \in L$, either $[0, x]$ is semiatomic or $x$ has a complement and $L$ is not semiatomic.

**Proof.**

$(1) \Rightarrow (2)$. In view of lemma 2.7 $s(L)$ is the unique essential element of $L$. Let $s(L) \leq m$, then $m$ is also an essential element, therefore $s(L)$ is a maximal element.

$(2) \Rightarrow (3)$. Let $x$ be an element of $L$ and $y$ be a pesudue-complement of it. Then $x \lor y$ is essential in $L$, clearly if $x \lor y \neq 1$, then $x \lor y$ is the socle of $L$, i.e in this case $[0, x]$ is semiatomic, as every sublattice of a semiatomic is semiatomic, see [3] section 1.8.4. It is evident that $L$ is not semiatomic.

$(3) \Rightarrow (1)$. Let $e$ be an essential element of $L$ then $[0, e]$ must be semiatomic, therefore $e$ must be the socle.

**Corollary 3.7** Every compactly generated ue-lattice is a uem-lattice.
Proof. Evident.

Recall that a lattice with 0 and 1 is of finite length if there exists a (Jordan-Holder) composition series between 0 and 1. It is well-known that a modular lattice with 0 and 1 is of finite length if and only if it is both Artinian and Noetherian.

**Corollary 3.8** A compactly generated ue-lattice with finite Goldie dimension has finite length.

Proof. The lattice \([0, s(L)]\) is semiatomic and so there exists a join-independent set \(\{s_i: i \in I\}\) consisting of atoms such that \(s(L) = \bigvee_{i \in I} s_i\). Now the finiteness of the Goldie dimension of \(L\) implies that \(I\) is finite, now since \(s(L)\) is maximal in \(L \setminus \{1\}\), it follows that \(L\) has a composition series and we are through.

**Corollary 3.9** Every compactly generated ue-lattice has Loewy length equal to 2.

Proof. Evident.

**Proposition 3.10** If \(L\) is a compactly generated ue-lattice then for any nonzero \(x\) in \(L\), either \([x, 1]\) is semiatomic or \([x, 1]\) is a ue-lattice and \([0, x]\) is not semiatomic.

Proof. If \([x, 1]\) is not semiatomic, then it has an essential element \(y\), clearly \(y\) is essential in \([0, 1]\) i.e \(y = s(L)\). This means that \([x, 1]\) has a unique essential element namely \(s(L)\). Thus \([x, 1]\) is a ue-lattice and \([0, x]\) is semiatomic.

**Theorem 3.11** A compactly generated non semiatomic lattice \(L\) is a ue-lattice if and only if for each \(x \in L\), either \(x\) has a complement \(y\) or \([0, x]\) is semiatomic. Moreover if \([0, x]\) is not semiatomic, it is a ue-lattice and \([0, y]\) is semiatomic.

Proof. The first part is proposition 3.10. Now let \(x \lor y = 1\) and \(x \land y = 0\), where \([0, x]\) is not semiatomic clearly \([0, x]\) is a ue-lattice, for otherwise \(L\) contains more than one essential element, which is not possible. Now it remains to show that \([0, y]\) is semiatomic. To this end assume by way of contradiction that \(y \succ s(L)\), then \(y \lor s(L) = 1\). Let \(z\) be a complement of \(y \land s(L)\) in \([0, s(L)]\), i.e \(s(L) = (y \land s(L)) \lor z\) and \((y \land s(L)) \land z = 0\), thus \(y \lor z = 1\) and \(y \land z = 0\). This shows that \([y, 1] \cong [0, z] \cong [0, x]\) is semiatomic, a contraction and therefore \([0, y]\) must be semiatomic.
Example 3.12.

1. A chain is a $ue$-lattice if and only if it consists only of 3 elements.

2. Let $p$ and $q$ be distinct prime numbers. and consider the cyclic group $G = \mathbb{Z}_{pq}$. Let $A, B, C, D$ be respectively subgroups of $G$ of orders, $pq, q^2, p$ and $q$. The lattice of all subgroups of $G$ is represented below,

A is the only essential element of this lattice different from $G$, and therefore the lattice of subgroups of $\mathbb{Z}_{pq}$ is a $ue$-lattice.

References


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