Pre-Strongly Solid and Left-Edge(Right-Edge)-Strongly Solid Varieties of Semigroups

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Abstract

Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language do not necessarily preserve the arities. Strong hyperidentities are identities which are closed under these generalized hypersubstitutions and a strongly solid variety is a variety which every its identity is a strong hyperidentity. In this paper we consider $M$-strongly solid varieties for some submonoids $M$ of the monoid of all generalized hypersubstitutions and we also characterize all pre-strongly solid varieties of semigroups as well as the least and the greatest left-edge(right-edge)-strongly solid varieties of semigroups.

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1 Introduction

The concept of generalized hypersubstitutions, strong hyperidentities and strongly solid varieties were introduced by S. Leeratanavalee and K. Denecke in [6]. Let $\{f_i \mid i \in I\}$ be an indexed set of operation symbols of type $\tau$ where $f_i$ is $n_i$-ary, $n_i \in \mathbb{N}$, and let $W_\tau(X)$ be the set of all terms built up by elements

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of the alphabet \(X\) and operation symbols from \(\{f_i \mid i \in I\}\). Generalized hypersubstitutions are mappings \(\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)\) which do not necessarily preserve the arities. To define the extension \(\hat{\sigma}\) of \(\sigma\) to a mapping defined on terms we defined inductively the concept of superposition of terms \(S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X)\) by the following steps:

(i) If \(t = x_i, 1 \leq i \leq m\), then
\[
S^m(x_i, t_1, \ldots, t_m) := t_i \text{ where } t_1, \ldots, t_m \in W_\tau(X).
\]

(ii) If \(t = x_i, m < i \in \mathbb{N}\), then
\[
S^m(x_i, t_1, \ldots, t_m) := x_i.
\]

(iii) If \(t = f_i(s_1, \ldots, s_n)\), then
\[
S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_n, t_1, \ldots, t_m)).
\]

Then the generalized hypersubstitution \(\sigma\) will be extended to a mapping \(\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)\) by the following steps:

(i) \(\hat{\sigma}[x_i] := x_i \in X\),

(ii) \(\hat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).\)

Let \(Hyp_G(\tau)\) be the set of all generalized hypersubstitutions of type \(\tau\) and let \(Hyp(\tau)\) be the set of all usual hypersubstitutions of type \(\tau\). We define a binary operation \(\circ_G\) on \(Hyp_G(\tau)\) by \(\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \hat{\sigma}_2\) where \(\circ\) denotes the usual composition of mappings and \(\sigma_1, \sigma_2 \in Hyp_G(\tau)\). Let \(\sigma_{id}\) be the identity hypersubstitution mapping which maps each \(n_i\)-ary operation symbol \(f_i\) to the term \(f_i(x_1, \ldots, x_{n_i})\). Then we have:

**Proposition 1.1** (\([6]\)) For arbitrary terms \(t, t_1, \ldots, t_n \in W_\tau(X)\) and for arbitrary generalized hypersubstitutions \(\sigma, \sigma_1, \sigma_2\) we have

(i) \(S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]\),

(ii) \((\hat{\sigma}_1 \circ \hat{\sigma}_2) = \hat{\sigma}_1 \circ \hat{\sigma}_2\).

Now, we are able to prove:

**Theorem 1.2** (\([6]\)) \(Hyp_G(\tau) = (Hyp_G(\tau); \circ_G, \sigma_{id})\) is a monoid and the monoid \(Hyp(\tau) = (Hyp(\tau); \circ_h, \sigma_{id})\) of all arity-preserving hypersubstitutions of type \(\tau\) forms a submonoid of \(Hyp_G(\tau)\).
Varieties of semigroups

Identities which are closed under all generalized hypersubstitutions from $Hyp_G(\tau)$ are called strong hyperidentities and a variety is called strongly solid if every identity in it is a strong hyperidentity. If $M$ is a submonoid of $Hyp_G(\tau)$ then $s \approx t$ is called an $M$-strong hyperidentity if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities for every $\sigma \in M$. We denote the sets of all identities, strong hyperidentities and $M$-strong hyperidentities satisfied in the variety $V$ by $Id_V$, $HSId_V$ and $H_MGId_V$, respectively. A variety $V$ is called $M$-strongly solid if every identity satisfied in $V$ is an $M$-strong hyperidentity. In this paper we want to study $M$-strongly solid varieties of semigroups for different monoids $M$ of generalized hypersubstitutions. For more background of generalized hypersubstitutions and strongly solid varieties see [6].

2 V-proper generalized hypersubstitutions and normal forms

Let $V$ be a variety of type $\tau$ and $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions. To test whether an identity $s \approx t$ of $V$ is a strong hyperidentity of $V$, our definition requires to check that for each generalized hypersubstitution $\sigma$ in $Hyp_G(\tau)$, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of $V$. In this section we restrict our testing to certain “special” generalized hypersubstitutions $\sigma$, those which correspond to normal form terms. Now we make this precise generalizing the concept of a $V$-proper hypersubstitution introduced by J. P/ónka in [8] and the concept of a normal form hypersubstitution introduced by Sr. Arworn and K. Denecke in [1] to $V$-proper generalized hypersubstitutions and to normal forms of generalized hypersubstitutions.

Definition 2.1 Let $V$ be a variety of type $\tau$. A generalized hypersubstitution $\sigma$ of type $\tau$ is called a $V$-proper generalized hypersubstitution if for every identity $s \approx t$ of $V$, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in $V$. We use $P_G(V)$ for the set of all $V$-proper generalized hypersubstitutions of type $\tau$.

Proposition 2.2 For any variety $V$ of type $\tau$, $(P_G(V); \circ_G, \sigma_{id})$ is a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

Proof. Let $\sigma_1, \sigma_2 \in P_G(V)$ and let $s \approx t \in IdV$. Consider

$$(\sigma_1 \circ_G \sigma_2)[s] = \hat{\sigma}_1[\hat{\sigma}_2[s]] = \hat{\sigma}_1[\hat{\sigma}_2[t]] = (\sigma_1 \circ_G \sigma_2)[t].$$

So $(P_G(V); \circ_G, \sigma_{id})$ is closed and clearly that $\sigma_{id} \in P_G(V)$. Hence $(P_G(V); \circ_G, \sigma_{id})$ is a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$. 

\[\blacksquare\]
Definition 2.3 Let $V$ be a variety of type $\tau$. Two generalized hypersubstitutions $\sigma_1$ and $\sigma_2$ of type $\tau$ are called $V$-generalized equivalent if $\sigma_1(f_i) \approx \sigma_2(f_i)$ are identities in $V$ for all $i \in I$. In this case we write $\sigma_1 \sim_{V_G} \sigma_2$.

Theorem 2.4 Let $V$ be a variety of type $\tau$, and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following are equivalent:

(i) $\sigma_1 \sim_{V_G} \sigma_2$.

(ii) For all $t \in W_\tau(X)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity in $V$.

(iii) For all $A \in V, \sigma_1[A] = \sigma_2[A]$ where $\sigma_k[A] = (A; (\sigma_k(f_i))^A)_{i \in I}; k = 1, 2$.

Proof. (i)⇒(ii): We give the proof by induction on the complexity of terms. If $t = x_i, i \in \mathbb{N}$, then $\hat{\sigma}_1[x_i] = x_i = \hat{\sigma}_2[x_i]$. If $t = f_i(t_1, \ldots, t_n)$, and assume that $\hat{\sigma}_1[t_j] = \hat{\sigma}_2[t_j] \in \mathcal{I}d_V$ for all $j \in \{1, 2, \ldots, n_i\}$, then $\hat{\sigma}_1[t] = S^{n_i}(\sigma_1(f_i), \hat{\sigma}_1[t_1], \ldots, \hat{\sigma}_1[t_n]) \approx S^{n_i}(\sigma_2(f_i), \hat{\sigma}_2[t_1], \ldots, \hat{\sigma}_2[t_n]) = \hat{\sigma}_2[t]$.

(ii)⇒(iii): Let $A \in V$, let $i \in I$ and let $t = f_i(t_1, \ldots, t_n)$, From (ii) we get $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity in $V$. Then $A \models \hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$. Thus $\sigma_1(f_i)^A = \sigma_2(f_i)^A$ for all $i \in I$. Therefore $\sigma_1[A] = \sigma_2[A]$.

(iii)⇒(i): Let $A \in V$ and $\sigma_1[A] = \sigma_2[A]$. Then $\sigma_1(f_i)^A = \sigma_2(f_i)^A$ for all $i \in I$. Thus $\sigma_1(f_i) \approx \sigma_2(f_i) \in \mathcal{I}d_V$. So $\sigma_1 \sim_{V_G} \sigma_2$.

Proposition 2.5 Let $V$ be a variety of type $\tau$. Then the following hold:

(i) For all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{V_G} \sigma_2$, then $\sigma_1$ is a $V$-proper generalized hypersubstitution iff $\sigma_2$ is a $V$-proper generalized hypersubstitution.

(ii) For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{V_G} \sigma_2$, then the equation $\hat{\sigma}_1[s] \approx \hat{\sigma}_2[t]$ is an identity in $V$ iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in $V$.

Proof. (i) Let $\sigma_1$ be a $V$-proper generalized hypersubstitution. Then for all identities $s \approx t$ in $\mathcal{I}d_V$ the equation $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in $V$. Since $\sigma_1 \sim_{V_G} \sigma_2$, we have $V \models \hat{\sigma}_2[s] \approx \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$. Thus $\sigma_2$ is a $V$-proper generalized hypersubstitution. The other direction can be proved in the same way.

(ii) Suppose that $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in $V$. Since $\sigma_1 \sim_{V_G} \sigma_2$, the equations $\hat{\sigma}_1[s] \approx \hat{\sigma}_2[s]$ and $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in $V$ by Theorem 2.4. Thus $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in $V$. The converse follows in the same way.
The relation \(\sim_{VG}\) is an equivalence relation on \(hyp_G(\tau)\), but it is not necessary a congruence. We factorize \(hyp_G(\tau)\) by \(\sim_{VG}\) and consider the submonoid \(P_G(V)\) of \(hyp_G(\tau)\) is the union of equivalence classes of the relation \(\sim_{VG}\). This is also true for a submonoid \(M\) of \(hyp_G(\tau)\) and the relation \(\sim_{VG_M}\).

**Lemma 2.6** Let \(M\) be a submonoid of \(hyp_G(\tau)\) and let \(V\) be a variety of type \(\tau\). Then the monoid \(P_G(V) \cap M\) is the union of all equivalence classes of the restricted relation \(\sim_{VG_M}\).

**Proof.** Let \(\sigma\) be a generalized hypersubstitution in \(P_G(V) \cap M\), and let \(\rho \in M\) satisfies \(\sigma \sim_{VG_M} \rho\). To see that \(\rho\) is also in \(P_G(V)\), we let \(s \approx t\) be any identity of \(V\). Since \(\sigma \sim_{VG} \rho\) we have \(\tilde{\sigma}[s] \approx \tilde{\rho}[s]\) and \(\tilde{\sigma}[t] \approx \tilde{\rho}[t]\) both in \(Id V\) by Theorem 2.4. Since \(\sigma\) is a \(V\)-proper generalized hypersubstitution, we also have \(\tilde{\sigma}[s] \approx \tilde{\sigma}[t]\) in \(Id V\). It follows that \(\tilde{\rho}[s] \approx \tilde{\rho}[t]\) is in \(Id V\). This proves that any generalized hypersubstitution \(\rho\) from \(M\) which is \(\sim_{VG}\)-equivalent to a generalized hypersubstitution \(\sigma\) in \(P_G(V) \cap M\) is also in \(P_G(V) \cap M\).

**Definition 2.7** Let \(M\) be a monoid of generalized hypersubstitutions of type \(\tau\), and let \(V\) be a variety of type \(\tau\). Let \(\phi\) be a choice function which chooses from \(M\) one generalized hypersubstitution from each equivalence class of the relation \(\sim_{VG_M}\), and let \(N^M_\phi(V)\) be the set of generalized hypersubstitutions which are chosen. Thus \(N^M_\phi(V)\) is a set of distinguished generalized hypersubstitutions from \(M\), which we might call \(V\)-normal form generalized hypersubstitutions. We will say that the variety \(V\) is \(N^M_\phi(V)\)-strongly solid if for every identity \(s \approx t \in Id V\) and for every generalized hypersubstitution \(\sigma \in N^M_\phi(V)\), the identity \(\tilde{\sigma}[s] \approx \tilde{\sigma}[t] \in Id V\).

**Theorem 2.8** Let \(M\) be a monoid of generalized hypersubstitutions of type \(\tau\) and let \(V\) be a variety of type \(\tau\). For any choice function \(\phi\), \(V\) is \(M\)-strongly solid if and only if \(V\) is \(N^M_\phi(V)\)-strongly solid.

**Proof.** It is clear that if \(V\) is \(M\)-strongly solid then it is certainly also \(N^M_\phi(V)\)-strongly solid. Conversely, suppose that \(V\) is \(N^M_\phi(V)\)-strongly solid. This means that all the members of the set \(N^M_\phi(V)\) are also members of \(P_G(V) \cap M\). Since by Lemma 2.6, \(P_G(V) \cap M\) is a union of \(\sim_{VG_M}\)-classes, anything equivalent to an element of \(N^M_\phi(V)\) is also in \(P_G(V) \cap M\). But by construction any element of \(M\) is equivalent to an element of \(N^M_\phi(V)\). Thus \(M \subseteq P_G(V)\), and \(V\) is \(M\)-strongly solid.

All strongly solid varieties of semigroups were described by S. Leeratanavalee and K. Denecke([6]), so we will bring some important results.
Proposition 2.9 ([6]) There is no non-trivial strongly solid variety of bands.

Proposition 2.10 ([6]) The variety \( Rec := \text{Mod}\{x(yz) \approx (xy)z \approx xz\} \) is strongly solid.

Theorem 2.11 ([6]) The variety \( V_{\text{big}} := \text{Mod}\{x(yz) \approx (xy)z, xy^2 \approx x^2y \approx xy, xyzu \approx xzyu\} \) is the greatest strongly solid variety of semigroups.

A consequence of Proposition 2.9, Proposition 2.10 and Theorem 2.11, the varieties \( V_{\text{big}} \) and \( Rec \) are the only non-trivial strongly solid varieties of semigroups.

3 Pre-strongly solid varieties of semigroups

From now on we assume that the type is \( \tau = (2) \). So we have only one binary operation symbol, say \( f \). The concept of presolid variety was introduced by K. Denecke and Sh. L. Wismath in [3]. A variety \( V \) is called a presolid variety if every identity \( s \approx t \) in \( V, \hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id} \ V \) for all \( \sigma \in \text{Hyp}(2) \setminus \{\sigma_{x_1}, \sigma_{x_2}\} \). In this section we generalize the concept of presolid to pre-strongly solid.

Definition 3.1 A generalized hypersubstitution \( \sigma \in \text{Hyp}_G(2) \) is called a pre-generalized hypersubstitution if \( \sigma \in \text{Hyp}_G(2) \setminus \{\sigma_{x_1}, \sigma_{x_2}\} \) where \( \sigma_{x_1} \) and \( \sigma_{x_2} \) we denote the generalized hypersubstitutions which map \( f \) to \( x_1 \) and to \( x_2 \), respectively. We denote the set of all pre-generalized hypersubstitutions by \( \text{Pre}_G \).

The reason to delete the generalized hypersubstitutions which map \( f \) to \( x_1 \) and to \( x_2 \) from \( \text{Hyp}_G(2) \) is if we apply the generalized hypersubstitution which maps \( f \) to \( x_1 \) (or \( x_2 \)) on both sides of the commutative law \( xy \approx yx \) we obtain the equation \( x \approx y \) which satisfied only in a one-element semigroup.

Definition 3.2 An equation \( s \approx t \) is called a pre-strong hyperidentity in a variety \( V \) if \( \hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id} \ V \) for all \( \sigma \in \text{Pre}_G \).

A variety \( V \) is called a pre-strongly solid variety if every identity in \( V \) is a pre-strong hyperidentity.

Proposition 3.3 \( \text{Pre}_G \) is a submonoid of \( \text{Hyp}_G(2) \).
Proof. Let $\sigma_1, \sigma_2 \in \text{Pre}_G$. We have to prove that $\sigma_1 \circ \sigma_2$ belongs also to $\text{Pre}_G$. We use the inductive definition of the extension $\hat{\sigma}$ where $\hat{\sigma}[x] := x$ for all variables $x \in X$, and $\hat{\sigma}[f(t_1, t_2)] := S^2(\sigma_1(f), \hat{\sigma}[t_1], \hat{\sigma}[t_2])$ where $t_1, t_2 \in W(2)(X)$. We consider the following two cases:

Case 1: If $\sigma_2(f) = x_i(i > 2)$, then

$$(\sigma_1 \circ \sigma_2)(f) = \hat{\sigma}_1[\sigma_2(f)] = \hat{\sigma}_1[x_i] = x_i = \sigma_{x_i}(f) \in \text{Pre}_G.$$ 

Case 2: If $\sigma_2(f)$ is not $x_i(i > 2)$ and since $\sigma_2(f) \notin \{\sigma_{x_1}, \sigma_{x_2}\}$, then there exist $s_1, s_2 \in W(2)(X)$ such that $\sigma_2(f) = f(s_1, s_2)$. Consider

$$(\sigma_1 \circ \sigma_2)(f) = (\hat{\sigma}_1 \circ \sigma_2)(f) \in \text{Pre}_G.$$ 

(2.1) If $\sigma_1(f) = x_i(i > 2)$, then

$$(\sigma_1 \circ \sigma_2)(f) = S^2(\sigma_1(f), \hat{\sigma}_1[s_1], \hat{\sigma}_1[s_2]) = x_i = \sigma_{x_i}(f) \in \text{Pre}_G.$$

(2.2) If $\sigma_1(f)$ is not $x_i(i > 2)$ and since $\sigma_1(f) \notin \{\sigma_{x_1}, \sigma_{x_2}\}$, then there exist $t_1, t_2 \in W(2)(X)$ such that $\sigma_1(f) = f(t_1, t_2)$ and

$$(\sigma_1 \circ \sigma_2)(f) = S^2(f(t_1, t_2), \hat{\sigma}_1[s_1], \hat{\sigma}_2[s_2]) \notin \{\sigma_{x_1}, \sigma_{x_2}\}.$$ 

Thus $\sigma_1 \circ \sigma_2 \in \text{Pre}_G$. Obviously, $\sigma_{id} \in \text{Pre}_G$. Hence $\text{Pre}_G$ is a submonoid of $Hyp_G(2)$. 

Remark. 3.4 Every strongly solid variety $V$ of semigroups is a pre-strongly solid variety.

Proof. Let $V$ be a strongly solid variety of semigroups, and $s \approx t$ be an identity in $V$. Then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id}V$ for all generalized hypersubstitutions $\sigma$. Let $\sigma' \in \text{Pre}_G$. Since $\text{Pre}_G \subseteq Hyp_G(2)$, then $\hat{\sigma'}[s] \approx \hat{\sigma'}[t] \in \text{Id}V$. Thus $V$ is a pre-strongly solid variety. 

As a consequence of Remark 3.4, Proposition 2.10 and Theorem 2.11 we have the varieties $Rec$ and $V_{big}$ are pre-strongly solid.

Remark. 3.5 Every pre-strongly solid variety of semigroups is a presolid variety of semigroups.
Proof. Let $V$ be a pre-strongly solid variety of semigroups, and $s \approx t$ be an identity in $V$. Then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id} V$ for all $\sigma \in \text{Pre}_G$. Let $\sigma' \in \text{Hyp}(2) \setminus \{\sigma_{x_1}, \sigma_{x_2}\}$. Since $\text{Hyp}(2) \setminus \{\sigma_{x_1}, \sigma_{x_2}\} \subseteq \text{Pre}_G, \hat{\sigma'}[s] \approx \hat{\sigma'}[t] \in \text{Id} V$. Therefore $V$ is a presolid variety.

Proposition 3.6 The variety of zero semigroups $Z = \text{Mod}\{xy \approx uv\}$ is pre-strongly solid.

Proof. Since arbitrary terms over the variety $Z$ have the form $x_i \in X$ or $x_i x_j$, where $i, j \in \mathbb{N}$. Therefore we have only to consider the pre-generalized hypersubstitutions $\sigma_{x_i}$ where $i > 2$ and $\sigma_{x_1 x_2}$. For a pre-generalized hypersubstitution $\sigma_{x_i}$ where $i > 2$, we have

\[ \hat{\sigma}_{x_i}[f(x, y)] = S^2(\sigma_{x_i}(f), \hat{\sigma}_{x_i}[x], \hat{\sigma}_{x_i}[y]) = S^2(x_i, x, y) = x_i, \]

and $\hat{\sigma}_{x_i}[f(u, v)] = S^2(x_i, u, v) = x_i$. Thus $\hat{\sigma}_{x_i}[f(x, y)] = \hat{\sigma}_{x_i}[f(u, v)]$.

For a pre-generalized hypersubstitution $\sigma_{x_1 x_2}$, we have $\hat{\sigma}_{x_1 x_2}[f(x, y)] = S^2(x_1 x_2, x, y) = xy$ and $\hat{\sigma}_{x_1 x_2}[f(u, v)] = S^2(x_1 x_2, u, v) = uv$. Then we have $\hat{\sigma}_{x_1 x_2}[f(x, y)] = \hat{\sigma}_{x_1 x_2}[f(u, v)]$. This proves that the variety $Z$ is a pre-strongly solid variety of semigroups.

Since every pre-strongly solid variety of semigroups is a presolid variety. So it may help us to find out all pre-strongly solid varieties of semigroups if we know all presolid varieties of semigroups. We recall the following result:

Theorem 3.7 ([3]) For every non-trivial variety $V$ of semigroups the following propositions are equivalent:

(i) $V$ is presolid.

(ii) $V$ is solid, or $V$ is dual solid and $Z \subseteq V \subseteq V_{PS} = \text{Mod}\{(xy)z \approx x(yz),\ xyzzxyx \approx xyzyx, x^2 \approx y^2, x^3 \approx y^3\}$.  

\[ \text{Lemma 3.8} \] The variety $Z = \text{Mod}\{xy \approx uv\}$ is the least non-trivial pre-strongly solid variety of semigroups.

\[ \text{Theorem 3.9} \] The greatest non-trivial pre-strongly solid variety of semigroups which is not strongly solid is $Z$. 

Proof. The greatest pre-strongly solid variety of semigroups which is not strongly solid, is the class of all semigroups for which the associative law is satisfied as pre-strong hyperidentity, i.e. the class \( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \} \setminus H_{\text{SMod}} \{ \text{Ass.} \} \). If we apply \( \sigma_{x,x_1} \) and \( \sigma_{x_1,x} \), where \( i > 2 \) on both sides then from the identity \( x^2 \approx y^2 \) we have \( \sigma_{x,x_1}[x^2] = x;x \approx x; y = \sigma_{x_1,x}[y^2] \) and \( \sigma_{x_1,x}[x^2] = xx \approx yy \approx \sigma_{x,x_1}[y^2] \). If we substitute for \( x ; i \) a new variable, then we have the identities \( ux \approx uy \) and \( xu \approx yu \). Thus \( uz \approx ux \approx vx \in \text{Id}( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \} \setminus H_{\text{SMod}} \{ \text{Ass.} \}) \). Hence \( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \} \setminus H_{\text{SMod}} \{ \text{Ass.} \} = Z \).

Theorem 3.10 The variety \( V_{\text{big}} = \text{Mod} \{ \text{Ass.}, x^2y \approx xy^2 \approx xy, xyzu \approx xzyu \} \) is the greatest pre-strongly solid variety of semigroups.

Proof. The greatest pre-strongly solid variety of semigroups is the class of all semigroups for which the associative law is satisfied as pre-strong hyperidentity, i.e. the class \( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \} \). Applying \( \sigma_{x,x_1}, \sigma_{x_1,x}, \sigma_{x_1,x_1} \ (i > 2) \in \text{PreG} \) on the associative law, \( \sigma_{x,x_2} \) gives \((xy)z \approx x(yz)\), \( \sigma_{x,x_1} \) gives \( xi \approx x^2i \), and \( \sigma_{x_1,x_i} \) gives \( x_1i \approx x^2 \). If we substitute for \( x_i \) a new variable, then we have the identities \( yr \approx y^2r, xyr \approx y^2r \). That means \( x^2y \approx xy^2 \approx xy \in \text{Id}( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \}) \). Since \( x^2 \approx y^2, x^3 \approx y^3 \in \text{Id}( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \}) \), then \( xy^2 \approx xx^2 \approx x^3 \approx xy \) and \( y^2x \approx x^2x \approx x^3 \approx yx \). Thus \( xy \approx yx \). So \( xyzu \approx xzyu \) by \( xy \approx yx \). Hence \( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \} \) satisfies all identities of \( V_{\text{big}} \), i.e., \( H_{\text{PreG}} \text{Mod} \{ \text{Ass.} \} \subseteq V_{\text{big}} \). To prove the converse inclusion we have to check the associative law using all pre-generalized hypersubstitutions. We can restrict our checking to the following pre-generalized hypersubstitutions \( \sigma_t \) where \( t \in \{ x | i > 2 \} \cup \{ x_i | i, j \in \mathbb{N} \} \cup \{ x_i x_j x_k | i, j, k \in \mathbb{N}, i \neq j, j \neq k \} \cup \{ x_{i_1} x_{i_2} \cdots x_{i_k} | k, i_1, \ldots, i_k \in \mathbb{N}, k > 3, (i_m \neq i_n \text{ for } m, n \in \{ 2, 3, \ldots, k - 1 \}, m \neq n \text{ and } i_1 = i_k \text{ or } (i_m \neq i_n \text{ for } m, n \in \{ 1, 2, \ldots, k \}, m \neq n) \} \). Applying these \( \sigma_t \) in the associative law.

Obviously, if we apply \( \sigma_{x,i} \), \( i > 2 \) on both sides of the associative law, we get the term \( x_i \).

If we apply \( \sigma_{x,j}, i, j \in \mathbb{N} \) on both sides of the associative law, we have the following table.

<table>
<thead>
<tr>
<th>( i, j \in \mathbb{N} )</th>
<th>( \sigma_{x,xj}[(xy)z] )</th>
<th>( \sigma_{x,xj}[x(yz)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = j = 1 )</td>
<td>( xxxx )</td>
<td>( xx )</td>
</tr>
<tr>
<td>( i = j = 2 )</td>
<td>( zz )</td>
<td>( zzzz )</td>
</tr>
<tr>
<td>( i = 1, j = 2 )</td>
<td>( (xy)z )</td>
<td>( x(yz) )</td>
</tr>
<tr>
<td>( i = 2, j = 1 )</td>
<td>( z(yx) )</td>
<td>( (zy)x )</td>
</tr>
<tr>
<td>( i = 1, j &gt; 2 )</td>
<td>( xx_j x_j )</td>
<td>( xx_j )</td>
</tr>
<tr>
<td>( i = 2, j &gt; 2 )</td>
<td>( z x_j )</td>
<td>( z x_j x_j )</td>
</tr>
<tr>
<td>( i &gt; 2, j = 1 )</td>
<td>( x_i x_i x )</td>
<td>( x_i x )</td>
</tr>
<tr>
<td>( i &gt; 2, j = 2 )</td>
<td>( x_i z )</td>
<td>( x_i x_i z )</td>
</tr>
<tr>
<td>( i, j &gt; 2 )</td>
<td>( x_i x_j )</td>
<td>( x_i x_j )</td>
</tr>
</tbody>
</table>
Because of the associative law and the identity \( x^2y \approx xy^2 \approx xy \) we have both sides are equal.

If we apply \( \sigma_{x_i,x_j,x_k} \), \( i, j, k \in \mathbb{N} \), \( i \neq j, j \neq k \) on both sides of the associative law, we have the following table.

<table>
<thead>
<tr>
<th>( i,j,k \in \mathbb{N} )</th>
<th>( \sigma_{x_i,x_j,x_k}(xy)z )</th>
<th>( \sigma_{x_i,x_j,x_k}(x)yz )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 ), ( j = 2 )</td>
<td>( xyxyx )</td>
<td>( xyxyx )</td>
</tr>
<tr>
<td>( i = 1 ), ( j &gt; 2 )</td>
<td>( xxjxxjxxjx )</td>
<td>( xxjx )</td>
</tr>
<tr>
<td>( i = 2 ), ( j = 1 )</td>
<td>( zyxz )</td>
<td>( zyxzxyz )</td>
</tr>
<tr>
<td>( i = 2 ), ( j &gt; 2 )</td>
<td>( xxzjzxxjz )</td>
<td>( xxzjzxxjz )</td>
</tr>
<tr>
<td>( i = 1 ), ( j = 2 ), ( k &gt; 2 )</td>
<td>( xyykxxk )</td>
<td>( xyykxxk )</td>
</tr>
<tr>
<td>( i = 2 ), ( j = 1 ), ( k &gt; 2 )</td>
<td>( zyxkxxk )</td>
<td>( zyxkxxk )</td>
</tr>
<tr>
<td>( i = 1 ), ( j = 2 ), ( k = 2 )</td>
<td>( xx_jyx_jz )</td>
<td>( xx_jyx_jz )</td>
</tr>
</tbody>
</table>

Using the associative law, the medial law and identity \( x^2y \approx xy^2 \approx xy \) we have both sides are equal.

If we apply \( \sigma_t \) where \( t = x_{i_1}x_{i_2}\ldots x_{i_k} \) and \( k, i_1, \ldots, i_k \in \mathbb{N} \), \( k > 3, (i_m \neq i_n) \) for \( m, n \in \{2, 3, \ldots, k-1\} \), \( m \neq n \) and \( i_1 = i_k \) or \( (i_m \neq i_n) \) for \( m, n \in \{1, 2, \ldots, k\}, m \neq n \) on both sides of the associative law, we have \( \hat{\sigma}_t(xy)z = S^2(t, S^2(t, x, y), z) \) and \( \hat{\sigma}_t(x)yz = S^2(t, x, S^2(t, y, z)) \).

(i) If \( i_1 = i_k = 1, i_m > 2 \) for all \( m \in \{2, 3, \ldots, k-1\} \), then
\[
\hat{\sigma}_t(xy)z = x_{i_2}\ldots x_{i_{k-1}}xx_{i_2}\ldots x_{i_{k-1}}x_{i_k-1}x, \\
\hat{\sigma}_t(x)yz = x_{i_2}\ldots x_{i_{k-1}}x.
\]

(ii) If \( i_1 = i_k = 2, i_m > 2 \) for all \( m \in \{2, 3, \ldots, k-1\} \), then
\[
\hat{\sigma}_t(xy)z = z_{x_{i_2}\ldots x_{i_{k-1}}}z, \\
\hat{\sigma}_t(x)yz = z_{x_{i_2}\ldots x_{i_{k-1}}}z_{x_{i_2}\ldots x_{i_{k-1}}}z_{x_{i_2}\ldots x_{i_{k-1}}}z.
\]

(iii) If \( i_1 = i_k = 1 \) and there exists a unique \( n \in \{2, 3, \ldots, k-1\} \) such that \( i_n = 2 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, k-1\} \), \( m \neq n \), then
\[
\hat{\sigma}_t(xy)z = x_{i_2}\ldots x_{i_{n-1}}y_{i_{n+1}}x_{i_k-1}xx_{i_2}\ldots x_{i_{n-1}}z_{x_{i_2}\ldots x_{i_{k-1}}}z_{x_{i_2}\ldots x_{i_{k-1}}}z_{x_{i_2}\ldots x_{i_{k-1}}}x_{i_k-1}x_{i_k}y.
\]
\[
\sigma_t[x(yz)] = xx_{i_2} \cdots x_{i_{n-1}} y x_{i_2} \cdots x_{i_{n-1}} z x_{i_1} y x_{i_{n+1}} \cdots x_{i_{k-1}} x.
\]

(iv) If \( i_1 = i_k = 2 \) and there exists a unique \( n \in \{2, 3, \ldots, k-1\} \) such that \( i_n = 1 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, k-1\}, m \neq n \), then

\[
\hat{\sigma}_t[(xy)z] = zx_{i_2} \cdots x_{i_{n-1}} y x_{i_2} \cdots x_{i_{n-1}} x_{i_k-1} y x_{i_{n+1}} \cdots x_{i_{k-1}} z,
\]
\[
\hat{\sigma}_t[x(yz)] = zx_{i_2} \cdots x_{i_{n-1}} y x_{i_{n+1}} \cdots x_{i_{k-1}} z x_{i_2} \cdots x_{i_{n-1}} y x_{i_{k-1}} z.
\]

(v) If \( i_1 = 1, i_k = 2 \) and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, k-1\} \), then

\[
\hat{\sigma}_t[(xy)z] = xx_{i_2} \cdots x_{i_{k-1}} y x_{i_2} \cdots x_{i_{k-1}} z,
\]
\[
\hat{\sigma}_t[x(yz)] = xx_{i_2} \cdots x_{i_{k-1}} y x_{i_2} \cdots x_{i_{k-1}} z.
\]

(vi) If \( i_1 = 2, i_k = 1 \) and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, k-1\} \), then

\[
\hat{\sigma}_t[(xy)z] = zx_{i_2} \cdots x_{i_{k-1}} y x_{i_2} \cdots x_{i_{k-1}} x,
\]
\[
\hat{\sigma}_t[x(yz)] = zx_{i_2} \cdots x_{i_{k-1}} y x_{i_2} \cdots x_{i_{k-1}} x.
\]

(vii) If \( i_1 = 1 \) and there exists a unique \( n \in \{2, 3, \ldots, k-1\} \) such that \( i_n = 2 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, k\}, m \neq n \), then

\[
\hat{\sigma}_t[(xy)z] = xx_{i_2} \cdots x_{i_{n-1}} y x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k} x_{i_2} \cdots x_{i_{k-1}} x_{i_k},
\]
\[
\hat{\sigma}_t[x(yz)] = xx_{i_2} \cdots x_{i_{n-1}} y x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k} x_{i_2} \cdots x_{i_{k-1}} x_{i_k}.
\]

(viii) If \( i_1 = 2 \) and there exists a unique \( n \in \{2, 3, \ldots, k-1\} \) such that \( i_n = 1 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, k\}, m \neq n \), then

\[
\hat{\sigma}_t[(xy)z] = zx_{i_2} \cdots x_{i_{n-1}} y x_{i_2} \cdots x_{i_{n-1}} x_{i_k} x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k},
\]
\[
\hat{\sigma}_t[x(yz)] = zx_{i_2} \cdots x_{i_{n-1}} y x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k} x_{i_2} \cdots x_{i_{k-1}} x_{i_k}.
\]

(ix) If \( i_k = 1 \) and there exists a unique \( n \in \{2, 3, \ldots, k-1\} \) such that \( i_n = 2 \), and \( i_m > 2 \) for all \( m \in \{1, 2, \ldots, k-1\}, m \neq n \), then

\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{n-1}} z x_{i_{n+1}} \cdots x_{i_k},
\]
\[
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{n-1}} x_{i_1} \cdots x_{i_{n-1}} z x_{i_{n+1}} \cdots x_{i_k} x_{i_{n+1}} \cdots x_{i_k}.
\]

(x) If \( i_k = 2 \) and there exists a unique \( n \in \{2, 3, \ldots, k-1\} \) such that \( i_n = 1 \), and \( i_m > 2 \) for all \( m \in \{1, 2, \ldots, k-1\}, m \neq n \), then

\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{n-1}} x_{i_1} \cdots x_{i_{n-1}} x_{i_k} x_{i_{n+1}} \cdots x_{i_k-1} y x_{i_{n+1}} \cdots x_{i_k-1} z,
\]
\[
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{n-1}} x x_{i_{n+1}} \cdots x_{i_k-1} x_{i_1} x_{i_{n-1}} y x_{i_{n+1}} \cdots x_{i_k-1} z.
\]
(xi) If there exists a unique $n \in \{2, 3, \ldots, k-1\}$ such that $i_n = 1$, and $i_m > 2$ for all $m \in \{1, 2, 3, \ldots, k\}, m \neq n$, then
\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{n-1}} x_{i_1} \cdots x_{i_{n-2}} x_{i_{n-1}} x_{i_n} x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}, \\
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{n-1}} x_{i_1} \cdots x_{i_{n-2}} x_{i_{n-1}} x_{i_n} x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}.
\]

(xii) If there exists a unique $n \in \{2, 3, \ldots, k-1\}$ such that $i_n = 2$, and $i_m > 2$ for all $m \in \{1, 2, \ldots, k\}, m \neq n$, then
\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{n-1}} z x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}, \\
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{n-1}} x_{i_1} \cdots x_{i_{n-2}} x_{i_{n-1}} z x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}.
\]

(xiii) If there exist a unique $m$ and a unique $n$ in $\{2, 3, \ldots, k-1\}$ such that $m < n, i_m = 1, i_n = 2$, and $i_l > 2$ for all $l \in \{1, 2, \ldots, k\}, l \neq m, l \neq n$, then
\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{m-1}} x_{i_1} \cdots x_{i_{n-1}} x_{i_{m-1}} y x_{i_{m+1}} \cdots x_{i_{n-1}} x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}, \\
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{m-1}} x_{i_1} \cdots x_{i_{n-1}} x_{i_{m-1}} y x_{i_{m+1}} \cdots x_{i_{n-1}} z x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}.
\]

(xiv) If there exist a unique $m$ and a unique $n$ in $\{2, 3, \ldots, k-1\}$ such that $m > n, i_m = 1, i_n = 2$, and $i_l > 2$ for all $l \in \{1, 2, \ldots, k\}, l \neq m, l \neq n$, then
\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{m-1}} z x_{i_{m+1}} \cdots x_{i_{n-1}} x_{i_{m-1}} y x_{i_{m+1}} \cdots x_{i_{n-1}} x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}, \\
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{m-1}} x_{i_1} \cdots x_{i_{n-1}} z x_{i_{m+1}} \cdots x_{i_{n-1}} y x_{i_{m+1}} \cdots x_{i_{k-1}} x_{i_k} x_{i_{n+1}} \cdots x_{i_{k-1}} x_{i_k}.
\]

(xv) If $i_m > 2$ for all $m \in \{1, \ldots, k\}$, then
\[
\hat{\sigma}_t[(xy)z] = x_{i_1} \cdots x_{i_{k}}, \\
\hat{\sigma}_t[x(yz)] = x_{i_1} \cdots x_{i_{k}}.
\]

Using the associative law, the medial law and the identity $x^2y \approx xy^2 \approx xy$, we have $\hat{\sigma}_t[(xy)z] \approx \hat{\sigma}_t[x(yz)]$. This finishes the proof.

A consequence of Lemma 3.8, Theorem 3.9 and Theorem 3.10, there is exactly one pre-strongly solid but not strongly solid variety of semigroups, namely $Z$. Altogether we have three pre-strongly solid varieties of semigroups: $Z, Rec, V_{big}$. 


4 Left-edge(Right-edge)-strongly solid

In this section we study another important class of generalized hypersubstitutions.

**Definition 4.1** A generalized hypersubstitution $\sigma \in \text{Hyp}_G(2)$ is said to be a leftmost(rightmost) generalized hypersubstitution if $\sigma(f)$ starts with $x_1$ (ends with $x_2$). Let $\text{Left}_G(\text{Right}_G)$ be the set of all leftmost(rightmost) generalized hypersubstitutions of type $\tau = (2)$.

An identity $s \approx t$ is called a leftmost(rightmost)-strong hyperidentity in a variety $V$ if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id}_V$ for all $\sigma \in \text{Left}_G(\sigma \in \text{Right}_G)$.

If every identity is a leftmost(rightmost)-strong hyperidentity in $V$ then $V$ is called a left-edge(right-edge)-strongly solid variety.

Then we have:

**Lemma 4.2** The set $\text{Left}_G(\text{Right}_G)$ forms a submonoid of the monoid $\text{Hyp}_G(2)$.

**Proof.** Clearly $\sigma_{id} = \sigma_{x_1x_2} \in \text{Left}_G$. Assume that $\sigma_1, \sigma_2 \in \text{Left}_G$, i.e. $\sigma_1(f), \sigma_2(f)$ start with $x_1$. We have to show that $(\sigma_1 \circ G \sigma_2)(f) \in \text{Left}_G$, i.e. $(\sigma_1 \circ G \sigma_2)(f)$ starts with $x_1$. Indeed, $(\sigma_1 \circ G \sigma_2)(f) = \hat{\sigma}_1[\sigma_2(f)]$. Since $\sigma_2(f)$ is a term which starts with $x_1$. Then also $\hat{\sigma}_1[\sigma_2(f)]$ starts with $x_1$ and thus $\sigma_1 \circ G \sigma_2 \in \text{Left}_G$. For $\text{Right}_G$ we conclude in the same way.

All left-edge(right-edge)-strongly solid varieties of type $\tau$ (of semigroups) form a complete lattice which contains the lattice of all strongly solid varieties of semigroups. Clearly, every left-edge-strongly solid variety is a left-edge-solid class of semigroups.

**Proposition 4.3** The variety $LZ = \text{Mod}\{xy \approx x\}$ $(RZ = \text{Mod}\{xy \approx y\})$ is the least non-trivial left-edge(right-edge)-strongly solid variety of semigroups.

**Proof.** We will give the proof only for $LZ$. The proof for $RZ$ is similar. From every equivalence class of binary terms over $LZ$ we have only to check one representative of the corresponding equivalence class of generalized hypersubstitutions. Therefore we have only to consider the leftmost generalized hypersubstitution $\sigma_{x_1}$. Applying this leftmost generalized hypersubstitution to $xy \approx x$ we get $x \approx x$. Hence $LZ$ is left-edge-strongly solid. Since $\text{Id}_LZ$ is the set of all identities $s \approx t$ where $s$ and $t$ start with the same variable, we have $LZ \subseteq V$ for every non-trivial left-edge-strongly solid variety $V$ of semigroups.

Now we want to determine the greatest left-edge(right-edge)-strongly solid variety of semigroups.
Theorem 4.4 The variety \( L_{\text{big}} = \text{Mod}\{(xy)z \approx x(yz), xy \approx xy^2, xyzu \approx xyuzyx \} \)
\( \approx x(yzu) \)
\( \approx xyzuyx \}(R_{\text{big}} = \text{Mod}\{(xy)z \approx x(yz), xy \approx x^2y, xyzu \approx xyxzu, xyuzyx \approx x(yzu)yx \}) \) is the greatest left-edge (right-edge)-strongly solid variety of semi-groups.

Proof. We will give the proof only for \( L_{\text{big}} \). The proof for \( R_{\text{big}} \) is similar. The greatest left-edge-stongly solid variety of semigroups is the left-edge model class of the associative law, i.e. \( H_{\text{leftG}} \text{Mod}\{(xy)z \approx x(yz)\} \). We will show that \( L_{\text{big}} = H_{\text{leftG}} \text{Mod}\{(xy)z \approx x(yz)\} \). If we apply \( \sigma_{x_1x_2}, \sigma_{x_1x_1}, \sigma_{x_1x_2x_1} \in \text{Left}_G \) where \( i > 2 \) to the associative law we obtain: \( \sigma_{x_1x_2} \) gives \( (xy)z \approx x(yz) \), \( \sigma_{x_1x_1} \) gives \( i \approx x_i \), \( \sigma_{x_1x_2x_1} \) gives \( x_i \approx \sigma_{x_1x_2x_1}[x(yz)] \) and \( \sigma_{x_1x_2x_1}[x(yz)] = xyx_{i}x_i \approx xyx_i \). If we substitute for \( x_i \) a new variable, then we have identities \( xy \approx xy^2 \) and \( xyu \approx x(yzu) \). If we apply \( \sigma_{x_1x_2x_1} \in \text{Left}_G \) where \( i > 2 \) to the associative law and use the identity \( xyu \approx x(yzu) \) we obtain: \( \sigma_{x_1x_2x_1}[x(yz)] = xyx_{i}x_i \approx xyx_{i}. \) If we substitute for \( x_i \) a new variable, then we have the identity \( xyu \approx x(yzu)yx \). Therefore, \( H_{\text{leftG}} \text{Mod}\{(xy)z \approx x(yz)\} \subseteq L_{\text{big}} \). To show the converse, we have to show that the associative law is a leftmost strong hyperidentity in \( L_{\text{big}} \). We can restrict our checking to the following leftmost generalized hypersubstitutions \( \sigma_t \) where \( t \in \{x_1, x_i, x_i x_j \mid i \in \mathbb{N}, i \neq j\} \cup \{x_1 x_i x_j \mid i, j \in \mathbb{N}, i \neq j\} \cup \{x_1 x_i x_j x_k \mid i, j, k \in \mathbb{N}, i \neq j, j \neq k\} \cup \{x_1 x_i x_j \cdots x_n \mid n, i_n \in \mathbb{N}, n \geq 4 \text{ and } i_j \neq i_k \text{ for } j, k \in \{2, 3, \ldots, n\}, j \neq k\} \). Applying these \( \sigma_t \) in the associative law. Obviously, if we apply \( \sigma_{x_1} \) on both sides we get the term \( x \).

If we apply \( \sigma_{x_1x_1}, i \in \mathbb{N} \) on both sides of the associative law, we have the following table.

<table>
<thead>
<tr>
<th>( i \in \mathbb{N} )</th>
<th>( \sigma_{x_1x_1}[x(yz)] )</th>
<th>( \sigma_{x_1x_1}[x(yz)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i = 1 )</td>
<td>( xxxx )</td>
<td>( xx )</td>
</tr>
<tr>
<td>( i = 2 )</td>
<td>( (xy)z )</td>
<td>( x(yz) )</td>
</tr>
<tr>
<td>( i &gt; 2 )</td>
<td>( xx_i x_i )</td>
<td>( xx_i )</td>
</tr>
</tbody>
</table>

Because of the associative law and the identity \( xy \approx xy^2 \) we have both sides are equal.

If we apply \( \sigma_{x_1x_1x_i}, i, j \in \mathbb{N}, i \neq j \) on both sides of the associative law, we have the following table.
Using the associative law and the identities \(xy \approx xy^2\) and \(xyzu \approx xyuzu\) we have both sides are equal.

If we apply \(\sigma_{x_{1,2},x_{1,k}}\) where \(i, j, k \in \mathbb{N}, i \neq j\) and \(j \neq k\) on both sides of the associative law, we have the following table.

<table>
<thead>
<tr>
<th>(i, j, k \in \mathbb{N}, i \neq j, j \neq k)</th>
<th>(\sigma_{x_{1,i},x_{j,k}}[(xy)z] = S^2(x_{1,i}x_{j,k}, S^2(x_{1,i}x_{j,k}, x, y)z))</th>
<th>(\sigma_{x_{1,i},x_{j,k}}[x(yz)] = S^2(x_{1,i}x_{j,k}, x, S^2(x_{1,i}x_{j,k}, y, z)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i = k = 1, j = 2)</td>
<td>(xyxyxyz)</td>
<td>(xyyz)</td>
</tr>
<tr>
<td>(i = 1, j &gt; 2)</td>
<td>(xx_{i,2}xx_{j,k}x_{j})</td>
<td>(xx_{i,j})</td>
</tr>
<tr>
<td>(i = 2, j = 1)</td>
<td>(xyzzxyz)</td>
<td>(xyz)</td>
</tr>
<tr>
<td>(i = 2, j &gt; 2)</td>
<td>(xy_{i,j}x_{j,k}x_{j})</td>
<td>(xy_{j,k})</td>
</tr>
<tr>
<td>(i \neq 2, j = 1)</td>
<td>(xx_{i,2}xx_{j,k}x_{j})</td>
<td>(xx_{i,j})</td>
</tr>
<tr>
<td>(i = 2, j &gt; 2)</td>
<td>(xx_{i,j}x_{j,k}x_{j,k})</td>
<td>(xx_{i,j})</td>
</tr>
</tbody>
</table>

Using the associative law and the identities \(xy \approx xy^2, xyzu \approx xyuzu\) and \(xyyzxu \approx xyuzxu\) we have both sides are equal.

If we apply \(\sigma_t\) where \(t = x_{1,i_1}x_{i_2} \cdots x_{i_n}\) and \(n, i_n \in \mathbb{N}, n \geq 4\) and \(i_j \neq i_k\) for \(j, k \in \{2, 3, \ldots, n\}, j \neq k\) on both sides of the associative law, we have \(\hat{\sigma}_t[(xy)z] = S^2(t, S^2(t, x, y), z)\) and \(\hat{\sigma}_t[x(yz)] = S^2(t, x, S^2(t, y, z))\).
(i) If \( i_1 = 1 \) and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\} \), then
\[
\hat{\sigma}_t[(xy)z] = (xxx_{i_2} \cdots x_{i_m})^2 x_{i_2} \cdots x_{i_m}, \\
\hat{\sigma}_t[x(yz)] = xxx_{i_2} \cdots x_{i_m}.
\]

(ii) If \( i_1 = 1 \) and there exists a unique \( k \in \{3, 4, \ldots, n\} \) such that \( i_k = 1 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\} \), \( m \neq k \), then
\[
\hat{\sigma}_t[(xy)z] = (xxx_{i_2} \cdots x_{i_k} x_{i_{k-1}} x_{i_k+1} \cdots x_{i_m} x_{i_{k+1}} \cdots x_{i_m})^2 x_{i_2} \cdots x_{i_k} z x_{i_{k+1}} \cdots x_{i_m}, \\
\hat{\sigma}_t[x(yz)] = xxx_{i_2} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_m}.
\]

(iii) If \( i_1 = 1 \) and there exists a unique \( k \in \{2, 3, \ldots, n\} \) such that \( i_k = 2 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\} \), \( m \neq k \), then
\[
\hat{\sigma}_t[(xy)z] = (xxx_{i_2} \cdots x_{i_k} x_{i_{k-1}} y x_{i_k+1} \cdots x_{i_m})^2 x_{i_2} \cdots x_{i_k} z x_{i_{k+1}} \cdots x_{i_m}, \\
\hat{\sigma}_t[x(yz)] = xxx_{i_2} \cdots x_{i_k} y x_{i_{k+1}} \cdots x_{i_m} x_{i_{k+1}} \cdots x_{i_m}.
\]

(iv) If \( i_1 = 1 \) and there exists a unique \( k \in \{3, 4, \ldots, n-1\} \) such that \( i_k = 1 \), and there exists a unique \( l \in \{4, 5, \ldots, n\} \), \( l > k \) such that \( i_l = 2 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\} \), \( m \neq k, m \neq l \), then
\[
\hat{\sigma}_t[(xy)z] = (xxx_{i_2} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_{l-1}} y x_{i_{l+1}} \cdots x_{i_m})^2 x_{i_2} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_{l-1}} x_{i_{l+1}} \cdots x_{i_m}, \\
\hat{\sigma}_t[x(yz)] = xxx_{i_2} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_{l-1}} x_{i_{l+1}} \cdots x_{i_m}.
\]

(v) If \( i_1 = 1 \) and there exists a unique \( k \in \{2, 3, \ldots, n-1\} \) such that \( i_k = 2 \), and there exists a unique \( l \in \{3, 4, \ldots, n\} \), \( l > k \) such that \( i_l = 1 \), and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\} \), \( m \neq k, m \neq l \), then
\[
\hat{\sigma}_t[(xy)z] = (xxx_{i_2} \cdots x_{i_k} y x_{i_{k+1}} \cdots x_{i_{l-1}} x_{i_{l+1}} \cdots x_{i_m})^2 x_{i_2} \cdots x_{i_k} z x_{i_{k+1}} \cdots x_{i_{l-1}} x_{i_{l+1}} \cdots x_{i_m}, \\
\hat{\sigma}_t[x(yz)] = xxx_{i_2} \cdots x_{i_k} (y x_{i_{k+1}} \cdots x_{i_{l-1}} y x_{i_{l+1}} \cdots x_{i_m}) x_{i_{k+1}} \cdots x_{i_{l-1}} x_{i_{l+1}} \cdots x_{i_m}.
\]

(vi) If \( i_1 = 2 \) and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\} \), then
\[
\hat{\sigma}_t[(xy)z] = y x_{i_2} \cdots x_{i_m} z x_{i_2} \cdots x_{i_m}, \\
\hat{\sigma}_t[x(yz)] = xyz x_{i_2} \cdots x_{i_k} x_{i_{k+1}} \cdots x_{i_m}.
\]
(vii) If \( i_1 = 2 \) and there exists a unique \( k \in \{2, 3, \ldots, n\} \) such that \( i_k = 1, i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\}, m \neq k \), then
\[
\hat{\sigma}_t[(xy)z] = (xyx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n})zzx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}, \\
\hat{\sigma}_t[x(yz)] = (yzx_{i_2} \cdots x_{i_{k-1}}yyx_{i_k+1} \cdots xx_{i_n})xx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}.
\]

(viii) If \( i_1 = 2 \) and there exists a unique \( k \in \{3, 4, \ldots, n\} \) such that \( i_k = 2, i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\}, m \neq k \), then
\[
\hat{\sigma}_t[(xy)z] = (xyx_{i_2} \cdots x_{i_{k-1}}yx_{i_k+1} \cdots xx_{i_n})zx_{i_2} \cdots x_{i_{k-1}}(yxz_{i_2} \cdots x_{i_k-1})zxx_{i_2} \cdots x_{i_{k-1}}z \cdots xx_{i_n}, \\
\hat{\sigma}_t[x(yz)] = (yzx_{i_2} \cdots x_{i_{k-1}}yyx_{i_k+1} \cdots xx_{i_n})xx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}.
\]

(ix) If \( i_1 = 2 \) and there exists a unique \( k \in \{2, 3, \ldots, n - 1\} \) such that \( i_k = 1, \) and there exists a unique \( l \in \{3, 4, \ldots, n\}, l > k \) such that \( i_l = 2, \) and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\}, m \neq k, m \neq l \), then
\[
\hat{\sigma}_t[(xy)z] = (xyx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n})zzx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}, \\
\hat{\sigma}_t[x(yz)] = (yzx_{i_2} \cdots x_{i_{k-1}}yyx_{i_k+1} \cdots xx_{i_n})xx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}.
\]

(x) If \( i_1 = 2 \) and there exists a unique \( k \in \{3, 4, \ldots, n - 1\} \) such that \( i_k = 2, \) and there exists a unique \( l \in \{4, 5, \ldots, n\}, l > k \) such that \( i_l = 1, \) and \( i_m > 2 \) for all \( m \in \{2, 3, \ldots, n\}, m \neq k, m \neq l \), then
\[
\hat{\sigma}_t[(xy)z] = (xyx_{i_2} \cdots x_{i_{k-1}}yx_{i_k+1} \cdots xx_{i_n})zzx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}, \\
\hat{\sigma}_t[x(yz)] = (yzx_{i_2} \cdots x_{i_{k-1}}yyx_{i_k+1} \cdots xx_{i_n})xx_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}.
\]

(xi) If \( i_m > 2 \) for all \( m \in \{1, 2, \ldots, n\} \), then
\[
\hat{\sigma}_t[(xy)z] = xx_{i_1}x_{i_2} \cdots xx_{i_1}x_{i_2} \cdots xx_{i_n}, \\
\hat{\sigma}_t[x(yz)] = xx_{i_1}x_{i_2} \cdots xx_{i_n}.
\]

(xii) If there exists a unique \( k \in \{2, 3, \ldots, n\} \) such that \( i_k = 1, \) and \( i_m > 2 \) for all \( m \in \{1, 2, \ldots, n\}, m \neq k \), then
\[
\hat{\sigma}_t[(xy)z] = (xx_{i_1}x_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n})xx_{i_1}x_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}, \\
\hat{\sigma}_t[x(yz)] = xx_{i_1}x_{i_2} \cdots x_{i_{k-1}}xx_{i_k+1} \cdots xx_{i_n}.
\]
(xiii) If there exists a unique $k \in \{2, 3, \ldots, n\}$ such that $i_k = 2$, and $i_m > 2$ for all $m \in \{1, 2, \ldots, n\}, m \neq k$, then

$$
\hat{\sigma}_t[(xy)z] = (xx_1x_{i_2} \cdots x_{i_{k-1}}x_{i_{k+1}} \cdots x_{i_{l-1}}yx_{i_{l+1}} \cdots x_{i_l})x_1x_{i_2} \cdots x_{i_{k-1}}(xx_1x_{i_2} \cdots x_{i_{k-1}}yxx_{i_{k+1}} \cdots x_{i_{l-1}}yxx_{i_{l+1}} \cdots x_{i_l})x_{i_{k+1}} \cdots x_{i_l}.
$$

$$
\hat{\sigma}_t[x(yz)] = xx_1x_{i_2} \cdots x_{i_{k-1}}x_{i_{k+1}} \cdots x_{i_{l-1}}(yx_1x_{i_2} \cdots x_{i_{k-1}}yx_{i_{k+1}} \cdots x_{i_{l-1}}yxx_{i_{l+1}} \cdots x_{i_l})x_{i_{k+1}} \cdots x_{i_l}.
$$

(xiv) If there exists a unique $k \in \{2, 3, \ldots, n - 1\}$ such that $i_k = 1$, and there exists a unique $l \in \{3, 4, \ldots, n\}, l > k$ such that $i_l = 2$, and $i_m > 2$ for all $m \in \{1, 2, \ldots, n\}, m \neq k, m \neq l$, then

$$
\hat{\sigma}_t[(xy)z] = (xx_1x_{i_2} \cdots x_{i_{k-1}}x_{i_{k+1}} \cdots x_{i_{l-1}}yx_{i_{l+1}} \cdots x_{i_l})x_1x_{i_2} \cdots x_{i_{k-1}}(xx_1x_{i_2} \cdots x_{i_{k-1}}yxx_{i_{k+1}} \cdots x_{i_{l-1}}yxx_{i_{l+1}} \cdots x_{i_l})x_{i_{k+1}} \cdots x_{i_l}.
$$

$$
\hat{\sigma}_t[x(yz)] = xx_1x_{i_2} \cdots x_{i_{k-1}}x_{i_{k+1}} \cdots x_{i_{l-1}}(yx_1x_{i_2} \cdots x_{i_{k-1}}yx_{i_{k+1}} \cdots x_{i_{l-1}}yxx_{i_{l+1}} \cdots x_{i_l})x_{i_{k+1}} \cdots x_{i_l}.
$$

(xv) If there exists a unique $k \in \{2, 3, \ldots, n - 1\}$ such that $i_k = 2$, and there exists a unique $l \in \{3, 4, \ldots, n\}, l > k$ such that $i_l = 1$, and $i_m > 2$ for all $m \in \{1, 2, \ldots, n\}, m \neq k, m \neq l$, then

$$
\hat{\sigma}_t[(xy)z] = (xx_1x_{i_2} \cdots x_{i_{k-1}}yx_{i_{k+1}} \cdots x_{i_{l-1}}x_{i_{l+1}} \cdots x_{i_l})x_1x_{i_2} \cdots x_{i_{k-1}}zx_{i_{k+1}} \cdots x_{i_l}.
$$

$$
\hat{\sigma}_t[x(yz)] = xx_1x_{i_2} \cdots x_{i_{k-1}}(yx_1x_{i_2} \cdots x_{i_{k-1}}zyx_{i_{k+1}} \cdots x_{i_{l-1}}yxx_{i_{l+1}} \cdots x_{i_l})x_{i_{k+1}} \cdots x_{i_l}.
$$

Using the associative law and the identities $xy \approx xy^2, xyzu \approx xyuz$ and $xyzuyx \approx xyuzyx$, we have $\hat{\sigma}_t[(xy)z] \approx \hat{\sigma}_t[x(yz)]$. This finishes the proof. ■

Clearly, the varieties $V_{big}$ and $Rec$ are also left-edge-strongly solid and $V_{big} \subseteq L_{big}$. It arises the question whether there are more left-edge-strongly solid varieties of semigroups which are definable by identities. In fact, we have

**Theorem 4.5** The variety $L_1 = \text{Mod}\{x(yz) \approx (xy)z, xy \approx xy^2, xyzu \approx xyzu, xyz \approx xyxz\}$ $(R_1 = \text{Mod}\{x(yx) \approx (xy)z, xy \approx x^2y, xyz \approx xyz, xyzu \approx xxyzu\})$ is a left-edge(right-edge)-strongly solid variety of semigroups which is definable by identities.

**Proof.** We will give the proof only for $L_1$. The proof for $R_1$ is similar. Because of $xyzuyx \approx xyzyux \approx xyzyx$ by $xyzu \approx xyzu$ and
If we apply $\sigma_{x_i x_i}$, $i \in \mathbb{N}$ on both sides, we have the following table.

<table>
<thead>
<tr>
<th>$i \in \mathbb{N}$</th>
<th>$\hat{\sigma}_{x_i x_i}[(xy)(xz)]$</th>
<th>$\hat{\sigma}_{x_i x_i}[x(yz)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>$xxx$</td>
<td>$xx$</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>$(xy)(xz)$</td>
<td>$x(yz)$</td>
</tr>
<tr>
<td>$i &gt; 2$</td>
<td>$xx_i x x_i$</td>
<td>$x x_i$</td>
</tr>
</tbody>
</table>

Because of the associative law, the identity $xy \approx xy^2$ and $xyz \approx xyxz$ we have both sides are equal.

If we apply $\sigma_{x_i x_j}$, $i, j \in \mathbb{N}, i \neq j$ on both sides of the identity $(xy)(xz) \approx x(yz)$, we have the following table.

<table>
<thead>
<tr>
<th>$i, j \in \mathbb{N}, i \neq j$</th>
<th>$\hat{\sigma}_{x_i x_j}[(xy)(xz)] = S^2(x_1 x_i x_j, S^2(x_1 x_i x_j, x, y), S^2(x_1 x_i x_j, x, z))$</th>
<th>$\hat{\sigma}_{x_i x_j}[x(yz)] = S^2(x_1 x_i x_j, x, S^2(x_1 x_i x_j, y, z))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1, j = 2$</td>
<td>$xx_i y x y x_i z$, $x y x y x_i z$</td>
<td>$x x_i y x y_i z$, $x x_i y x i z$</td>
</tr>
<tr>
<td>$i = 1, j &gt; 2$</td>
<td>$xx_i x x_i x_j$</td>
<td>$x x_i$</td>
</tr>
<tr>
<td>$i = 2, j = 1$</td>
<td>$x x_i y x x_i x_j$</td>
<td>$x x_i$</td>
</tr>
<tr>
<td>$i = 2, j &gt; 2$</td>
<td>$x x_i y x x_i x_j$</td>
<td>$x x_i$</td>
</tr>
<tr>
<td>$i &gt; 2, j = 1$</td>
<td>$x x_i y x x_i x_i z$</td>
<td>$x x_i y x i z$</td>
</tr>
<tr>
<td>$i &gt; 2, j &gt; 2$</td>
<td>$x x_i y x i x_i z$</td>
<td>$x x_i y x z$</td>
</tr>
</tbody>
</table>

Using the associative law and the identities $xy \approx xy^2$, $xyz \approx xyzu$ and $xyz \approx xyxz$ we have both sides are equal.

If we apply $\sigma_{x_i x_j x_k}$ where $i, j, k \in \mathbb{N}, i \neq j$ and $j \neq k$ on both sides of the identity $(xy)(xz) \approx x(yz)$, we have the following table.
Using the associative law and the identities $xy \approx xy^2, xyzu \approx xyzu$, and $xyz \approx xyzx$ we have both sides are equal.

If we apply $\sigma_t$ where $t = x_1x_{i_1}x_{i_2}\cdots x_{i_n}$ and $n, i_n \in \mathbb{N}, n \geq 4, i_r \neq 1$ for all $r \in \{2, 3, \ldots, n-1\}$ and $i_j \neq i_k$ for $j, k \in \{2, 3, \ldots, n\}, j \neq k$ on both sides of the identity $(xy)(xz) \approx x(yz)$, we have $\hat{\sigma}_t[(xy)(xz)] = S^2(t, S^2(t, x, y), S^2(t, x, z))$ and $\hat{\sigma}_t[x(yz)] = S^2(t, x, S^2(t, y, z)).$

(i) If $i_1 = i_n - 1$ and $i_m > 2$ for all $m \in \{2, 3, \ldots, n-1\}$, then

$$\hat{\sigma}_t[(xy)(xz)] = (xxix_{i_2} \cdots x_{i_{n-1}}x_{i_n})^2x_{i_2} \cdots x_{i_{n-1}}(xxix_{i_2} \cdots x_{i_n})x_{i_2} \cdots x_{i_{n-1}}x_{i_n};$$

$$\hat{\sigma}_t[x(yz)] = xxix_{i_2} \cdots x_{i_{n-1}}x_{i_n}.$$

(ii) If $i_1 = i_n = 1$ and there exists a unique $k \in \{2, 3, \ldots, n-1\}$ such that $i_k = 2$, and $i_m > 2$ for all $m \in \{2, 3, \ldots, n-1\}, m \neq k$, then

$$\hat{\sigma}_t[(xy)(xz)] = (xxix_{i_2} \cdots x_{i_{k-1}}x_{i_k}x_{i_{k+1}} \cdots x_{i_{n-1}}x_{i_n})^2x_{i_2} \cdots x_{i_{k-1}}x_{i_k}x_{i_{k+1}} \cdots x_{i_{n-1}}x_{i_n};$$

$$\hat{\sigma}_t[x(yz)] = xxix_{i_2} \cdots x_{i_{k-1}}x_{i_k}x_{i_{k+1}} \cdots x_{i_{n-1}}x_{i_n}x_{i_k}x_{i_{k+1}} \cdots x_{i_{n-1}}x_{i_n}.$$

(iii) If $i_1 = 1$ and $i_m > 2$ for all $m \in \{2, 3, \ldots, n\}$, then

$$\hat{\sigma}_t[(xy)(xz)] = (xxix_{i_2} \cdots x_{i_n})^2x_{i_2} \cdots x_{i_n};$$

$$\hat{\sigma}_t[x(yz)] = xxix_{i_2} \cdots x_{i_n}.$$

(iv) If $i_1 = 1$ and there exists a unique $k \in \{2, 3, \ldots, n\}$ such that $i_k = 2$, and $i_m > 2$ for all $m \in \{2, 3, \ldots, n\}, m \neq k$, then

$$\hat{\sigma}_t[(xy)(xz)] = (xxix_{i_2} \cdots x_{i_{k-1}}x_{i_k}x_{i_{k+1}} \cdots x_{i_n})^2x_{i_2} \cdots x_{i_{k-1}}x_{i_k}x_{i_{k+1}} \cdots x_{i_{n-1}}x_{i_n};$$
\[
\sigma_t[x(yz)] = x_{i_1}x_{i_2} \cdots x_{i_{k-1}}x_{i_k}x_{i_{k+1}} \cdots x_{i_n}.
\]

(v) If \(i_n = 1\) and there exists a unique \(k \in \{1, 2, \ldots, n-1\}\) such that \(i_k = 2\), and \(i_m > 2\) for all \(m \in \{1, 2, \ldots, n-1\}\), \(m \neq k\), then

\[
\sigma_t[(xy)(xz)] = (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} x) x_{i_1} \cdots x_{i_{m-1}} (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} x),
\]

\[
\sigma_t[x(yz)] = xx_{i_1} \cdots x_{i_{m-1}} x.
\]

(vi) If \(i_n = 1\) and \(i_m > 2\) for all \(m \in \{1, 2, \ldots, n\}\), then

\[
\sigma_t[(xy)(xz)] = (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} x) x_{i_1} \cdots x_{i_{m-1}} (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} x),
\]

\[
\sigma_t[x(yz)] = xx_{i_1} \cdots x_{i_{m-1}} x.
\]

(vii) If there exists a unique \(k \in \{1, 2, \ldots, n\}\) such that \(i_k = 2\), and \(i_m > 2\) for all \(m \in \{1, 2, \ldots, n\}\), \(m \neq k\), then

\[
\sigma_t[(xy)(xz)] = (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} x) x_{i_1} \cdots x_{i_{m-1}} (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} x),
\]

\[
\sigma_t[x(yz)] = xx_{i_1} \cdots x_{i_{m-1}} x.
\]

(viii) If \(i_1 = i_n = 2\) and \(i_m > 2\) for all \(m \in \{2, 3, \ldots, n-1\}\), then

\[
\sigma_t[(xy)(xz)] = (xy_{i_2} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} z) x_{i_1} \cdots x_{i_{m-1}} (xx_{i_1} \cdots x_{i_{m-1}} yx_{i_m} \cdots x_{i_n-1} z),
\]

\[
\sigma_t[x(yz)] = xx_{i_1} \cdots x_{i_{m-1}} x.
\]

(ix) If \(i_m > 2\) for all \(m \in \{1, 2, \ldots, n\}\), then

\[
\sigma_t[(xy)(xz)] = xx_{i_1} \cdots x_{i_{m-1}} x.
\]

\[
\sigma_t[x(yz)] = xx_{i_1} \cdots x_{i_{m-1}} x.
\]

Using the associative law and the identities \(xy \approx xy^2, yzu \approx yzu,\) and \(xyz \approx xyz\), we have \(\sigma_t[(xy)(xz)] \approx \sigma_t[x(yz)]\). This finishes the proof. ■

The following subvariety of \(L_{big}(R_{big})\) is also left-edge(right-edge)-strongly solid: \(L_2 := \text{Mod}\{(xy)z \approx (xy)z, xy \approx xy^2, xzyt \approx xzyt\}\) \((R_2 := \text{Mod}\{(xy)z \approx (xy)z, xy \approx x^2y, xzyt \approx xzyt\}\). To show this one has only to show that the medial law \(xzyt \approx xzyt\) is a leftmost(rightmost)-strong hyperidentity by applying some generalized hypersubstitutions from proving \(L_{big}\) to the medial law and for \(R_2\) we conclude in the similar way.
References


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