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Compactly Generated Lattices with a Unique Essential Element

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Abstract

Lattices with a unique essential element are investigated. It is shown that, a compactly generated lattice with a unique essential element different from the top element, and with finite Goldie dimension has finite length.

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1 Introduction

In the literature an essential element of a lattice is not assumed to be different from the top element. Therefore compactly generated lattices with a unique essential element are nothing but the semiatomic ones, see [3] section 1.8. Modules with unique proper essential submodule, first introduced and studied in [2]. Our aim in this note is to find basic properties of lattices with a unique essential element.(ue-lattices). It is clear that every maximal element in a compactly generated lattice is either essential or is a complement. We also study lattices which has a unique essential maximal element.(uem-lattices). In the preliminary section definitions and results which are essential in the article are given. In the main section uem and ue-lattices will be investigated. Examples are given at the end of this section.

2 Preliminary Notes

Definition 2.1 *A lattices L is said to be upper-continuous if it is complete and such that for each subset A of L. we have:* $a \wedge (\bigvee_{x \in A} x) = \bigvee_{x \in A} (a \wedge x)$.

Definition 2.2 *An element a in an upper-continuous lattice L is said to be compact if for each directed subset A of L for which $a \leq \bigvee_{x \in A} x$, there exists an element $x_0 \in A$ such that $a \leq x_0$. L is compact if the top element is compact in L, and L in compactly generated if each element of L is compact.Throughout of this note L will always denotes an upper-continuous and modular,lattice with the top element 1 and bottom element $0.(0 \neq 1)$. for all $a \leq b$*

$$[a, b] = \{x \in L : a \leq x \leq b\}$$

is called the factor of b by a. obviously $[a, b]$ is a sublattice of L and $\phi_b : x \to x \vee b$ is an isomorphism of $[a, a \vee b]$ onto $[a \wedge b, b]$. the inverse isomorphism is $\Psi_a : x \to x \vee a$.

A subset I of $L \setminus \{0\}$ is called join-independent if for any finite subset X of I and $x \in I \setminus X$. we have $\bigvee X \wedge x = (0)$, where $\bigvee X$ denotes the join of all elements of X.

Definition 2.3 *The Goldie dimension of a lattice L, denoted by $G-\dim L$ is defined to be the supremum of all cardinal numbers $\aleph$ such that there exists a set X of join-independent set of cardinality $\aleph$.*

Definition 2.4 *A nonzero element a in a lattice L is essential in L, if for any nonzero element $x \in L$, $a \wedge x \neq 0$.*

Definition 2.5 *An element b in a lattice L is a complement of $a \in L$, if $a \vee b = 1$ and $a \wedge b = 0$. Any b such that $a \wedge b = 0$ and b is maximal with this property is called a pseudo-complement of a in L. It is known that if b is a pseduo-complement of a, then $a \vee b$ is essential in L.*

Lemma 2.6 *Suppose L is a compact lattice, given $a \neq 0$ in L, then there exists a maximal element m such that $a \leq m$.*

Proof. see [3] section 1.5.8.

An element $a \in L$ is called an atom if $b \leq a$ implies $b = 0$ or $b = a$. The join of all atom of L which in denoted by $s(L)$ is called the socle of L and whenever 1 is a join of atoms, we call L semiatomic.

For the proof of the following lemma see [3] section 1.9.

Lemma 2.7 *If L is compactly generated, then $s(L)$ is the meet of all essential elements of L.*

Definition 2.8 *The Loewy chain of a lattice L is defined inductively, If $\alpha = 0$, $s_\alpha(L) = 0$, if $\alpha = 1$ $s_1(L) = L$. If $\alpha = \beta + 1$, put $s_\alpha = s[s_\beta(L), 1]$ and if α in a limit ordinal, we define $s_\alpha(L) = \bigvee_{\beta<\alpha} s_\beta(L)$. so we obtain*

$$0 = s_\alpha(L) \leq s_1(L) \leq \cdots \leq s_\alpha(L) < \cdots$$

since L is a set, there exists a least ordinal α such that

$$s_\alpha(L) = s_{\alpha+1}(L) = \cdots.$$

This ordinal is called the Loewy length of L and is denoted by $l(L)$. If $s_\alpha(L) = 1$, L is called a Loewy lattice or a semiartinian lattice.

3 Uem-lattices and ue-lattices

It is well-known and easy to prove that if every maximal right ideal of a ring R is a direct summand, then R is Artin semisimple. The next proposition is the lattice theoretical formulation of this result. First we recall that a complemented lattice is a one, such that every element has a complement and a semiatomic lattice is complemented see [3] section 1.8.2. The converse is true if L is compactly generated, see [3] section 1.8.7. Next we observe that for a compactly generated lattice to be complemented (semiatomic) it is enough that all its maximal elements have a complement. First we need the following easy lemma.

Lemma 3.1 *Let L be a compactly generated lattice, if $m \in L$ has a complement n, then m is maximal if and only if n is an atom.*

Proof. Since L is modular $[m, m \vee n]$ is lattice isomorphic to $[m \wedge n, n]$ i.e $[m.1] \cong [0, n]$ and we are through.

Proposition 3.2 *A compactly generated lattice L is complemented if and only if every maximal element has a complement.*

Proof. It is sufficient to show that $s(L) = 1$, If $s(L) \neq 1$ then $s(L) \leq m$ for some maximal element m of $L \setminus \{1\}$ If n is a complement of m, then by the above lemma n is an atom and in view of lemma 2.7, $n \leq s(L) \leq m$, which is absurd. Therefore $s(L) = 1$, and L is complemented.

Next we investigate lattices in which all maximal elements, except one has a complement.

Definition 3.3 *A lattice L is a uem-lattice if it has a unique essential maximal element different from 1.*

Proposition 3.4 *A compactly generated lattice is a uem-lattice if and only if $[s(L), 1]$ has only one maximal element in $L \setminus \{1\}$.*

Proof. If R has only one maximal element, then the statement is clear. Hence we assume that L has more than one maximal element and that L is uem-lattice. Let m be the non-essential maximal element of L, then clearly $s(L) \leq m$, and m is the only element with this property. Therefore $[s(L), 1]$ has only one maximal element. Conversely if $[s(L), 1]$ has only one maximal element, namely m, clearly m is essential in L. Now we claim that each maximal element of L different from m, has a complement. So let n be such a maximal element, then n is not essential as otherwise $n \geq s(L)$ which implies $n = m$, a contradiction. Therefore n has a complement.

Definition 3.5 *A lattice L is said to be a ue-lattice if it contains only one essential element different from 1.*

Proposition 3.6 *Let L be a compactly generated lattice, then the following are equivalent:*

1. *L is a ue-lattice,*
2. *$s(L)$ is a maximal element of $L - \{1\}$,*
3. *For any $x \in L$, either $[0, x]$ is semiatomic or x has a complement and L is not semiatomic.*

Proof.

(1) $\Rightarrow$ (2). In view of lemma 2.7 $s(L)$ is the unique essential element of L. let $s(L) \leq m$, then m is also an essential element, therefore $s(L)$ is a maximal element.

(2) $\Rightarrow$ (3). Let x be an element of L and y be a pesudue-complement of it. Then $x \vee y$ is essential in L, clearly if $x \vee y \neq 1$, then $x \vee y$ is the socle of L, i.e in this case $[0, x]$ is semiatomic, as every sublattice of a semiatomic is semiatomic, see [3] section 1.8.4. It is evident that L is not semiatomic.

(3) $\Rightarrow$ (1). Let e be an essential element of L then $[0, e]$ must be semiatomic, therefore e must be the socle.

Corollary 3.7 *Every compactly generated ue-lattice is a uem-lattice.*

Proof. Evident.

Recall that a lattice with 0 and 1 is of finite length if there exists a (Jordan-Holder) composition series between 0 and 1. It is well-known that a modular lattice with 0 and 1 is of finite length if and only if it is both Artinian and Noetherian.

Corollary 3.8 *A compactly generated ue-lattice with finite Goldie dimension has finite length.*

Proof. The lattice $[0, s(L)]$ is semiatomic and so there exists a join - independent set $\{s_i : i \in I\}$ consisting of atoms such that $s(L) = \bigvee_{i \in I} s_i$. Now the finiteness of the Goldie dimension of L implies that I is finite, now since $s(L)$ is maximal in $L \setminus \{1\}$, it follows that L has a composition series and we are through.

Corollary 3.9 *Every compactly generated ue-lattice has Loewy length equal to 2.*

Proof. Evident.

Proposition 3.10 *If L is a compactly generated ue-lattice then for any nonzero x in L, either $[x, 1]$ is semiatomic or $[x, 1]$ is a ue-lattice and $[0, x]$ is not semiatomic.*

Proof. If $[x, 1]$ is not semiatomic, then it has an essential element y, clearly y is essential in $[0, 1]$ i.e $y = s(L)$. this means that $[x, 1]$ has a unique essential element namely $s(L)$. Thus $[x, 1]$ is a ue-lattice and $[0, x]$ is semiatomic.

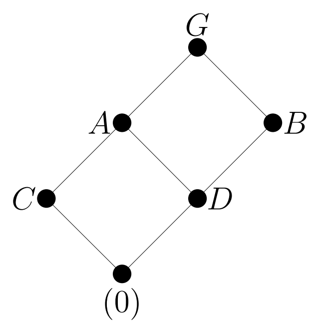
Theorem 3.11 *A compactly generated non semiatomic lattice L is a ue-lattice if and only if for each $x \in L$, either x has a complement y or $[0, x]$ is semiatomic. Moreover if $[0, x]$ is not semiatomic, it is a ue-lattice and $[0, y]$ is semiatomic.*

Proof. The first part is proposition 3.10. Now let $x \vee y = 1$ and $x \wedge y = 0$, where $[0, x]$ is not semiatomic clearly $[0, x]$ is a ue-lattice, for otherwise L contains more than one essential element, which is not possible. Now it remains to show that $[0, y]$ is semiatomic. To this end assume by way of contradiction that $y > s(L)$, then $y \vee s(L) = 1$. Let z be a complement of $y \wedge s(L)$ in $[0, s(L)]$, i.e $s(L) = (y \wedge s(L)) \vee z$ and $(y \wedge s(L)) \wedge z = 0$, thus $y \vee z = 1$ and $y \wedge z = 0$. this shows that $[y, 1] \cong [0, z] \cong [0, x]$ is semiatomic, a contraction and therefore $[0, y]$ must be semiatomic.

Example 3.12 .

1. A chain is a ue-lattice if and only if it consists only of 3 elements.

2. Let p and q be distinct prime numbers. and consider the cyclic group $G = \mathcal{Z}_{pq^2}$. Let A, B, C, D be respectively subgroups of G of orders, pq, q^2, p and q. The lattice of all subgroups of G is represented below,



A is the only essential element of this lattice different from G, and therefore the lattice of subgroups of $\mathcal{Z}_{pq^2}$ is a ue-lattice.

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