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Transcendental Continued Fractions over $\mathbb{K}_p(X)$

H. Benamar, A. Chandoul, M. Jellali and M. Mkaouar

Faculté des Sciences de Sfax, BP 802, Sfax 3018, Tunisie
helabenamare@yahoo.fr
amarachandoul@yahoo.fr
maneljellali@yahoo.fr
mohamed.mkaouar@fss.rnu.tn

Abstract

The purpose behind this work is to construct from a family of algebraic formal power series of degree more than 2, a family of transcendental fractions over $\mathbb{K}_p(X)$.

1 Introduction

Let $\mathbb{K}_p$ be a field of characteristic $p \geq 0$ and $\mathbb{K}_p((X^{-1}))$ be the field of formal power series,

$$\mathbb{K}_p((X^{-1})) = \{f = \sum_{n \geq n_0} f_n X^{-n} \ : \ f_n \in \mathbb{K}_p, \ n_0 \in \mathbf{Z}\}.$$

A formal power series $f = \sum_{n \geq n_0} f_n X^{-n}$ has a unique decomposition as $f = [f] + \{f\}$ where $[f]$ is the polynomial part of f and $\{f\}$ is the fractional part of f such that $[f]$ is a polynomial defined by $[f] = f_0 + f_{-1}X + \ldots + f_{n_0}X^{-n_0}$ if $n_0 \leq 0$, else $[f] = 0$. Let $\gamma(f) = n_0$ if $[f] \neq 0$, otherwise $\gamma(0) = +\infty$. We say that $\deg f = -\gamma(f)$. We define an absolute value on $\mathbb{K}_p((X^{-1}))$ by $\mid f \mid = \exp(-\gamma(f))$ for any f in $\mathbb{K}_p((X^{-1}))$. Let

$$M_p = \{f \in \mathbb{K}_p((X^{-1})) \ : \ \mid f \mid < 1\},$$

then for any f in M_p, we have

$$f = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [0\ ; a_1\ , a_2, \ldots],$$

where a_i is a polynomial in $\mathbb{K}_p[X]$ for any $i \geq 1$ of degree more than 1 and for any f in $\mathbb{K}_p((X^{-1}))$, we have

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0\ ; a_1\ , a_2, \ldots].$$

Let f in $\mathbb{K}_p((X^{-1}))$ such that $[a_0\ ; a_1\ , a_2, \ldots]$ is the continued fraction expansion of f, we define a sequence of best approximation $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ of f by requiring that $q_{-1} = p_{-2} = 0$, $q_{-2} = p_{-1} = 1$ and for any $n \in \mathbb{N}$, $q_n = a_n q_{n-1} + q_{n-2}$, $p_n = a_n p_{n-1} + p_{n-2}$. We can write $\frac{p_n}{q_n} = [a_0\ ; a_1\ , \ldots, a_n]$ and we refer to $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ as the convergent of f and to $(a_n)_{n\in\mathbb{N}}$ as the partial quotients.

Let f be an algebraic formal power series of minimal polynomial $P = \alpha_m Y^m + \ldots + \alpha_0$ where α_i are pairwise coprimes over $\mathbb{K}_p[X]$. Set $H(f) = \max\limits_{0\leq i\leq m} \mid \alpha_i \mid$ and $\sigma(f) = \alpha_m$. Let Λ be a polynomial in $\mathbb{K}_p[X][Y]$ such that

$$\Lambda(Y) = A_m Y^m + \ldots + A_0,$$

and $A_i \in \mathbb{K}_p[X]$ for any $i \in \{0, \ldots, m\}$. Λ is reduced if $\deg A_{m-1} > \deg A_i$ for any $i \neq m-1$. An algebraic formal power series is reduced if its minimal polynomial is reduced and $[f] \neq 0$.

In 1844, Liouville [4] was the first to prove that transcendental numbers existed. Moreover, he constructed explicit examples of such numbers. Liouville's theorem states that if x is algebraic of degree n then there exists a positive number c such that, for arbitrary integers p and q $(q > 0)$,

$$\mid x - \frac{p}{q} \mid > \frac{c}{\mid q \mid^n}.$$

In 1962, Baker [1] proved the following : {if $x = [B_0\ , B_1\ , B_2, \ldots]$ where B_n is a block of k_n consecutive partial quotients and if there are infinetely many n for which

$$B_n = B_{n+1} = \ldots = B_{n+\lambda(n)-1},$$

where $\lambda(n)$ is a sequence of integers verifying certain increasing properties, then x is transcendental}. The proof of this result has been based on Liouville's and Roth's theorems.

After this, in 1967, Shmidt [9] stipulated the following : {if a positif irrational number is approximate by quadratic numbers, it is either quadratic or transcendental}. In fact, it was used in most theorem's proofs.

In [3], the transcendence of continued fractions which have repetition properties has been deduced.

It was proved in [7] that the real number whose continued fraction expansion in the {1 - 2} valued Thue-Morse sequence is a transcendental number. Moreover, in [5] it improved Baker's results using Shmidt's theorem.

We alluded that Liouville's theorem had an analogous in the field of formal power series. But Roth's and Shmidt's theorems does not have an equivalent in the case of formal power series.

In [2], Baum and Sweet were the first to give an example of algebraic formal power series of degree 3 that had a bounded continued fraction expansion. This, remained an open problem in the real case. This work was pursued in [8] by Mills and Robbins who gave an example of formal power series over $\mathbb{K}_p(X)$ where its continued fractions expansion was explicitly given.

In this note, we improve the theorem published in [6], which was concerned by constructing a family of transcendental continued fractions over $\mathbb{K}_p(X)$ from an algebraic formal power series of degree more than 2.

2 Results

Theorem 1 *Let $\{g_j\}_{1\leq j\leq k}$ be a family of algebraic formal power series for any integer $j \in \{1,\ldots,k\}$ such that $g_j = [a_1^{(j)},\ a_2^{(j)},\ldots,a_n^{(j)},\ldots]$ and $f = [B_1, B_2, \ldots]$ where*

$$B_i = [a_1^{(1)},\ldots,a_{n_{i,1}}^{(1)},a_1^{(2)},\ldots,a_{n_{i,2}}^{(2)},\ldots,a_1^{(k)},\ldots,a_{n_{i,k}}^{(k)},\ldots].$$

We denote by $\delta_{i,j}$ the sum of degrees of the first $n_{i,j}$-terms of g_j, d_i the sum of degrees of B_i's terms and $M_i = \sup\limits_{1\leq j\leq k} \delta_{i,j}$.

If $\liminf\limits_{i\to+\infty} \dfrac{\sum_{l=1}^{i-1} d_l}{M_i} = 0$, *then f is transcendental or quadratic.*

Theorem 2 *Let f and g be two algebraic formal power series of degrees d and m respectively which having the same first s-terms of their sequences of partial quotients, let δ_s be the sum of degrees of these terms. If $\mid f \mid > 1$ and g is reduced then*

$$2\delta_s \leq m\log H(f) + d\log\mid g\mid + md\log\mid \sigma(g)\mid.$$

We need the following lemmas to prove these theorems.

Lemma 1 *Let f be an algebraic formal power series of degree d and P its minimal polynomial. We denote by $f_1, f_2, \ldots, f_{d-1}$ the conjugates of f in the algebraic closure of $\mathbb{K}_p((X^{-1}))$. Then f is reduced if and only if $\mid f \mid > 1$ and $\mid f_i \mid < 1$ for any $i \in \{1, 2, \ldots, d-1\}$.*

Proof. Suppose that f is a reduced formal power series, and $P(Y) = A_d Y^d + \ldots + A_0$ such that $A_i \in \mathbb{K}_p[X]$ for any $i \in \{0, 1, \ldots, d\}$. Let $Y = \lambda Z$ such that $\lambda = -\frac{A_{d-1}}{A_d}$. We have

$$\frac{-1}{A_{d-1}\lambda^{d-1}} P(\lambda Z) = Z^{d-1}(Z-1) + L(Z),$$

where $L(Z) = -\sum_{j=0}^{d-2} \frac{A_j}{\lambda^{d-j-1} A_{d-1}} Z^j$. We remark that the absolute value of the coefficient of $L(Z)$ is strictly less than 1. By Hensel's lemma [10], we state that $P(Y)$ has a unique root f in $\mathbb{K}_p((X^{-1}))$ such that $f = \lambda Z$ where $\mid Z - 1 \mid < 1$. Moreover, for this Z we have $\mid Z \mid = 1$, and since the coefficient's absolute value of $L(Z)$ is strictly less than $\frac{1}{\lambda}$, then $\mid Z - 1 \mid = \mid L(Z) \mid < \frac{1}{\lambda}$. Thus the unique root f of P verify $\mid f - \lambda \mid < 1$ then $\mid f \mid = \mid \lambda \mid$. The other roots Z_i of the polynomial $P(\lambda Z)$ in the algebraic closure over $\mathbb{K}_p((X^{-1}))$ satisfy $\mid Z_i \mid < 1$, then the conjugate $f_i = \lambda Z_i$ of f satisfy $\mid f_i \mid < \mid \lambda \mid$, it implies that $\mid f_i \mid < 1$. In fact, if $1 \leq \mid f_i \mid < \mid \lambda \mid$, we obtain for any $k \in \{0, 1, \ldots, d-1\}$, $\mid A_k \mid\mid f_i \mid^k < \mid A_{d-1} \mid\mid f \mid^{d-1}$ and we have $\mid A_d \mid\mid f_i \mid^d < \mid A_{d-1} \mid\mid f_i \mid^{d-1}$, then $\mid P(f_i) \mid = \mid A_{d-1} \mid\mid f_i \mid^{d-1}$. Thus it leads to a contradiction with $P(f_i) = 0$. Conversely, if $P(Y)$ has a simple root f such that $\mid f \mid > 1$ and the others f_i of P are simple and verify $\mid f_i \mid < 1$. Then, in this case, we have f reduced since P is unitary, then the coefficients of f are symmetrical polynomials in f, f_i. ♢

Lemma 2 *Let $Q \in \mathbb{K}_p[X][Y]$ and $F(Y_1, Y_2, \ldots, Y_n) = Q(Y_1) \ldots Q(Y_n)$. Then, there exists a polynomial T with coefficients over $\mathbb{K}_p[X]$ such that $F(Y_1, Y_2, \ldots, Y_n) = T(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where*

$$\begin{cases} \sigma_1 = \sum_{1 \leq i \leq n} Y_i \\ \sigma_2 = \sum_{1 \leq i < j \leq n} Y_i Y_j \\ \sigma_3 = \sum_{1 \leq i < j < k \leq n} Y_i Y_j Y_k \\ \vdots \\ \sigma_n = Y_1 Y_2 \ldots Y_n. \end{cases}$$

Moreover, the degree of T is less or equal to the degree of Q (as a polynomial in Y).

Proof. Let $\alpha_1 = \deg(Q)$. From the terms containing $Y_1^{\alpha_1}$, we denote by α_2 the highest exponent of Y_2, from the terms containing $Y_1^{\alpha_1}Y_2^{\alpha_2}$ we denote by α_3 the highest exponent of $Y_3 \ldots$ Then we define a dominant term such as $AY_1^{\alpha_1}Y_2^{\alpha_2}\ldots Y_n^{\alpha_n}$. As F is symmetrical, we have $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ (in fact, F contains all terms $AY_{\pi(1)}^{\alpha_1}Y_{\pi(2)}^{\alpha_2}\ldots Y_{\pi(n)}^{\alpha_n}$ where π is a permutation of $\{1, 2, \ldots, n\}$). Then, we remark that the dominant term of $\sigma_1^{\alpha_1-\alpha_2}\ldots\sigma_{n-1}^{\alpha_{n-1}-\alpha_n}\sigma_n^{\alpha_n}$ is $Y_1^{\alpha_1}Y_2^{\alpha_2}\ldots Y_n^{\alpha_n}$. Calculating $F(Y_1, \ldots, Y_n) - A\sigma_1^{\alpha_1-\alpha_2}\ldots\sigma_{n-1}^{\alpha_{n-1}-\alpha_n}\sigma_n^{\alpha_n}$, the lemma can be concluded by induction. ♢

Lemma 3 *Let f and g be two algebraic formal power series of degrees d and m respectively. If g is reduced, then*

$$\mid f - g \mid \geq \frac{1}{H(f)^m \mid g \mid^{d-2} \mid \sigma(g) \mid^{\max(m-1, m(d-m+2)+1)}}.$$

Proof. As g is algebraic of degree m then

$$\sigma(g)^{k+1}g_i^{m+k} = A_{m-1,k}g_i^{m-1} + \ldots + A_{0,k},$$

for any integer k and i such that $i \in \{1, \ldots, m\}$ and $A_{i,k} \in \mathbb{K}_p[X]$, where $g_1 = g, g_2, \ldots, g_m$ are the conjugates of g. We state for any $i \in \{1, \ldots, m\}$ that

$$\sigma(g)^{\max(0,d-m+1)}P(g_i) = B_{m-1}g_i^{m-1} + \ldots + B_0,$$

such that $B_j \in \mathbb{K}_p[X]$ for any $j \in \{0, \ldots, m-1\}$ and P the minimal polynomial of f. Consequently,

$$\sigma(g)^{\max(0,m(d-m+1))}\prod_{i=1}^{m} P(g_i) = \prod_{i=1}^{m}(B_{m-1}g_i^{m-1} + \ldots + B_0).$$

Then, lemma 2 implies that $\prod_{i=1}^{m}(B_{m-1}g_i^{m-1} + \ldots + B_0) = T(\sigma_1, \ldots, \sigma_m)$, where for any $i \in \{1, \ldots, m\}$, $Q_i \in \mathbb{K}_p[X]$ and $\sigma_i = \sum_{1\leq k_1<\ldots<k_i\leq m} g_{k_1}g_{k_i} = \dfrac{Q_i}{\sigma(g)}$. Thus,

$$\sigma(g)^{\max(0,m(d-m+1))}\prod_{i=1}^{m} P(g_i)\sigma(g)^{m-1} \in \mathbb{K}_p[X],$$

and

$$|\prod_{i=1}^{m} P(g_i)| \geq \frac{1}{|\sigma(g)|^{\max(m-1,m(d-m+2)-1)}}.$$

As g is reduced, then using lemma 1, we get $|g_i| < 1$ and $|P(g_i)| < H(f)$ for any $i \in \{2,\ldots,m\}$. Whence

$$|P(g)| \geq \frac{1}{|\sigma(g)|^{\max(m-1,m(d-m+2)-1)} H(f)^{m-1}}.$$

If $P(Y) = A_d Y^d + \ldots + A_0$ such that $A_i \in \mathbb{K}_p[X]$ for any $i \in \{0,\ldots,d\}$, then

$$|A_d f^d| = |A_{d-1} f^{d-1} + \ldots + A_0| \text{ and } |A_d f^{d-1}| \leq H(f) |f|^{d-2}.$$

On the other hand, we can suppose that $|f-g| < |g|$, it implies that $|f| = |g|$. Consequently,

$$|P(g)| = |P(f) - P(g)| \leq H(f) |g|^{d-2} |f-g|,$$

and

$$|f-g| \geq \frac{P(g)}{H(f) |g|^{d-2}}.$$

Finally,

$$|f-g| \geq \frac{1}{H(f)^m |g|^{d-2} |\sigma(g)|^{\max(m-1,m(d-m+2)-1)}},$$

and lemma 1 is proved. ♢

Lemma 4 *Let f be an algebraic formal power series such that $f = [a_1,\ldots,a_i,\ldots]$, then there exists a formal power series g which is reduced such that $f = [a_1,\ldots,a_n,g]$.*

Proof. Let $g = -\frac{Q_{n-1}f - P_{n-1}}{Q_n f - P_n}$, where $(\frac{P_n}{Q_n})_{n\in\mathbb{N}}$ is the sequence of convergent in the continued fraction expansion of f. Since for any integer n, $|Q_n f - P_n| < |Q_{n-1}f - P_{n-1}|$, then $|g| > 1$. It clearly appears that $f = [a_1,\ldots,a_n,g]$, we can conclude that g is algebraic. On the other hand, the conjugate of g satisfy $|g_i| < 1$ for n sufficiently large. Therefore from lemma 1 g is reduced . ♢

Lemma 5 *Let f be an algebraic formal power series of degree d such that $f = [a_1,\ldots,a_t,h]$ where $a_1,\ldots,a_t \in \mathbb{K}_p[X]$, $h \in \mathbb{K}_p((X^{-1}))$. If $|f| \geq 1$ and $|h| > 1$ then h is algebraic of degree d and $H(h) \leq H(f) |\prod_{i=1}^{t} a_i|^{d-2}$.*

Proof. Let us begin by the particular case where $f = \alpha + \frac{1}{h}$, $\alpha \in \mathbb{K}_p[X]$ and $\mid h \mid > 1$. We have $\mid \alpha \mid \geq 1$, then $\mid f \mid = \mid \alpha \mid$. If $A_d f^d + \ldots + A_0 = \sum_{j=0}^{d} A_j f^j = 0$, then

$$h^d \sum_{j=0}^{d} A_j (\alpha + \frac{1}{h})^j = 0,$$

we get $\sum_{k=0}^{d} B_{d-k} h^{d-k} = 0$, where $B_{d-k} = \sum_{j=k}^{d} C_j^k A_j \alpha^{j-k}$. It clearly appears that

$$\mid B_{d-1} \mid \leq \max(\mid A_d \mid\mid \alpha \mid^{d-1},\ H(f) \mid f \mid^{d-2}),$$

and for $k \geq 2$, we have $\mid B_{d-k} \mid \leq H(f) \mid f \mid^{d-2}$. Since $A_d f^d + \ldots + A_0 = 0$, we get

$$\mid A_d f^d \mid = \mid A_{d-1} f^{d-1} + \ldots + A_0 \mid \leq H(f) \mid f \mid^{d-1},$$

then $\mid A_d \mid\mid \alpha \mid^{d-1} \leq H(f) \mid f \mid^{d-2}$. Hence we get $\mid B_{d-1} \mid \leq H(f) \mid f \mid^{d-2}$. Finally,

$$B_d = A_d \alpha^d + \ldots + A_0 = A_d(\alpha^d - f^d) + \ldots + A_1(\alpha - f),$$

and we have $\mid B_d \mid = \max_{1 \leq j \leq d} (\mid A_j \mid\mid f \mid^{j-1}) \leq H(f) \mid f \mid^{d-2}$. We conclude that $H(h) \leq H(f) \mid f \mid^{d-2}$. In the general case, if $f = [a_1, \ldots, a_t, h]$ and $f_i = [a_i, \ldots, a_t, h]$ for any $i \in \{1, \ldots, t\}$ then we deduce the result by iterating the particular case. ♢

Proof of theorem 2. Let $f = [b_1, \ldots, b_s, \ldots]$ and $\frac{P_k}{Q_k}$ its k-th convergent . As f and g have the same first s-terms of their continued fraction expansion, then

$$\mid f - g \mid \leq \frac{1}{\mid Q_{s-1} \mid^2}.$$

Thus, it follows from theorem 2 that

$$\mid Q_{s-1} \mid^2 \leq H(f)^m \mid g \mid^{d-2} \mid \sigma(g) \mid^{\max(m-1, m(d-m+2)+1)}.$$

Since $\deg Q_{s-1} = \sum_{j=1}^{s-1} \deg b_j = \delta_s - \log \mid g \mid$ then, after taking logarithms, we obtain $2\delta_s - 2 \deg g \leq m \log H(f) + (d-2) \deg g + md \log \mid \sigma(g) \mid$. ♢

Proof of theorem 1. We may assume that f is algebraic not quadratic of degree d.

Case 1. For any $j \in \{1, \ldots, k\}$, g_j is reduced.
Let $B_{i,j} = a_1^{(j)}, \ldots, a_{n_{i,j}}^{(j)}, \ldots, a_1^{(k)}, \ldots, a_{n_{i,k}}^{(k)}, C_i$ and $f_{i,j} = [B_{i,j}, B_{i+1}, \ldots]$, so, there exists $i_0 \in \mathbb{N}$ such that, for any $i \geq i_0$, $f_{i,j} \neq g_i$. (If not, there exists two integers i and l $(i \leq l)$ such that $f_{i,j} = f_{l,j} = g_j$, then

$$\begin{aligned} g_j = & \ [B_{i,j}, B_{i+1}, \ldots, B_{l-1}, a_1^{(1)}, \ldots, a_{n_{l,1}}^{(1)}, \ldots, a_1^{(j-1)}, \ldots, a_{n_{l,j-1}}^{(j-1)}, f_{l,j} \ ], \\ = & \ [\overline{B_{i,j}, B_{i+1}, \ldots, B_{l-1}, a_1^{(1)}, \ldots, a_{n_{l,1}}^{(1)}, \ldots, a_1^{(j-1)}, \ldots, a_{n_{l,j-1}}^{(j-1)}} \ ], \end{aligned}$$

it implies that g_j is quadratic and f too. Such a fact is absurd).
Since

$$f = [B_1, \ldots, B_{i-1}, a_1^{(1)}, \ldots, a_{n_{i,1}}^{(1)}, \ldots, a_1^{(j-1)}, \ldots, a_{n_{i,j-1}}^{(j-1)}, f_{i,j}],$$

it follows from lemma 5 that $f_{i,j}$ is algebraic of degree d and

$$\log H(f_{i,j}) \leq \log H(f) + (d-2) \sum_{l=1}^{i-1} d_l + (d-2) \sum_{l=1}^{j-1} \delta_{i,l}.$$

As g_j and $f_{i,j}$ have the same first $n_{i,j}$-terms, it follows from theorem 2 that

$$2\delta_{i,j} \leq m_j \log H(f_{i,j}) + d \log |\ g_j\ | + m_j\ d \log |\ \sigma(g_j)\ |,$$

with $\delta_{i,j}$ as the sum of degrees of the first $n_{i,j}$ terms of g_j and m_j as the algebricity's degree of g_j, it implies that

$$2\delta_{i,j} \leq \alpha \log H(f_{i,j}) + \beta,$$

where $\alpha = \max\limits_{1 \leq j \leq k} m_j$ and $\beta = \max\limits_{1 \leq j \leq k} (\log |\ g_j\ | + \alpha \log |\ \sigma(g_j)\ |) d$, then

$$2\delta_{i,j} \leq \alpha \log H(f) + \alpha(d-2) \sum_{l=1}^{i-1} d_l + \alpha(d-2) \sum_{l=1}^{j-1} \delta_{i,l} + \beta.$$

Since $1 \leq j \leq k$, $\delta_{i,l} < d_l$ and $M_i = \sup\limits_{1 \leq j \leq k} \delta_{i,j}$, then

$$2M_i \leq \alpha \log H(f) + \alpha(d-2) \sum_{l=1}^{i-1} d_l + \alpha(d-2) \sum_{l=1}^{k-1} d_l + \beta,$$

and $\liminf\limits_{i \to +\infty} \frac{\sum_{l=1}^{i-1} d_l}{M_i} \geq 2$, which contradicts the fact that $\liminf\limits_{i \to +\infty} \frac{\sum_{l=1}^{i-1} d_l}{M_i} = 0$.

Case 2. We assume that there exists $j \in \{1, \ldots, k\}$ such that g_j is not reduced.

By lemma 4, there exists h reduced such that $g_j = [a_1, \ldots, a_t, h]$. For any $i \in \{1, \ldots, k\}$, we have $B_{i,j} = a_1, \ldots, a_t, b_1, \ldots, b_{n_{i,j}-t}, A_{i,j}$, therefore

$$f_{i,j} =$$

$$[a_1, \ldots, a_t, b_1, \ldots, b_{n_{i,j}-t}, A_{i,j}, a_1^{(1)}, \ldots, a_{n_{i+1,1}}^{(1)}, \ldots, a_1^{(j-1)}, \ldots, a_{n_{i+1,j-1}}^{(j-1)}, a_1, \ldots, a_t, \ldots].$$

Then $f_{i,j} = [a_1, \ldots, a_t, C_{i,j}, C_{i+1,j}, \ldots]$, where

$$C_{i,j} = b_1, \ldots, b_{n_{i,j}-t}, A_{i,j}, a_1^{(1)}, \ldots, a_{n_{i+1,1}}^{(1)}, \ldots, a_1^{(j-1)}, \ldots, a_{n_{i+1,j-1}}^{(j-1)}, a_1, \ldots, a_t.$$

The sum of $C_{i,j}$'s degrees is equal to $d_i + \sum_{l=1}^{j-1} (\delta_{i+1,l} - \delta_{i,l})$. h and $C_{i,j}$ have the same first $(n_{i,j} - t)$ terms, which sum of degrees is equal to $\delta_{i,j} - \alpha$, where $\alpha = \sum_{l=1}^{j} \deg(a_l)$. From the first case, we deduce that

$$\liminf_{i \to +\infty} \frac{\sum_{t=1}^{i-1} (d_t + \sum_{l=1}^{j-1} (\delta_{t+1,l} - \delta_{t,l}))}{\sup_{1 \le l \le k} (\delta_{i,l} - \alpha)} \neq 0,$$

hence

$$\liminf_{i \to +\infty} \frac{\sum_{t=1}^{i-1} (d_t + \sum_{l=1}^{j-1} (\delta_{t+1,l} - \delta_{t,l}))}{M_i} \neq 0.$$

- If $\sum_{l=1}^{j-1} (\delta_{t+1,l} - \delta_{t,l}) \geq 0$ then

$$\liminf_{i \to +\infty} \frac{\sum_{t=1}^{i-1} d_t}{M_i} \neq 0.$$

- If $\sum_{l=1}^{j-1} (\delta_{t+1,l} - \delta_{t,l}) < 0$ then

$$\liminf_{i \to +\infty} \frac{\sum_{t=1}^{i-1} (d_t + \sum_{l=1}^{j-1} (\delta_{t+1,l} - \delta_{t,l}))}{M_i} < \liminf_{i \to +\infty} \frac{\sum_{t=1}^{i-1} d_t}{M_i} \text{ and } \liminf_{i \to +\infty} \frac{\sum_{t=1}^{i-1} d_t}{M_i} \neq 0.$$

In both cases, we contradict with the hypothesis and we conclude that f is transcendental or quadratic. $\diamondsuit$

Example 1 : Let g_1 and g_2 be two formal power series over $\mathbb{K}_2((X^{-1}))$ such that $g_1 = [\overline{X}]$, $g_2 = [\overline{X+1}]$ and $f = [B_1,\ B_2, \ldots, B_n, \ldots]$ where $B_k = U_k V_k$ such that $U_k = [X, \ldots, X]$ of length $k+1$, $V_k = [X+1, \ldots, X+1]$ of length $(k+1)^{k+1}$ quotients. For any $k \geq 1$, B_k and g_1 have the same first $(k+1)$ terms. Let us have d_l as the sum of degrees of B_l's terms, $\delta_{l,1}$ as the sum of degrees of the first $\mid U_l \mid$ terms of g_1 and $\delta_{l,2}$ as the sum of degrees of the first $\mid V_l \mid$ terms of g_2. We will have the following :

$$\liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} d_l}{\delta_{i,1}} = \liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} (l+1) + \sum_{l=1}^{i-1} (l+1)^{l+1}}{i+1} \neq 0,$$

then we can't use the theorem in [6]. However,

$$\liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} d_l}{\sup(\delta_{i,1}, \delta_{i,2})} = \liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} (l+1) + \sum_{l=1}^{i-1} (l+1)^{l+1}}{(i+1)^{i+1}} = 0,$$

it follows from theorem 1 that f is transcendental.

Example 2 : In this example we will treat the case where g_1 and g_2 are two algebraic formal power series over $\mathbb{K}_5((X^{-1}))$.
Let $g_1 = [aX,\ L_0(c),\ bX,\ L_0(-1),\ aX,\ L_1(c),\ bX,\ L_1(-1), \ldots]$ where $L_k(\delta)$ is the following sequence

$$\frac{X}{\delta},\ \delta X,\ \frac{X}{\delta}, \ldots, \frac{X}{\delta},\ \delta X,$$

of length $5^k - 1$, $L_0(\delta)$ is the empty sequence and $g_2 = [\overline{X}]$. it is clear that g_2 is quadratic and refers to [8], g_1 is cubic. Now, we define for any $k \geq 1$

$$U_k = [aX,\ L_0(c),\ bX,\ L_0(-1),\ aX,\ L_1(c),\ bX,\ L_1(-1), \ldots, aX,\ L_k(c),\ bX,\ L_k(-1)],$$

$V_k = [X, \ldots, X]$ of length $(k+4)^{k+1}$, $B_k = U_k V_k$ and $f = [B_1, B_2, \ldots]$. For any $k \geq 1$, B_k and g_1 have the same first $\mid U_k \mid$ terms where $\mid U_k \mid = \dfrac{5^{k+1}-1}{2}$. Let us have d_l as the sum of degrees of B_l's terms, $\delta_{l,1}$ as the sum of degrees of the first $\mid U_l \mid$ terms of g_1 and $\delta_{l,2}$ as the sum of degrees of the first $\mid V_l \mid$ terms of g_2. We will have the following :

$$\liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} d_l}{\sup(\delta_{i,1}, \delta_{i,2})} = \liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} d_l}{\delta_{i,2}} = \liminf_{i \to +\infty} \frac{\sum_{l=1}^{i-1} \frac{5^{l+1}-1}{2} + (l+4)^{l+1}}{(i+4)^{i+1}} = 0.$$

Referring to theorem 1, we can conclude that f is transcendental.

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