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On the Mutually Commuting n-tuples in Compact Groups

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Abstract

Let $\mathrm{Pr}_n\mathrm{Com}(G)$ denotes the probability that a randomly ordered n-tuples of elements in a finite group G be a mutually commuting n-tuples. We aim to generalize the above concept to a compact topological group which generally not only finite but also even uncountable. The results are mostly new or improvements of known results in finite case given in [1], [3] and [5].

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1 Introduction

One of the fundamental concepts in every area of mathematics is the idea of commutativity. Although, some of the most important mathematical structures such as the integers and real numbers that one may encounter are commutative, but there are many non-commutative cases such as topological groups, Banach algebras, modules and so on. It is natural that the commutative structure is simple and comprehensive. In topological group theory, many groups are not commutative. One of the important matter in such groups is to find some ways of qualifying the commutativity. In finite groups (which are, of course, topological groups), the usual way of doing this is to find the probability of two randomly chosen group elements commute, which is $\frac{\#com(G)}{|G|^2}$, where $\#com(G)$ is the number of pairs $(x, y) \in G \times G$ with $xy = yx$.

But, for infinite groups (that most of topological groups are infinite), this ratio is no longer meaningful. In this case, compact groups with normalized Haar measure which are a subclass of topological groups, are a good candidate for this procedure (see [6]).

One way to generalize this idea is to consider n-tuples $(x_1, x_2, \ldots, x_n)$ of elements in a compact group G with the property that $x_i x_j = x_j x_i$ for all $1 \leq i, j \leq n$. Such n-tuples are called mutually commuting n-tuples. So, we may investigate the probability that randomly chosen ordered n-tuples of the group elements are mutually commuting n-tuples which we denote it by $\mathrm{Pr}_n\mathrm{Com}(G)$. Note that for $n = 2$, this probability is exactly the same as the case of finite groups.

In the next section, we will give some definitions and known results, which are necessary for our purpose. A concrete example will be also given at the end.

2 Some Definitions and Basic Results

We begin with some definitions and preliminaries.

Definition 2.1. let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$, such measure is called a probability measure. In this case X is considered as the set of all possible of outcomes of some experiments and the measure of a set E in $\mathcal{M}$, $\mu(E)$ is the probability outcome lines in E.

Definition 2.2. Let G be a group with a locally compact and Hausdorff topology. Then G is called a *locally compact topological group* if the mapping $G \times G \longrightarrow G$, defined by $(a, b) \longmapsto ab^{-1}$ is continuous.

It is known that every locally compact topological group G admits a left Haar-measure μ, which is a positive Radon measure on a σ-algebra containing

Borel sets with the property that $\mu(xE) = \mu(E)$ (see [2] and [6]).

It is also known that the support of μ is G and it is usually unbounded. In fact, μ is bounded if and only if G is compact. In this case we may assume that $\mu(G) = 1$ i.e. μ is normalized.

Now, for a compact group G we have a unique probability measure space $(G, \mathcal{M}, \mu)$. In this note, we generalize some results on the number of mutually commuting n-tuples in finite groups to compact groups (not necessarily finite even uncountable).

Before we proceed with the exposition of this result, it will be useful to recall the following definition on finite groups.

Definition 2.3. For every $n \geq 2$, let

$$\mathrm{Pr}_n\mathrm{Com}(G) = \frac{|\{(x_1, x_2, \ldots, x_n) \in G^n \ ; \ \ x_i x_j = x_j x_i \ \text{ for all } \ 1 \leq i, j \leq n\}|}{|G|^n}.$$

If $n = 2$, then $\mathrm{Pr}_2\mathrm{Com}(G)$ is the probability that two randomly selected elements of group G commute, and generally $\mathrm{Pr}_n\mathrm{Com}(G)$ is the probability that n ordered randomly selected elements of a group G mutually commute. The following theorem on finite groups, has been known for a long time. A proof can be found in [5].

Theorem 2.4. If G is a finite non-abelian group, then $\mathrm{Pr}_n\mathrm{Com}(G) \leq \dfrac{5}{8}$. Furthermore, this bound is achieved if and only if $\dfrac{G}{Z(G)} \cong Z_2 \times Z_2$.

Now, let us state the following definition which is a generalization of the above probability on compact groups. First, we define $\mathrm{Pr}_2\mathrm{Com}(G)$ when G is a compact group.

Definition 2.5. Let G be a compact group with the normalized Haar measure μ. On the product measure space $G \times G$, we impose the product measure $\mu \times \mu$ which is a probability measure. Let

$$C_2 = \{(x, y) \in G \times G \mid xy = yx\},$$

then $C_2 = f^{-1}(1_G)$, where $f : G \times G \longrightarrow G$ is defined by $f(x, y) = x^{-1}y^{-1}xy$. It is clear that f is continuous and so C_2 is a compact subset of $G \times G$, and hence is measurable. Therefore we can define

$$\mathrm{Pr}_2\mathrm{Com}(G) = (\mu \times \mu)(C_2).$$

Similarly, with the above notations, we may define $\mathrm{Pr}_n\mathrm{Com}(G)$ for all positive integers $n \geq 2$, as the following. Suppose that G^n is the product of n-copies of G and $\mu^n = \mu \times \mu \times \ldots \times \mu$ (n-copies). Then, we define $\mathrm{Pr}_n\mathrm{Com}(G) = \mu^n(C_n)$, where

$$C_n = \{(x_1, \ldots, x_n) \in G^n \mid x_i x_j = x_j x_i \ , \ \text{for all} \ \ 1 \leq i, j \leq n\}.$$

We can see that if G is finite, then G is a compact group with the discrete topology and so the Haar measure of G is the counting measure. Therefore, $\mathrm{Pr}_n\mathrm{Com}(G) = \mu^n(C_n) = \dfrac{|C_n|}{|G|^n}$, which is the same as Definition 2.3 in finite case.

Now, we state our main results of this paper :

Theorem A let G be a non-abelian compact (not necessarily finite even uncountable) group. Then

$$\frac{G}{Z(G)} \cong Z_2 \times Z_2, \quad \text{if and only if} \ \ \mathrm{Pr}_n\mathrm{Com}(G) = \frac{3(2^{n-1}) - 1}{2^{2n-1}} \ \ \text{for all} \ \ n \geq 2.$$

Theorem B For every non-abelian compact $p-$group G, and every positive integer $n \geq 2$

$$\mathrm{Pr}_n\mathrm{Com}(G) \leq \frac{p^n + p^{n-1} - 1}{p^{2n-1}}.$$

3 Main Results

Throughout this section, we assume G is a non-abelian compact group that its Haar measure μ is normalized. First, we state some simple lemmas.

Lemma 3.1. Let $C_G(x)$ be the centralizer of element x in G. Then

$$\mathrm{Pr_2Com(G)} = \int_G \mu(C_G(x)) d\mu(x) \quad , where \ \ \mu(C_G(x)) = \int_G \chi_{C_2}(x, y) d\mu(y).$$

Proof. By Definition 2.5, it is obvious that $\mu(C_G(x)) = \int_G \chi_{C_2}(x,y)d\mu(y)$. So, by the Fobini Theorem we have

$$\begin{aligned} \mathrm{Pr_2Com(G)} = (\mu \times \mu)(C_2) &= \int_{G\times G} \chi_{C_2} d(\mu \times \mu) \\ &= \int_G \int_G \chi_{C_2}(x,y)d\mu(x)d\mu(y) \\ &= \int_G \mu(C_G(x))d\mu(x). \end{aligned}$$

Lemma 3.2. Let $Z(G)$ be the center of G. Then

$$x \in \mathrm{Z(G)} \quad \Longleftrightarrow \quad \mu(C_G(x)) > \frac{1}{2}\ .$$

Proof. It is clear that if $x \in Z(G)$ then $C_G(x) = G$ and therefore $\mu(C_G(x)) = 1 > \frac{1}{2}$. Conversely, assume that $\mu(C_G(x)) > \frac{1}{2}$ and $x \notin Z(G)$. Then there exists an element $t \in G$ such that $xt \neq tx$. Thus $C_G(x) \cup tC_G(x) \subseteq G$ and it would imply that

$$1 = \mu(G) \geq \mu(C_G(x)) + \mu(tC_G(x)) = 2\mu(C_G(x)).$$

which is a contradiction. Hence, $x \in Z(G)$ as required.

Lemma 3.3. $\mu(Z(G)) \leq \frac{1}{4}$.

Proof. Since G is not abelian, so there are elements x and y such that $xy \neq yx$. Thus, the cosets $Z(G)$, $xZ(G)$, $yZ(G)$, $xyZ(G)$ are pairwise disjoint and we have

$$G \supseteq Z(G) \cup xZ(G) \cup yZ(G) \cup xyZ(G).$$

Therefore,

$$\begin{aligned} 1 &= \mu(G) \geq \mu(Z(G)) + \mu(xZ(G)) + \mu(yZ(G)) + \mu(xyZ(G)) \\ &= 4 + \mu(Z(G)). \end{aligned}$$

Lemma 3.4. Assume that H is a closed subgroup of G. If $[G : H] \geq n$, then $\mu(H) \leq \frac{1}{n}$, where $[G : H]$ denoted the index of H in G.

Proof. Since $[G : H] \geq n$, so there are at least n distinct cosets $x_1H, x_2H, \ldots, x_nH$. Thus, we have

$$1 = \mu(G) \geq \mu(\bigcup_{i=1}^{n} \mu(x_iH)) = n\mu(H)$$

by Definition 2.2. Therefore $\mu(H) \leq \frac{1}{n}$.

The next two theorems are special case of Theorem A, when $n = 2$ and is also an improvement of Theorem 2.4.

Theorem 3.5. For every non-abelian compact group G, $\mathrm{Pr}_2\mathrm{Com}(G) \leq \frac{5}{8}$.

Proof. By Definition 2.5, and the previous lemmas, we have

$$\begin{aligned}
\mathrm{Pr}_n\mathrm{Com}(G) &= (\mu \times \mu)(C_2) \\
&= \int_G \mu(C_G(x))d\mu(x) \\
&= \textstyle\int_{Z(G)} \mu(C_G(x))d\mu(x) + \int_{G-Z(G)} \mu(C_G(x))d\mu(x) \\
&\leq \mu(Z(G)) + \mu(G - Z(G))(\frac{1}{2}) \\
&= \mu(Z(G)) + [\mu(G) - \mu(Z(G))]\,(\frac{1}{2}) \\
&= \frac{1}{2}\mu(Z(G)) + \frac{1}{2} \\
&\leq \frac{1}{8} + \frac{1}{2} \\
&= \frac{5}{8}.
\end{aligned}$$

Theorem 3.6. For any non-abelian compact group G, $\mathrm{Pr}_2\mathrm{Com}(G) = \frac{5}{8}$ if and only if $\frac{G}{Z(G)} \cong Z_2 \times Z_2$.

Proof. Assume that $\frac{G}{Z(G)} \cong Z_2 \times Z_2$, then G can be written as the union of four distinct cosets say

$$G = Z(G) \cup x_1Z(G) \cup x_2Z(G) \cup x_3Z(G).$$

Therefore

$$\begin{aligned}
1 &= \mu(G) = \mu(Z(G)) + \mu(x_1Z(G)) + \mu(x_2Z(G)) + \mu(x_3Z(G)) \\
&= 4\mu(Z(G)).
\end{aligned}$$

Since μ is a left Haar-measure, so $\mu(Z(G)) = \frac{1}{4}$. We can also see that if $a, b \in x_iZ(G)$, for $i = 1, 2, 3$, then $ab = ba$. Because we will have $a = x_iz_1$, $b = x_iz_2$ for some $z_1, z_2 \in Z(G)$. Therefore

$$ab = x_iz_1x_iz_2 = x_ix_iz_1z_2 = x_ix_iz_2z_1 = x_iz_2x_iz_1 = ba.$$

Thus, if $a \in x_iZ(G)$ then $C_G(a) = Z(G) \cup aZ(G)$ and so

$$\mu(C_G(a)) = \mu(Z(G)) + \mu(aZ(G)) = 2\mu(Z(G)) = 2(\frac{1}{4}) = \frac{1}{2}.$$

Hence, we have

$$\begin{aligned}
\mathrm{Pr}_2\mathrm{Com}(G) &= \textstyle\int_G \mu(C_G(x))d\mu(x) \\
&= \textstyle\int_{Z(G)} \mu(C_G(x))d\mu(x) + \displaystyle\sum_{i=1}^{3} \int_{x_iZ(G)} \mu(C_G(x))d\mu(x) \\
&= \mu(Z(G)) + \sum_{i=1}^{3} \frac{1}{2}\mu(x_iZ(G)) \\
&= \mu(Z(G)) + \frac{1}{2}\sum_{i=1}^{3} \mu(Z(G)) \\
&= \frac{5}{2}\mu(Z(G)) \\
&= \frac{5}{8}.
\end{aligned}$$

Conversely, Suppose that $\mathrm{Pr}_2\mathrm{Com}(G) = \frac{5}{8}$ and $\frac{G}{Z(G)} \ncong Z_2 \times Z_2$. If $[G : Z(G)] = 1, 2$ or 3, then $\frac{G}{Z(G)}$ is cyclic and so G is abelian which is a contradiction. Thus, we should have $[G : Z(G)] \geq 5$ and by Lemma 3.4, $\mu(Z(G)) \leq \frac{1}{5}$. Therefore

$$\begin{aligned}
\frac{5}{8} = \mathrm{Pr}_2\mathrm{Com}(G) &= \int_G \mu(C_G(x))d\mu(x) \\
&= \int_{Z(G)} \mu(C_G(x))d\mu(x) + \int_{G-Z(G)} \mu(C_G(x))d\mu(x) \\
&\leq \mu(Z(G)) + (1 - \mu(Z(G)))\frac{1}{2} \quad \text{(by Lemma 3.2)} \\
&= \frac{1}{2}\mu(Z(G)) + \frac{1}{2} \\
&\leq \frac{1}{2}(\frac{1}{5}) + \frac{1}{2} \\
&= \frac{3}{5}\ ,
\end{aligned}$$

which is a contradiction. Hence $\frac{G}{Z(G)} \cong Z_2 \times Z_2$ as required.

The following lemma will prove the necessary part of Theorem A.

Lemma 3.7. Let G be a non-abelian compact group. Then, if $\frac{G}{Z(G)} \cong Z_2 \times Z_2$ then $\mathrm{Pr}_n\mathrm{Com}(\mathrm{G})= \frac{3(2^{n-1})-1}{2^{2n-1}}$ for all $n \geq 2$.

Proof. We may proceed by induction on n. If $n=2$, then $\frac{3(2^{n-1})-1}{2^{2n-1}} = \frac{5}{8}$ and the proof is clear by Theorem 3.6. Now, assume that the result holds for $n-1$. Then by Theorem 3.6 and the hypothesis induction we have

$$
\begin{aligned}
&\mathrm{Pr}_n\mathrm{Com}(G) = \\
&= \int_{G^n} \chi_{C_n}(x_1,\dots,x_n)d\mu^n(x_1,\dots,x_n) \\
&= \int_{G^n} \chi_{C_{n-1}}(x_2,\dots,x_n)\chi_{C_2}(x_1,x_2)\chi_{C_2}(x_1,x_3)\dots\chi_{C_2}(x_1,x_n)d\mu^n(x_1,\dots,x_n) \\
&= \int_G \left[\int_{G^{n-1}} \chi_{C_{n-1}}(x_2,\dots,x_n)\chi_{C_2}(x_1,x_2)\chi_{C_2}(x_1,x_3)\dots\chi_{C_2}(x_1,x_n)d\mu^{n-1}(x_2,\dots,x_n)\right] d\mu(x_1) \\
&= \int_{Z(G)} \left[\int_{G^{n-1}} \chi_{C_{n-1}}(x_2,\dots,x_n)\chi_{C_2}(x_1,x_2)\chi_{C_2}(x_1,x_3)\dots\chi_{C_2}(x_1,x_n)d\mu^{n-1}(x_2,\dots,x_n)\right] d\mu(x_1) \\
&+ \int_{G-Z(G)} \left[\int_{G^{n-1}} \chi_{C_{n-1}}(x_2,\dots,x_n)\chi_{C_2}(x_1,x_2)\chi_{C_2}(x_1,x_3)\dots\chi_{C_2}(x_1,x_n)d\mu^{n-1}(x_2,\dots,x_n)\right] d\mu(x_1) \\
&= \int_{Z(G)} \mathrm{Pr}_{n-1}\mathrm{Com}(G)d\mu(x_1) + \int_{G-Z(G)} \left[\int_{[C_G(x_1)]^{n-1}} \chi_{C_{n-1}}(x_2,\dots,x_n)d\mu^{n-1}(x_2,\dots,x_n)\right] d\mu(x_1) \\
&= \mu(Z(G))\mathrm{Pr}_{n-1}\mathrm{Com}(G) + \int_{G-Z(G)} \left[\int_{[C_G(x_1)]^{n-1}} \chi_{C_{n-1}}(x_2,\dots,x_n)d\mu^{n-1}(x_2,\dots,x_n)\right] d\mu(x_1) \\
&= \mu(Z(G))\mathrm{Pr}_{n-1}\mathrm{Com}(G) + \mu(G-Z(G))\mu(C_G(x_1))^{n-1} \\
&= \frac{1}{4}\left(\frac{3(2^{n-2})-1}{2^{2n-3}}\right) + \frac{3}{4}(\frac{1}{2})^{n-1} \\
&= \frac{3(2^{n-1})-1}{2^{2n-1}}.
\end{aligned}
$$

Lemma 3.8. For any non-abelian compact group G and $n \geq 2$,

$$\mathrm{Pr}_n\mathrm{Com}(G) \leq \frac{3(2^{n-1})-1}{2^{2n-1}}.$$

Proof. We can proceed by induction on n by the same argument given as in the Lemma 3.7. So, one can easily see that

$$\begin{aligned}
\Pr_n\mathrm{Com}(G) &= \mu(Z(G))\Pr_{n-1}\mathrm{Com}(G) + \mu(G - Z(G))[\mu(C_G)]^{n-1} \\
&< \mu(Z(G))\left(\frac{3(2^{n-2})-1}{2^{2n-3}}\right) + (1-\mu(Z(G))(\frac{1}{2})^{n-1} \\
&= \mu(Z(G))\left[\frac{3(2^{n-2})-1}{2^{2n-3}} - (\frac{1}{2})^{n-1}\right] + (\frac{1}{2})^{n-1} \\
&\leq \frac{1}{4}\left[\frac{3(2^{n-1})-1-2^{n-2}}{2^{2n-3}}\right] + \frac{1}{2^{n-1}} \\
&= \frac{(2^{n-1})-1}{2^{2n-1}} + \frac{1}{2^{n-1}} \\
&= \frac{2^{n-1}(1+2)-1}{2^{2n-1}} \\
&= \frac{3(2^{n-1})-1}{2^{2n-1}},
\end{aligned}$$

by Lemma 3.2 and the hypothesis induction. Hence the proof of the lemma is completed.

Now, we are able to proof Theorem A.

Proof of Theorem A. The necessary part follows directly by Lemma 3.7. For the sufficient part, we assume that $\Pr_n\mathrm{Com}(G) = \frac{3(2^{n-1})-1}{2^{2n-1}}$. So, if $\frac{G}{Z(G)} \ncong Z_2 \times Z_2$, then $\mu(Z(G)) < \frac{1}{4}$ by Theorem 3.6. A similar argument in the proof of Lemma 3.7 implies that

$$\begin{aligned}
\Pr_n\mathrm{Com}(G) &= \mu(Z(G))\Pr_{n-1}\mathrm{Com}(G) + (1-\mu(Z(G)))[\mu(C(x_1))]^{n-1} \\
&\leq \mu(Z(G))\frac{3(2^{n-2})-1}{2^{2n-3}} + (1-\mu(Z(G)))(\frac{1}{2})^{n-1} \\
&= \mu(Z(G))\left[\frac{3(2^{n-2})-1}{2^{2n-3}} - (\frac{1}{2})^{n-1}\right] + (\frac{1}{2})^{n-1} \\
&< \frac{3(2^{n-1})-1}{2^{2n-1}},
\end{aligned}$$

which is a contradiction and this completes the proof.

To prove Theorem B, we need to state the following two lemmas.

Lemma 3.9. let G be any non-abelian $p-$group. Then $|\dfrac{G}{Z(G)}| \geq p^2$.

Proof. The proof is clear when the index of $Z(G)$ in G is infinity. So, assume that G is a non-abelian $p-$group and its centre $Z(G)$ has finite index. Since G is $p-$group, so $|\dfrac{G}{Z(G)}|$ is a power of p (see [4]). If $|\dfrac{G}{Z(G)}| < p^2$, then the only possibilities are that the index is 1 or p. If the index is 1, then $G = Z(G)$ is abelian. But the index can not be p, since if $|\dfrac{G}{Z(G)}| = p$, then $Z(G)$ is a maximal subgroup, and then G is generated by elements that all commute with each other, and therefore again G is abelian which is a contradiction. Thus, for any non-abelian $p-$group $|\dfrac{G}{Z(G)}| \geq p^2$.

Lemma 3.10. For every non-abelian compact $p-$group G,

$$\mathrm{Pr}_2\mathrm{Com}(G) \leq \frac{p^2+p-1}{p^3} .$$

Proof. It is obvious that if $x \notin Z(G)$, then $[G : C_G(x)] \geq p$. Thus, by Definition 2.5, and Lemma 3.9, we have

$$\begin{aligned}
\mathrm{Pr}_2\mathrm{Com}(G) &= (\mu \times \mu)(C_2) \\
&= \int_G \mu(C_G(x))d\mu(x) \\
&= \textstyle\int_{Z(G)} \mu(C_G(x))d\mu(x) + \int_{G-Z(G)} \mu(C_G(x))d\mu(x) \\
&\leq \mu(Z(G)) + \mu(G - Z(G))(\frac{1}{p}) \\
&= \mu(Z(G)) + [\mu(G) - \mu(Z(G))]\,(\frac{1}{p}) \\
&= \frac{p-1}{p}\mu(Z(G)) + \frac{1}{p} \\
&\leq (\frac{p-1}{p})(\frac{1}{p^2}) + \frac{1}{p} \\
&= \frac{p^2+p-1}{p^3}.
\end{aligned}$$

Proof of Theorem B.

As the proof of Theorem A, we can proceed by induction on n. If $n = 2$, then it is clear by Lemma 3.10. Now, assume that the result holds for $n-1$, then one can see that

$$\begin{aligned}
\Pr_n\mathrm{Com}(G) &= \mu(Z(G))\Pr_{n-1}\mathrm{Com}(G)+\mu(G-Z(G))[\mu(C_G(x))]^{n-1}\\
&\leq \mu(Z(G))(\frac{p^{n-1}+p^{n-2}-1}{p^{2n-3}}-\frac{1}{p^{n-1}})+\frac{1}{p^{n-1}}\\
&\leq \frac{1}{p^2}(\frac{p^{n-1}-1}{p^{2n-3}})+\frac{1}{p^{n-1}}\\
&= \frac{p^n+p^{n-1}-1}{p^{2n-1}},
\end{aligned}$$

by Lemma 3.9 and the hypothesis induction. Hence, the proof of Theorem B is completed.

Examples 3.11.
(i) Let G be a direct product of Dihedral group of order 8, D_8 and the group of unit circle. Then, we can see that the centre of G has index 4 and so by Theorem A, we have $\Pr_n\mathrm{Com}(G)=\dfrac{3(2^{n-1})-1}{2^{2n-1}}$.

(ii) Assume that $G=<x,y|x^2y=yx^2,xy^2=y^2x>$. Then, it is obvious that $Z(G)=<x^2,y^2>$, and therefore $|G:Z(G)|=4$. Since G is not abelian, so $\dfrac{G}{Z(G)}\cong Z_2\times Z_2$. Thus, again by Theorem A, we have $\Pr_n\mathrm{Com}(G)=\dfrac{3(2^{n-1})-1}{2^{2n-1}}$.

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