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W-type and H-type Non-Associative Algebras using Additive Maps I

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Abstract

A W-type Lie algebra is defined using a map from an additive group to a field of characteristic zero in the paper (see [10]). We define Lie-admissible W-type and H-type non-associative algebras over a field of any characteristic in this work (see [9]). We show that those algebras are simple.

Keywords: simple, W-type algebra, H-type algebra, graded algebra

1 Preliminaries

It is an interesting problem to find a simple algebra (see [6]). Let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}_p$ a field of characteristic p. Throughout this paper,

$\mathbb{N}$ and $\mathbb{Z}$ will denote the non-negative integers and the integers, respectively. For a given algebra A, we can define its anti-symmetric algebra A^- using the commutator $[,]$ (see [1], [10], and [11]). The W-type non-associative algebra $NW(n)$ has the standard basis

$$\{x_1^{a_1}\cdots x_n^{a_n}\partial_u | a_1,\cdots,a_n\in\mathbb{N}, 1\le u\le n\}$$

with the multiplication on basis elements is defined as follows:

$$x_1^{a_1}\cdots x_n^{a_n}\partial_u * x_1^{b_1}\cdots x_n^{b_n}\partial_v = b_u x_1^{a_1+b_1}\cdots x_n^{a_n+b_n}x_u^{-1}\partial_v \tag{1}$$

and it can be linearly extended to $NW(n)$ where ∂_u, $1\le u\le n$, is partial derivative of $\mathbb{F}[[x_1,\cdots,x_{n+m}]]$ with respect to x_u (see [6], [7], [8], and [10]). Let us define the W-type non-associative algebra $NW(n,m)$ with the standard basis

$$\{e^{a_1x_1}\cdots e^{a_nx_n}x_1^{u_1}\cdots x_{n+m}^{u_{n+m}}\partial_k | a_1,\cdots,a_n,u_1,\cdots,u_{n+m}\in\mathbb{Z}, 1\le k\le n+m\}$$

and similarly its multiplication $*$ are defined as (1) on its basis elements. We know that the algebra $NW(n)$ is a subalgebra of $NW(n,m)$; please refer to the papers (see [1] and [10]) for more details on a W-type algebra. Let g_r and h_r, $1\le r\le n$, be additive maps from $\mathbb{Z}$ into $\mathbb{F}$. We define the non-associative algebra $NW(g_n,h_n,n)$ with the standard basis

$$B_W = \{\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}\partial_k | a_1,\cdots,a_n,i_1,\cdots,i_n\in\mathbb{Z}, 1\le k\le n\} \tag{2}$$

and the multiplication $*$ on basis elements is defined as follows:

$$\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}\partial_k * (\binom{b_1}{j_1}\cdots\binom{b_n}{j_n}\partial_r = g_k(b_k)\binom{a_1+b_1}{i_1+j_1}\cdots \binom{a_n+b_n}{i_n+j_n}\partial_r + h_k(j_k)\binom{a_1+b_1}{i_1+j_1}\cdots\binom{a_k+b_k}{i_k+j_k-1}\binom{a_{k+1}+b_{k+1}}{i_{k+1}+j_{k+1}}\cdots \binom{a_n+b_n}{i_n+j_n}\partial_r \tag{3}$$

and it can be linearly extended to the algebra $NW(g_n,h_n,n)$ (see [2], [3], [7], [8], and [10]). Also, it is not hard to show that the above bracket satisfies the Jacobi identity with respect to the commutator of the algebra (see [4] and [5]). Throughout the paper, A_r, $1\le r\le n$, denotes an additive subgroup of $\mathbb{F}$ or $\mathbb{F}_p$. If h_r, $1\le r\le n$, are inclusions, then the algebra $NW(g_n,h_n,n)$ is the algebra $NW(h_n,n)$ in the paper (see [10]). Similarly to $NW(g_n,h_n,n)$ (resp.

$NW(g_n, h_n, n)^-$), we can define the algebra $NW(g_n, h_n, n)_p$ (resp. $NW(g_n, h_n, n)^-{}_p$) over $\mathbb{F}_p$ by taking the standard basis

$$B_{W\,p} = \{\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}\partial_k | a_1, \cdots, a_n, i_1, \cdots, i_n \in \mathbb{Z}_p, 1 \leq k \leq n\} \quad (4)$$

and appropriate mappings g_r and h_r from A_r to $\mathbb{F}_p$, $1 \leq r \leq n$, where A_r, $1 \leq r \leq n$, are additive subgroups of $\mathbb{F}_p$. Similarly, for additive maps g_r and h_r, $1 \leq r \leq n$, from A_r to $\mathbb{F}$, then we are able to the H-type non-associative algebra $NH(g_{2n}, h_{2n}, 2n)$ with the standard basis

$$B_H = \{(\binom{a_{11}}{i_{11}}\cdots\binom{a_{1n}}{i_{1n}})(\binom{b_{11}}{j_{11}}\cdots\binom{b_{1n}}{j_{1n}}) | a_{11}, \cdots, b_{1n}, i_{11}, \cdots, j_{1n} \in \mathbb{Z}\}$$

and the multiplication $*$ is defined as follows:

$$\begin{aligned}
&(\binom{a_{11}}{i_{11}}\cdots\binom{a_{1n}}{i_{1n}})(\binom{b_{11}}{j_{11}}\cdots\binom{b_{1n}}{j_{1n}}) * (\binom{a_{21}}{i_{21}}\cdots\binom{a_{2n}}{i_{2n}})\\
&(\binom{b_{21}}{j_{21}}\cdots\binom{b_{2n}}{j_{2n}}) = \sum_{k=1}^{n} g_k(a_{1k})g_k(b_{2k})(\binom{a_{11}+a_{21}}{i_{11}+i_{21}}\cdots\binom{a_{1n}+a_{2n}}{i_{1n}+i_{2n}})\\
&(\binom{b_{11}+b_{21}}{j_{11}+j_{21}}\cdots\binom{b_{1n}+b_{2n}}{j_{1n}+j_{2n}}) + \sum_{k=1}^{n} g_k(a_{1k})h_k(j_{2k})(\binom{a_{11}+a_{21}}{i_{11}+i_{21}}\cdots\\
&\binom{a_{1n}+a_{2n}}{i_{1n}+i_{2n}})(\binom{b_{11}+b_{21}}{j_{11}+j_{21}}\cdots\binom{b_{1k}+b_{2k}}{j_{1k}+j_{2k}-1}\binom{b_{1,k+1}+b_{2,k+1}}{j_{1,k+1}+j_{2,k+1}}\cdots\\
&\binom{b_{1n}+b_{2n}}{j_{1n}+j_{2n}}\binom{b_{1n}+b_{2n}}{j_{1n}+j_{2n}})
\end{aligned}$$

$$\begin{aligned}
+ \quad & \sum_{k=1}^{n} h_k(i_{1k})g_k(b_{2k})(\binom{a_{11}+a_{21}}{i_{11}+i_{21}}\cdots\binom{a_{1k}+a_{2k}}{i_{1k}+i_{2k}-1}\\
& \binom{a_{1,k+1}+a_{2,k+1}}{i_{1,k+1}+i_{2,k+1}}\cdots\binom{a_{1n}+a_{2n}}{i_{1n}+i_{2n}})(\binom{b_{11}+b_{21}}{j_{11}+j_{21}}\cdots\binom{b_{1n}+b_{2n}}{j_{1n}+j_{2n}})\\
+ \quad & \sum_{k=1}^{n} h_k(a_{1k})h_k(j_{2k})(\binom{a_{11}+a_{21}}{i_{11}+i_{21}}\cdots\binom{a_{1k}+a_{2k}}{i_{1k}+i_{2k}-1}\\
& \binom{a_{1,k+1}+a_{2,k+1}}{i_{1,k+1}+i_{2,k+1}}\cdots\binom{a_{1n}+a_{2n}}{i_{1n}+i_{2n}})(\binom{b_{11}+b_{21}}{j_{11}+j_{21}}\cdots\binom{b_{1k}+b_{2k}}{j_{1k}+j_{2k}-1}\\
& \binom{b_{1,k+1}+b_{2,k+1}}{j_{1,k+1}+j_{2,k+1}}\cdots\binom{b_{1n}+b_{2n}}{j_{1n}+j_{2n}}\binom{b_{1n}+b_{2n}}{j_{1n}+j_{2n}}) \qquad (5)
\end{aligned}$$

It is not hard to check that the Jacobi identity holds with respect to the commutator of the algebra $H(g_{2n}, h_{2n}, 2n)$ (see [10]). Thus the algebra is

Lie admissible. Since $NH(g_{2n}, h_{2n}, 2n)$ has the 1-dimensional maximal ideal $< 1 >$ generated by 1, we define the quotient algebra $\overline{NH(g_{2n}, h_{2n}, 2n)} = \frac{NH(g_{2n},h_{2n},2n)}{<1>}$. From now on, all the maps g_k and h_l, $1 \leq k, l \leq n$, will be injective for the simplicity of the algebra $\overline{H(g_{2n}, h_{2n}, 2n)}$. Similarly, we can define the anti-symmetric algebra $\overline{H(g_{2n}, h_{2n}, 2n)}^-$. Similarly to $NH(g_{2n}, h_{2n}, 2n)$ (resp. $NH(g_{2n}, h_{2n}, 2n)^-$), we can define the algebra $NH(g_{2n}, h_{2n}, 2n)_p$ (resp. $NH(g_{2n}, h_{2n}, 2n)_p{}^-$) over $\mathbb{F}_p$ with the standard basis

$$B_{H_p} = \{(\binom{a_{11}}{i_{11}} \cdots \binom{a_{1n}}{i_{1n}})(\binom{b_{11}}{j_{11}} \cdots \binom{b_{1n}}{j_{1n}}) \mid a_{11}, \cdots, b_{1n}, i_{11}, \cdots, j_{1n} \in \mathbb{Z}_p\} \quad (6)$$

and appropriate mappings g_r and h_r from A_r to $\mathbb{F}_p$, $1 \leq r \leq n$. Throughout the paper, given maps g_r and h_r will be additive and injective maps.

2 Simplicity of $NW(g_n, h_n, n)$

The algebra $NW(g_n, h_n, n)$ is $\mathbb{Z}^n$-graded as follows:

$$NW(g_r, h_r, n) = \bigoplus_{(a_1, \cdots, a_n) \in \mathbb{Z}^n} NW_{(a_1, \cdots, a_n)} \quad (7)$$

where $NW_{(a_1,\cdots,a_n)}$ is the vector subspace of $NW(g_n, h_n, n)$ with the standard basis

$$\bar{B} = \{\binom{a_1}{i_1} \cdots \binom{a_n}{i_n} \partial_u | a_1, \cdots, a_n, i_1, \cdots, i_n \in \mathbb{Z}, 1 \leq u \leq n\}$$

Let $NW_{(a_1,\cdots,a_n)}$ denote the $(a_1, \cdots, a_n)$-homogeneous component of $NW(g_n, h_n, n)$ and refer to the elements in $NW_{(a_1,\cdots,a_n)}$ as $(a_1, \cdots, a_n)$-homogeneous elements. Note that the $(0, \cdots, 0)$-homogeneous component is isomorphic to the algebra $NW(n)$. From now on the $(0, \cdots, 0)$-homogeneous component will be called the 0-homogeneous component. Using the lexicographic order on the set $\mathbb{Z}^{2n} \times \{1, \cdots, n\}$, we introduce the order $>_o$ of basis elements of $NW(g_n, h_n, n)$ as follows : for any two basis elements $\binom{a_1}{i_1} \cdots \binom{a_n}{i_n} \partial_u$ and $\binom{b_1}{j_1} \cdots \binom{b_n}{j_n} \partial_k$ of $NW(g_n, h_n, n)$, we define the order $>_o$ as follows: if $(a_1, \cdots, a_n, i_1, \cdots, i_n, \partial_u) > (b_1, \cdots, b_n, j_1, \cdots, j_n, \partial_k)$, then

$$\binom{a_1}{i_1} \cdots \binom{a_n}{i_n} \partial_u >_o \binom{b_1}{j_1} \cdots \binom{b_n}{j_n} \partial_k$$

where $\partial_u >_o \partial_k$ if $u > k$. Thus we can naturally define the order $>_o$ on $NW(g_n, h_n, n)$. For any element $l \in NW(g_n, h_n, n)$, l can be written as follows using the order and the gradation:

$$l = \sum_{i_1,\cdots,i_n,p} C(i_1,\cdots,i_n,p)\binom{a_{11}}{i_1}\cdots\binom{a_{1n}}{i_n}_p + \cdots$$

$$+ \sum_{j_1,\cdots,j_n,q} C(j_1,\cdots,j_n,q)\binom{a_{t1}}{j_1}\cdots\binom{a_{tn}}{j_n}_q$$

where $C(i_1,\cdots,i_n,p),\cdots, C(j_1,\cdots,j_n,q) \in \mathbb{F}$, $1 \le p,\cdots,q \le n$. Next, we define the string number $st(l)$ of l as the number of distinct homogeneous components of l (see [5, 6]), and $l_p(l)$ as $max\{i_1,\cdots,i_n,\cdots,j_1,\cdots,j_n\}$. For any basis element $\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}_r$ in $\bar{B}$, let us refer to $a_1,\cdots,a_n$ as the upper indices and $i_1,\cdots,i_n$ as the lower indices of $\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}_r$.
Remark 2.1 If g_r and h_r, $1 \le r \le n$, are inclusions, then $NW(g_n, h_n, n)^-$ is

the anti-symmetric (i.e., Lie) algebra which is studied in the paper (see [8]). □

Lemma 1 *For a non-zero element l of $NW(g_n, h_n, n)$, the ideal $< l >$ generated by l contains an element such that its all the lower indices are positive.*

Proof. Let l be a nonzero element of $NW(g_n, h_n, n)$ such that $\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}_u$ is the maximal term of l with respect to the order $>_o$. We can assume that either a_1 or i_1 is non-zero scalar. If we take an element $l_1 = \binom{0}{j_1}\cdots\binom{0}{j_n}_t$ such that $j_1 >> \cdots >> j_n >> 0$, then $l_1 * l \neq 0$ is the required element of the lemma where $a >> b$ means a is sufficiently larger than b. □

Lemma 2 *Let I be an ideal of the algebra $NW(g_n, h_n, n)$. If I contains an element $\binom{0}{0}\cdots\binom{0}{0}_u$, $1 \le u \le n$, then $I = NW(g_n, h_n, n)$.*

Proof. Since $NW(n)$ is a simple subalgebra $NW(g_n, h_n, n)$ of the algebra $NW(g_n, h_n, n)$, for any element $l \in NW(n)$, the ideal $< l >$ contains $NW(n)$ where $< l >$ is an ideal of $NW(g_n, h_n, n)$ and it is generated by the element l. Let I be a non-zero ideal of $NW(g_n, h_n, n)$ in the lemma. Let $\binom{0}{j_1}\cdots\binom{0}{j_n}_t$ be the element in the lemma. Thus the ideal $< \binom{0}{j_1}\cdots\binom{0}{j_n}_t >$ of $NW(g_n, h_n, n)$ contains the algebra $NW(n)$ where $< \binom{0}{j_1}\cdots\binom{0}{j_n}_t >$ is the ideal generated by the element $\binom{0}{j_1}\cdots\binom{0}{j_n}_t$. Let $\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_u$ be any basis element of in the standard basis of $NW(g_n, h_n, n)$. Without loss of generality, we can assume that $a_1 \neq 0$. If $i_1 = 0$, the we have that

$$\binom{0}{m}\binom{0}{0}\cdots\binom{0}{0}\partial_1 * \binom{a_1}{0}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t$$
$$= g_1(a_1)\binom{a_1}{m}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t$$

Since $g_1(a_1) \neq 0$, we have that $\binom{a_1}{0}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t \in I$. This implies that by induction on $m \in \mathbb{N}$ of $\binom{a_1}{m}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t$, we can prove that $\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t \in I$ where $i_1 \in \mathbb{N}$. Similarly, by induction on $m \in \mathbb{Z}$ of $\binom{a_1}{m}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t$, we can also prove that $\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}\partial_t \in I$ where $i_1 \in \mathbb{Z}$. Therefore we have proven the lemma. □

Theorem 1 *The algebra $NW(g_n, h_n, n)$ is simple.*

Proof. Let I be any non-zero ideal of $NW(g_n, h_n, n)$. Note that $NW(g_n, h_n, n)$ is $\mathbb{Z}^n$-graded and we have the following non-zero element in $NW_{0,\cdots,0}$

$$\binom{a_1}{i_1}\cdots\binom{a_n}{i_n}\partial_r * \binom{-a_1}{j_1}\cdots\binom{-a_n}{j_n}\partial_s \in NW_{0,\cdots,0} \tag{8}$$

where $i_1, \cdots, i_n, j_1, \cdots, j_n \in \mathbb{Z}$ and $NW_{0,\cdots,0}$ is the $(0, \cdots, 0)$-homogeneous component of $NW(g_n, h_n, n)$. Let l be a non-zero element of I. Let us prove the theorem by induction on $st(l)$ of the element l. If $st(l) = 1$, then there is nothing to prove by (8) and Lemma 2. If $st(l) = k \geq 1$, then by induction we can assume that $NW(g_n, h_n, n) =< l >= I$. Let us assume that $st(l) = k+1$. Either l has a term in the $NW_{(0,\cdots,0)}$ homogeneous component or not, by Lemma 2 and (8), we have an element l_1 of $< l >$ such that $st(l_1) < k+1$. This implies that by induction, $NW(g_n, h_n, n) =< l_1 >\subset I$, i.e., $NW(g_n, h_n, n) = I$. This implies that $NW(g_n, h_n, n)$ is simple. Therefore we have proven the theorem. □

Corollary 1 *The algebra $NW(n, 0)$ is simple.*

Proof. If we take additive embedding $g_r, \quad h_r : \mathbb{Z}$ to $\mathbb{F}$, $1 \leq r \leq n$, then we get the required results of the corollary (see [1] and [10]). □

Corollary 2 *The anti-symmetric algebra $NW(g_n, h_n, n)^-$ is simple.*

Proof. The proof of the well known (see [10]), but it is also straightforward by the similar proof of Theorem 1. So it is omitted. □

Theorem 2 *The algebra* $NW(g_n, h_n, n)_p$ *(resp.* ${NW(g_n, h_n, n)_p}^-$*) is simple.*

Proof. The proof of the theorem is similar to the proof of Theorem 1 and Corollary 1, so omitted. □

It is an interesting problem to find all the automorphisms of the subalgebra $NW(h, 1)$ of the algebra $NW(h_n, n)$.

Theorem 3 *For any algebra automorphism* $\theta \in Aut(NW(h, 1))$*, we have that*

$$\theta(\binom{0}{1}_1) = \sum_j C_j \binom{0}{j}_1$$

where $C_j \in F$.

Proof. Let θ be the automorphism of $NW(h, 1)$ in the lemma. Note that for any elements $l_1 \in NW_a$ and $l_2 \in NW_b$ such that $a \neq b$, we have that $l_1 * l_2 \neq 0$. Since $\binom{0}{1}_1$ is a right identity element of NW_0 with respect to the standard basis, the theorem follows from the above comment and the fact that θ is a homomorphism. □

It is easy to prove that there is an injective algebra homomorphism θ of $NW(g, h, 1)$ such that θ is not surjective. Thus every non-zero endomorphism of $NW(g, h, 1)$ is not an automorphism of $NW(g, h, 1)$. An element $l \in NW(g_n, h_n, n)$ is left stable (resp. right stable) with respect to a basis B of $NW(g_n, h_n, n)$, if $l * m = \alpha(m)m$ (resp. $m * l = \alpha(m)m$) for all $m \in B$ where $\alpha(m)$ is a scalar depending on l and m.

3 H-type Algebras

In this section, we prove that the algebra $\overline{NH(g_{2n}, h_{2n}, 2n)}$ is simple. The Lie algebra $\overline{NH(g_{2n}, h_{2n}, 2n)}$ is $\mathbb{Z}^{2n}$-graded as follows:

$$\overline{NH(g_{2n}, h_{2n}, 2n)} = \bigoplus_{(a_1,\cdots,a_n,b_1,\cdots,b_n)\in\mathbb{Z}^{2n}} NH_{(a_1,\cdots,a_n,b_1,\cdots,b_n)}$$

where $NH_{(a_1,\cdots,a_n,b_1,\cdots,b_n)}$ is the $(a_1, \cdots, a_n, b_1, \cdots, b_n)$-homogeneous component of $\overline{NH(g_{2n}, h_{2n}, 2n)}$ as $NW(h_p, g_p, n)$ (see Section 2). Using the commutator of $NH(g_{2n}, h_{2n}, 2n)$, we are able to define the anti-symmetric algebra $NH(g_{2n}, h_{2n}, 2n)^-$ and its subalgebra $\overline{NH(g_{2n}, h_{2n}, 2n)}^-$. Note that these anti-symmetric algebras are Lie algebras.

Lemma 3 *If* $l \in \overline{NH(g_{2n}, h_{2n}, 2n)}$ *is a non-zero element, then the ideal* $< l >$ *generated by the element* l *contains an element* l_1 *whose lower indices are positive integers.*

Proof. Let l be the element in the lemma. The element

$$l * (\binom{0}{s_1}\binom{0}{s_2}\cdots\binom{0}{s_n})(\binom{0}{t_1}\cdots\binom{0}{t_n})$$

is the required non-zero element in the lemma, where we took $s_1 >> \cdots >> s_n >> t_1 >> \cdots >> t_n >> 0$ and $a >> b$ means that a is a sufficiently large number than b. Therefore, we have proven the lemma. $\square$

Lemma 4 *If an ideal* I *of* $\overline{NH(g_{2n}, h_{2n}, 2n)}$ *contains* $\delta_1 := (\binom{0}{1}\binom{0}{0}\cdots\binom{0}{0})(\binom{0}{0}\cdots\binom{0}{0})$ *or,* $\cdots$ *, or,* $\delta_{2n} := (\binom{0}{0}\cdots\binom{0}{0})(\binom{0}{0}\cdots\binom{0}{0}\binom{0}{1})$*, then* $I = \overline{NH(g_{2n}, h_{2n}, 2n)}$.

Proof. Let I be the ideal of $\overline{NH(g_{2n}, h_{2n}, 2n)}$ in the lemma. Without loss of generality, we can assume that I contains $\delta_1 = (\binom{0}{1}\binom{0}{0}\cdots\binom{0}{0})(\binom{0}{0}\cdots\binom{0}{0})$. Since g_i, and h_i are one to one mappings, $1 \leq i \leq n$, we have that

$$\{(\binom{0}{i_1}\binom{0}{i_2}\cdots\binom{0}{i_n})(\binom{0}{j_1}\cdots\binom{0}{j_n})|i_1, \cdots, i_n, j_1, \cdots, j_n \in \mathbb{Z}\} \subset I.$$

Let us take an element $\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n}(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n})$ of $\overline{NH(g_{2n}, h_{2n}, 2n)}$ such that $b_1 \neq 0$. Then we have that

$$\begin{aligned} & \delta_1 * (\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n}) \\ = \; & g_{n+1}(b_1)(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n}) \\ + \; & h_{n+1}(j_1)(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1-1}\cdots\binom{b_1}{j_n}) \in I \end{aligned} \tag{9}$$

By taking $j_1 = 0$ in (9), we also have that

$$(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{0}\cdots\binom{b_1}{j_n}) \in I$$

for all $a_1, \cdots, a_n, b_1, \cdots, b_n, i_1, \cdots, i_n, j_2, \cdots, j_n \in \mathbb{Z}$. Let us prove this lemma by induction on j_1 of the element $(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n})$. For j_1, we can assume that $(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1-1}\cdots\binom{b_1}{j_n}) \in I$. This implies that the element $(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n})$ is in the ideal I from (9). For $j_1 < 0$, if

we assume that $(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})\,(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n})$ is in the ideal I, then by (9), we can prove that $(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1-1}\cdots\binom{b_1}{j_n}) \in I$. Therefore for any $j_1 \in \mathbb{Z}$, we have that
$\{(\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_n}{i_n})(\binom{b_1}{j_1}\cdots\binom{b_1}{j_n})\,|a_1,\cdots,a_n,b_1,\cdots,b_n,i_1,\cdots,i_n,j_2,\cdots,j_n \in \mathbb{Z}\}$ is a subset of I. This implies that all the basis elements of $\overline{NH(g_{2n},h_{2n},2n)}$ are in the ideal I. This implies $I = \overline{NH(g_{2n},h_{2n},2n)}$. Therefore we have proven the lemma. □

Theorem 4 *The algebra* $\overline{NH(g_{2n},h_{2n},2n)}$ *is simple.*

Proof. Let I be a non-zero ideal of $\overline{NH(g_{2n},h_{2n},2n)}$. Let l be a non-zero element of I. Note that every element l of $\overline{NH(g_{2n},h_{2n},2n)}$ can be written as the terms in its appropriate components using the order of the algebra. Let us prove this theorem by induction on $st(l)$ of the element l. Let us assume that l is a non-zero element in I such that $st(l)=1$. If l is the sum of terms in the $(0,\cdots,0)$-homogeneous component, then we have proven the theorem by Lemma 5. If l is the sum of terms in the $(a_1,\cdots,a_{2n})$-homogeneous component, then

$$(\binom{-a_1}{t_1}\binom{-a_2}{0}\cdots\binom{-a_n}{0})(\binom{-a_{n+1}}{0}\cdots\binom{-a_{2n}}{0}) * l$$

is in I such that l is the sum of terms in $(0,\cdots,0)$-homogeneous component. Since I contains a term of $(0,\cdots,0)$-homogeneous component, we are able to prove that $I = \overline{NH(g_{2n},h_{2n},2n)}$. Therefore, we have proven the theorem. Inductively, we can assume that if l is the sum of terms in the (k) homogeneous components, then $I = \overline{NH(g_{2n},h_{2n},2n)}$. Assume l is the sum of elements in the $(k+1)$ homogeneous components. If l has a term in the $(0,\cdots,0)$-homogeneous component and a_j, $1 \le j \le n$, is the first non-zero upper index and p be the maximal first lower index. We have the following non-zero element

$$l_3 = (\cdots(l * \delta_{j+n})*)\delta_{j+n} * \cdots * \delta_{j+n}) \tag{10}$$

and it is in I where we applied $(p+1)$-times of the multiplication in (10). Then the element l_3 is in I such that it has terms in the (k) homogeneous components. Therefore, we have proven the theorem by induction. If l has no term in the $(0,\cdots,0)$-homogeneous component and it has a term in the $(a_1,\cdots,a_{2n})$-homogeneous component, then the element

$$l_4 = (\binom{-a_1}{0}\binom{-a_2}{0}\cdots\binom{-a_n}{0})(\binom{-a_{n+1}}{0}\cdots\binom{-a_{2n}}{0}) * l \tag{11}$$

is a non-zero element in I and l_4 has terms in at most $(k+1)$ homogeneous components. It may have a term in the $(0,\cdots,0)$-homogeneous component or not. But we have already proven theorem on that case. Therefore, we have proven the theorem. □

Corollary 3 *The anti-symmetric algebra* $\overline{NH(g_{2n}, h_{2n}, 2n)}^{-}$ *is simple.*

Proof. The proof of the well known (see [8]), but it is straightforward by Theorem 3. □

Theorem 5 *The algebra* $\overline{NH(g_{2n}, h_{2n}, 2n)_p}$ *(resp.* $\overline{NH(g_{2n}, h_{2n}, 2n)_p}^{-}$ *) is simple.*

Proof. The proof of theorem is similar to the proof of Theorem 3 (resp. Corollary 3), so it is omitted. □

By reviewing the proofs of Theorem 2 and Theorem 4-5, the algebras $NW(g_p, h_p, n)$, $NW(g_p, h_p, n)_p$, $\overline{NH(g_{2n}, h_{2n}, 2n)}$ and $\overline{NH(g_{2n}, h_{2n}, 2n)_p}$ have a class of simple subalgebras by taking appropriate additive monoids instead of integers or $\mathbb{Z}_p$ on lower indices in the definitions of the given appropriate algebras.

4 Open problems on the algebras in the paper

This is a good place to suggest some open problems on the simple Lie algebras which are defined in this paper to compare with properties of the well-known Lie algebras.The following is related to the Jacobian conjecture on the W-type algebra $NW(g_p, h_p, n)$, $n > 1$.

Question 1. Is every non-zero endomorphism of $NW(g_p, h_p, n)$ (resp. $\overline{NH(g_{2n}, h_{2n}, 2n)}$) is an automorphism of $NW(g_p, h_p, n)$ (resp. $\overline{NH(g_{2n}, h_{2n}, 2n)}$)? □

The answer is no. Since every non-zero endomorphism of $NW(g_p, h_p, n)$ (resp. $\overline{NH(g_{2n}, h_{2n}, 2n)}$) is injective, it is enough to show that every non-zero endomorphism is surjective for the Jacobi conjecture on $NW(g_p, h_p, n)$ (resp. $\overline{NH(g_{2n}, h_{2n}, 2n)}$).

Question 2. Find the automorphism group $Aut(NW(g_p, h_p, n))$ (resp. $Aut(\overline{NH(g_{2n}, h_{2n}, 2n)})$) of the algebra $NW(g_p, h_p, n)$ (resp. $\overline{NH(g_{2n}, h_{2n}, 2n)}$). □

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