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The Weak Version of Huppert's ρ-σ-Conjecture

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Abstract

Suppose that G is a finite group, we prove that if $G/F(G)$ is solvable of odd order or supersolvable, then $|\rho(G)| \leq 3\sigma(G)$.

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1 INTRODUCTION

Let G be a finite solvable group and $Irr(G)$ denote the set of irreducible complex characters of G. Write $\rho(\chi)$ or $\pi(\chi(1))$ to denote the set of prime divisors of degree of character χ. And define $\rho(G) := \{\rho(\chi) | \chi \in Irr(G)\}$ and $\sigma(G) := max\{|\rho(\chi)| \mid \chi \in Irr(G)\}$. Huppert's conjecture claims that $|\rho(G)| \leq 2\sigma(G)$. Espuelas prove [3, Theorem 4.1] that the conjecture is true if G is of odd order and every normal Sylow subgroup is abelian. In fact, it is easily proved via [8, Lemma 18.1] that the conjecture is valid for a nilpotent-by-nilpotent group. Pálfy [10] verifies the conjecture when the degree graph is disconnected. Recently, Moretó prove [9] a weak version of the conjucture that the inequality $\rho(G) \leq 4\sigma(G)^2 + 6.5\sigma(G) + 13$ holds for any finite group.

Gluck and Manz [4] prove that $|\rho(G)| \leq 3\sigma(G)+32$; and the result is slightly improved that $|\rho(G)| \leq 3\sigma(G) + 2$ (for example, see [8, Corollary 17.8]). If the solvablity hypothesis is removed, it is proved via the classification theorem of finite simple groups that if G is a simple group, then $|\rho(G)| \leq 3\sigma(G)$ (see [7, 1]). In this note, we prove that if $G/F(G)$ is solvable of odd order or supersolvable, then $|\rho(G)| \leq 3\sigma(G)$.

In the following, we list some of useful facts. $\pi(G)$ denotes the set of prime divisors of the order of G.

Lemma 1.1. *Let $F(G)$ be the Fitting subgroup of a solvable group G, and $\mathcal{S}(G) = \{p \mid O_p(G) \in Syl_p(G)$ nonabelian$\}$, then $\rho(G) = \pi(G/F(G)) \dot{\cup} \mathcal{S}(G)$, where $\dot{\cup}$ denotes the disjointed union.*

Proof. See the proof of [8, Theorem 17.7] □

Lemma 1.2. *Assume that A acts on an abelian group V. Let $V^* = Irr(V)$, then*

1. *The action of A on V is faithful if and only if the action of A on V^* is faithful;*
2. *The action of A on V is irreducible if and only if the action of A on V^* is irreducible.*

Proof. This is Proposition 12.1 of [8]. □

Theorem 1.3. *Assume that G is a solvable group of odd order and V is a faithful and completely reducible G-module, then there exist $v, w \in V$ such that $C_G(v) \cap C_G(w) = 1$.*

Proof. This is [2, Theorem 3]. □

2 MAIN RESULTS

Theorem 2.1. *Suppose that $G/F(G)$ is solvable of odd order where $F(G)$ denotes Fitting subgroup of G, then $|\rho(G)| \leq 3\sigma(G)$.*

Proof. We firstly prove that there are two characters $\lambda, \chi \in Irr(G)$ such that $\pi(G/F(G)) = \pi(\chi(1)) \cup \pi(\lambda(1))$. Gaschütz's theorem 1.12 of [8] and Lemma 1.2 imply that $Irr(F(G)/\Phi(G))$ is a faithful and completely reducible $\overline{G} = G/F(G)$-module, let

$$Irr(F(G)/\Phi(G)) = V = V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

V_i's are faithful and irreducible $G/C_G(V_i)$-module, where $C_G(V_i) = \{v \in V_i | \ vg = v \text{ for any } g \in G\}$. Using Theorem 1.3 and Clifford's theorem 6.11 of [5], there exist $\chi = \theta^G \in Irr(G)$ with $\theta \in Irr(C_G(v))$, $\lambda = \eta^G$ with $\eta \in Irr(C_G(w))$, and $C_G(v) \cap C_G(w) = F(G)$, where $v, w \in V$. Also since

$$\begin{aligned} |G/F(G)| &= |G : C_G(v) \cap C_G(w)| \\ &= |G : C_G(v)||C_G(v) : C_G(v) \cap C_G(w)| \\ &= |G : C_G(v)||C_G(v)C_G(w) : C_G(w)| \end{aligned}$$

we get that $\pi(G/F(G)) = \pi(\chi(1)) \cup \pi(\lambda(1))$, as required.

Now if $\mathcal{S}(G)$ is empty, then the desired result follows from Lemma 1.1. Otherwise, let N be a normal Hall subgroup of nilpotent such that $\pi(N) = \mathcal{S}(G)$. It is easy to find $\alpha \in Irr(G)$ such that $\pi(N) = \pi(\alpha(1))$ by Theorem 6.2 of [5], again Lemma 1.1 shows $|\rho(G)| \leq 3\sigma(G)$, the proof is of complete. □

Theorem 2.2. *Suppose that $G/F(G)$ is supersolvable, where $F(G)$ denotes Fitting subgroup of G, then $|\rho(G)| \leq 3\sigma(G)$.*

Proof. Mimic the proof of Theorem 2.1, note that replace Theorem 1.3 in the proof of Theorem 2.1 with [11, Theorem A], which asserts that a supersolvable group has a semi-regular orbit in its faithful and completely reducible module. □

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