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On Semigroups whose Bi-ideals are Strongly Prime[1]

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Abstract

Let S be a semigroup with a zero element. A subsemigroup B of S is called a bi-ideal if $BSB \subseteq B$. A bi-ideal B of S is said to be strongly prime if for any bi-ideals C and D of S, $CD \cap DC \subseteq B$ implies that either $C \subseteq B$ or $D \subseteq B$. The characterizations for the semigroups whose bi-ideals are all strongly prime are to be given.

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1 Introduction

Throughout this paper, all semigroups are assumed to have a zero element. For the terminology and notation without explanation, the reader is referred to Howie [5].

Let S be a semigroup. A subsemigroup B of S is called *a bi-ideal* if $BSB \subseteq B$. Then, of course, each bi-ideal of S contains the zero element 0. A bi-ideal B of a semigroup S is said to be *semiprime* if it satisfies the condition

$$(\forall C \in \mathsf{B}(S)) \qquad C^2 \subseteq B \Longrightarrow C \subseteq B.$$

A bi-ideal B of a semigroup S is said to be *prime* if it satisfies the condition

$$(\forall C, D \in \mathsf{B}(S)) \qquad CD \subseteq B \Longrightarrow (C \subseteq B) \vee (D \subseteq B).$$

A bi-ideal B of a semigroup S is said to be *strongly prime* if it satisfies the condition

$$(\forall C, D \in \mathsf{B}(S)) \qquad \emptyset \neq CD \cap DC \subseteq B \Rightarrow (C \subseteq B) \vee (D \subseteq B).$$

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The set of all [semiprime, prime and strongly prime] bi-ideals of S is denoted by $\mathsf{B}(S)$ [$\mathsf{SePB}(S)$, $\mathsf{PB}(S)$ and $\mathsf{StPB}(S)$]. It is clear that

$$\mathsf{StPB}(S) \subseteq \mathsf{PB}(S) \subseteq \mathsf{SePB}(S) \subseteq \mathsf{B}(S).$$

The notion of a bi-ideal is a generalization of the notation of a (left and right) ideal. Steinfeld [9] characterized the semigroups whose bi-ideals are all semiprime. Li and He [7] described the semigroups whose bi-ideals are all prime. The investigation of strongly prime bi-ideals is firstly proposed by Shabir and Kanwalf[2]. The aim of this paper is to determined the semigroups whose bi-ideals are all strongly prime.

2 Preliminaries

Let S be a semigroup. For any $a \in S$, since the intersection of a family bi-ideals of S is also a bi-ideal, the intersection $(a)_b$ of all bi-ideals of S containing a is the minimum bi-ideal of S containing a. We call $(a)_b$ *the principle bi-ideal* of S generated by a. If a is a group element of S, then the $\mathcal{H}$-class of S containing a forms a group. We denote the identity of the subgroup H_a of S by a^0, and denote the inverse of a in the group H_a by a^{-1}. The following lemmas are to be used.

Lemma 2.1 [1] *Let S be a semigroup and $a \in S$. Then $(a)_b = \{a\} \cup aS^1a$. In particular, if a is a regular element, then $(a)_b = aSa$.*

It is a routine matter to show that, with respect to the operation

$$(\forall A, B \in \mathsf{B}(S)) \qquad A \circ B = \{ab | a \in A, b \in B\},$$

the set $\mathsf{B}(S)$ forms a semigroup $(\mathsf{B}(S), \circ)$, which is called *the semigroup of bi-ideals of S* (see Miccoli [6]). The following results due to Miccoli [6].

Lemma 2.2 [6] *If S is an orthogroup, then $(\mathsf{B}(S), \circ)$ is isomorphic to $(\mathsf{B}(E(S)), \circ)$.*

Lemma 2.3 [6] *A semigroup S is a Clifford semigroup (i.e., an orthogroup with semilattice of idempotents) if and only if $(\mathsf{B}(S), \circ)$ is a semilattice.*

In the sequel, by "a band $B = [Y; B_\alpha]$" we mean that "B is a band with the greatest semilattice decomposition $[Y; B_\alpha]$". A band B is called *a pure band* (see Cvetko-Vah [2]) if it satisfies the condition

$$(\forall e, f \in B) \qquad (efe = e) \vee (fef = f).$$

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[2] M. Shabir and N. Kanwalf, Prime bi-ideals in semigroups, unpublished.

A band B is called *a N-covered band* (see He [4]) if the minimum semilattice congruence N on B is a covering congruence, i.e.,

$$(\forall e, f \in B) \qquad eN^{\natural} < fN^{\natural} \Longrightarrow e < f.$$

A band B is called *a LR-band* (see Guo, Shum and Sen [3]) if it satisfies the condition

$$(\forall e, f \in B) \qquad (efe = ef) \vee (efe = fe).$$

A band B is called *a quasi-chain* (see Li and He [7]) if it satisfies the condition

$$(\forall e, f \in B) \qquad (ef = e) \vee (ef = f).$$

The following two results come from Li and He [7].

Lemma 2.4 [7] *Let $B = [Y; B_\alpha]$ be a band. Then the following statements are true:*

(1) *B is a pure band if and only Y is a chain;*

(2) *B is a N-covered pure band if it satisfies the condition*

$$(\forall e, f \in B) \qquad (efe = e) \vee (efe = f);$$

(3) *B is a LR-band if and only if each B_α is either a left or a right zero band;*

(4) *B is a quasi-chain if and only if it is a N-covered, pure LR-band.*

Lemma 2.5 [7] *Let S be a semigroup. Then the following statements are equivalent:*

(1) *$\mathsf{PB}(S) = \mathsf{B}(S)$;*

(2) *for any $a, b \in S$, either $a \in (a)_b(b)_b$ or $b \in (a)_b(b)_b$;*

(3) *S is a orthogroup whose band of idempotents is a quasi-chain.*

3 The main result

We now give the main theorem of this paper as bellow.

Theorem 3.1 *Let S be a semigroup. Then the following statements are equivalent:*

(1) *$\mathsf{StPB}(S) = \mathsf{B}(S)$;*

(2) *S is a chain of groups;*

(3) $(\mathsf{B}(S), \circ)$ *is a chain;*

(4) $(\mathsf{B}(S), \circ)$ *is a band and the partial ordered set* $(\mathsf{B}(S), \subseteq)$ *is a chain.*

Moreover, if one of the above statements is true, then

$$(\mathsf{B}(E(S)), \circ) \cong (\mathsf{B}(S), \circ) \cong (\mathsf{B}(S), \subseteq).$$

Proof. (1)⇒(2). We first assume that the statement (1) holds. Then, of course, $\mathsf{PB}(S) = \mathsf{B}(S)$. It follows from Lemma 2.5 that S is a orthogroup whose band $E(S)$ of idempotents is a quasi-chain. Suppose that the greatest semilattice decomposition of S is $[Y; S_\alpha]$. Then, Y is a chain and each S_α is either a left or a right group. Furthermore, for any idempotent e of S contained in some S_α, since $E(S)$ is a N-covered band, by using Lemma 2.1, we can easily see that

$$(e)_b = eSe = (\cup_{\beta<\alpha} S_\beta) \cup H_e.$$

Let α be an arbitrary element of Y. If S_α contains two distinct idempotents e, f, then α is not the minimum element of Y and

$$\begin{aligned}(e)_b \circ (f)_b \cap (f)_b \circ (e_b) &= \begin{cases} (e)_b \cap (f)_b & \text{if } S_\alpha \text{ is a left group,} \\ (f)_b \cap (e)_b & \text{if } S_\alpha \text{ is a right group} \end{cases} \\ &= \cup_{\beta<\alpha} S_\beta.\end{aligned}$$

Since neither $(e)_b$ nor $(f)_b$ is contained in $\cup_{\beta<\alpha} S_\beta$, we claim that $\cup_{\beta<\alpha} S_\beta \notin \mathsf{StPB}$. This is contrary to the assumption that $\mathsf{StPB}(S) = \mathsf{B}(S)$. Thus, S_α contains a unique idempotent. It follows that S_α is a group, and hence S is a chain of groups.

(2)⇒(3). Let S be a chain of groups. Then it follows from Lemma 2.3 that $(\mathsf{B}(S), \circ)$ is a semilattice. For any $a \in S_\alpha$ $(\alpha \in Y)$, since

$$(a)_b = aSa \supseteq aa^{-1}Sa^{-1}a = a^0Sa^0 \supseteq a^0aSaa^0 = aSa = (a)_a,$$

we claim that

$$B(a) = B(a^0) = a^0Sa^0 = \cup_{\beta\leq\alpha} S_\beta. \tag{3.1}$$

Thereby, for any bi-ideal B of S, we have

$$B = \cup_{a\in B}(a)_b = \cup_{a\in B}(a^0)_b = \cup_{e\in B\cap E(S)}(e)_b. \tag{3.2}$$

Let A and B be two arbitrary bi-ideals of S such that $A \backslash B \neq \emptyset$. Take a from $A\backslash B$. For any $b \in B$, if $b^0 \geq a^0$, then it follows by equation (3.1) that

$$(a)_b \subseteq (a^0)_b \subseteq (b^0)_b = (b)_b \subseteq B.$$

This is contrary to that $a \notin B$. Thus $b^0 < a^0$ so that $b \in A$. This yields that $B \subseteq A$, and hence $A \circ B = B$. Now, we claim that $(\mathsf{B}(S), \circ)$ is a chain. Furthermore, it follows from Lemma 2.2 that $(\mathsf{B}(S), \circ) \cong (\mathsf{B}(E(S)), \circ)$.

(3)⇒(4). Assume that $(\mathsf{B}(S), \circ)$ is a chain. Then, of course, $(\mathsf{B}(S), \circ)$ is a band. Furthermore, by Lemma 2.3, we can see that S is a Clifford semigroup. Suppose that the greatest semilattice decomposition of S is $[Y; G_\alpha]$. If Y is not a chain, then there exist α, β in Y such that $\alpha \nleq \beta$ and $\beta \nleq \alpha$. Then $A = \cup_{\gamma \leq \alpha} G_\gamma$ and $B = \cup_{\gamma \leq \beta} G_\gamma$ are bi-ideals of S such that $A \circ B = \cup_{\gamma \leq \alpha\beta} G_{\alpha\beta}$. This is contrary to the assumption that $(\mathsf{B}(S), \circ)$ is a chain. Thus, S is a chain of groups. By the above statements, we claim that $(\mathsf{B}(S), \subseteq)$ is a chain, which coincides with the chain $(\mathsf{B}(S), \circ)$.

(4)⇒(1). Suppose that the statement (4) is true. Let B be an arbitrary bi-ideal of S. If C and D are bi-ideals of S with $C \subseteq D$ such that $CD \cap DC \subseteq B$, then

$$CD \cap DC = (C \cdot D) \cap (D \circ C) \subseteq (C \circ C) \cap (C \circ C) = C \subseteq B.$$

Thus B is a strongly prime bi-ideal. This yields that $\mathsf{StPB}(S) = \mathsf{B}(S)$. The proof is then completed.

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