Some Remarks on Aperiodic Elements in Locally Nilpotent Groups

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Abstract

In this paper we point out some situations in which one can recognize the local nilpotence of a non-periodic group looking at properties of its aperiodic part.

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All through this paper, $T(G)$ will denote the set of all periodic elements of a group $G$. Let $G$ be a non-periodic group, and suppose that $T(G)$ is a subgroup of $G$. Then $G = T(G) \cup X$, where $X$ is the subgroup generated by all aperiodic elements of $G$. Therefore $G = X$ since $G$ is non-periodic. This trivial fact is used here in several situations to recognize the local nilpotence of a non-periodic group looking at properties of its aperiodic part. Clearly, the following results make some sense only when the group has both periodic and aperiodic elements.

Proposition 1. Let $G$ be a non-periodic group, and let $c$ be any positive integer. Then $G$ is nilpotent of class at most $c$ if and only if every aperiodic element of $G$ is contained in $Z_c(G)$.

Proof. Of course it suffices to prove that $T(G)$ is a subgroup of $G$, which implies $G = T(G) \cup Z_c(G)$ and $G = Z_c(G)$ since $G$ is non-periodic. Suppose not. Hence there exist periodic elements $a$ and $b$ in $G$ with $ab$ aperiodic. Thus $ab \in Z_c(G)$. Then the subgroup $H = \langle a, ab \rangle = \langle a, b \rangle$ is nilpotent (of class at most $c$). Therefore $H$ is finite and $ab$ is periodic, a contradiction. \[\square\]
Corollary 1. Let $G$ be a non-periodic group. Then $G$ is nilpotent if and only if there exists a positive integer $c$ such that every aperiodic element of $G$ is contained in $Z_c(G)$.

Let $\mathcal{X}$ be any class of groups. A group $G$ is said to be weakly $\mathcal{X}$ if every 2-generator subgroup of $G$ belongs to the class $\mathcal{X}$.

Proposition 2. Let $G$ be a non-periodic group which is weakly nilpotent or finite-by-nilpotent. Then $G$ is nilpotent if and only if the subgroup generated by its aperiodic elements is nilpotent.

Proof. It is very easy to see that if $G$ is weakly nilpotent or finite-by-nilpotent then the periodic elements of $G$ form a subgroup. Thus the result follows.

Theorem 1. Let $G$ be a non-periodic group. Then $G$ is hypercentral if and only if every aperiodic element of $G$ is contained in the hypercentre of $G$.

Proof. Let $\tilde{Z}(G)$ denote the hypercentre of $G$. Clearly, it suffices to prove that $T(G)$ is a subgroup of $G$. This implies $G = T(G) \cup \tilde{Z}(G)$ and $G = \tilde{Z}(G)$ since $G$ is non-periodic. Suppose not. Then there exist periodic elements $a$ and $b$ in $G$ with $ab$ aperiodic. Thus $ab, ba = (ab)^n \in \tilde{Z}(G)$, so $[a, b] \in \tilde{Z}(G)$. Let consider the subgroup $H = \langle a, b \rangle$. Then $H' = \langle [a, b] \rangle^H \leq \tilde{Z}(G)$. It follows that $H'$ is hypercentral, so it is locally nilpotent. Moreover $H/H'$ is finite, since it is an abelian group generated by two periodic elements. This implies that $H'$ is finitely generated and so it is polycyclic. Hence $H$ is polycyclic. Then $\tilde{Z}(H)$ is finitely generated and nilpotent. Since $H/\tilde{Z}(H)$ is abelian, $H = \tilde{Z}(H)$. Thus $H$ is nilpotent and therefore finite, and $ab$ is periodic, a contradiction.

An element $x$ of a group $G$ is said to be a right Engel element if for each $g$ in $G$ there is a non-negative integer $n$ such that $[x, n, g] = 1$.

Corollary 2. Let $G$ be a non-periodic group which satisfies the maximal condition on abelian subgroups. If every aperiodic element of $G$ is a right Engel element then $G$ is nilpotent.

Proof. Let $R(G)$ denote the set of all right Engel elements of $G$. Since $G$ satisfies the maximal condition on abelian subgroups, $R(G)$ coincides with the hypercentre of $G$ by a result due to T.A. Peng (see, for instance, Theorem 7.21 in [7], Part 2). Then $G$ is hypercentral by Theorem 1. Since a locally nilpotent group satisfying the maximal condition on abelian subgroups is nilpotent (see, for instance, Theorem 3.31 in [7], Part 1), the result follows.

Theorem 2. Let $G$ be a non-periodic group. If every aperiodic element of $G$ is contained in the $\omega$-hypercentre of $G$ then $G$ is locally nilpotent.
Proof. Let $Z_\omega(G) = \bigcup_n Z_n(G)$ denote the $\omega$-hypercentre of $G$. Arguing as in the proof of Proposition 1, one can prove that $T(G)$ is a subgroup of $G$, which implies $G = T(G) \cup Z_\omega(G)$ and $G = Z_\omega(G)$ since $G$ is non-periodic. 

It is well-known that there exist torsion-free locally nilpotent groups which are characteristically simple (see, for instance, the examples due to D.H. McLain in [7], Part 2, pages 14–16). Then the condition stated in the previous theorem is sufficient but not necessary for non-periodic groups to be locally nilpotent.

**Theorem 3.** Let $G$ be a non-periodic group, and suppose that all aperiodic elements of $G$ are contained in the Hirsch-Plotkin radical of $G$. Then either $G$ is locally nilpotent, or $G$ has an infinite polycyclic section $K/L$ such that $Z(K/L) = 1$ and $K$ is generated by two periodic elements of $G$.

**Proof.** Let $H(G)$ denote the Hirsch-Plotkin radical of $G$. If $T(G)$ is a subgroup of $G$, then $G$ is locally nilpotent since it is non-periodic. Otherwise, there exist periodic elements $a$ and $b$ in $G$ with $ab$ aperiodic. Thus $ab, ba \in H(G)$, so $[a, b] \in H(G)$. Let consider the subgroup $K = \langle a, b \rangle$. Then $K' = ([a, b])^K \leq H(G)$. It follows that $K'$ is locally nilpotent. Moreover $K/K'$ is finite, since it is an abelian group generated by two periodic elements. This implies that $K'$ is finitely generated and so it is polycyclic. Hence $K$ is a non-nilpotent infinite polycyclic group. Then there exists a positive integer $c$ such that $Z_c(K) = Z_{c+1}(K)$. Put $L = Z_c(K)$. If $K/L$ is finite then $\gamma_{c+1}(K)$ is also finite by a result due to R. Baer (see, for instance, Theorem 14.5.1 in [8]), and $K$ is nilpotent by Proposition 2. But this is a contradiction, so $K/L$ is infinite and the result follows.

Notice that a non-periodic group $G$ all whose aperiodic elements are contained in the Hirsch-Plotkin radical of $G$ need not be locally nilpotent, even if $G$ is supersoluble: consider, for example, the infinite dihedral group.

Let $R_2(G) = \{a \in G : [a, x, x] = 1, \text{ for all } x \in G\}$ denote the set of all **right 2-Engel elements** of a group $G$. It is known from [1] that $R_2(G)$ is a characteristic subgroup of $G$. A group $G$ is said to be **2-Engel** if $G = R_2(G)$.

**Theorem 4.** Let $G$ be a non-periodic group. Then $G$ is 2-Engel if and only if every aperiodic element of $G$ is contained in $R_2(G)$.

**Proof.** Of course it suffices to prove that $T(G)$ is a subgroup of $G$, which implies $G = T(G) \cup R_2(G)$ and $G = R_2(G)$ since $G$ is non-periodic. Suppose not. Then there exist periodic elements $a$ and $b$ in $G$ with $ab$ aperiodic. Thus $ab, ba \in R_2(G)$. Let consider the subgroup $H = \langle a, b \rangle$. Then $[b, a, a] = [ab, a, a] = 1$ and $[a, b, b] = [ba, b, b] = 1$. It follows that $H$ is nilpotent (of class at most 2). But this implies that $H$ is finite, and $ab$ is periodic, a contradiction.
Following W.P. Kappe, given any group $G$ and any positive integer $c$, let $B_c(G) = \{x \in G : [x, g, a_1, \ldots, a_c, g] = 1, \text{ for all } g, a_1, \ldots, a_c \in G\}$. It has been proved in [2] that $B_c(G)$ is a characteristic subgroup of $G$, and that $x \in B_c(G)$ if and only if $[x a_0, g, a_1, \ldots, a_c, g] = [a_0, g, a_1, \ldots, a_c, g]$, for all $g, a_0, a_1, \ldots, a_c \in G$. Moreover $B_c(G)$ is nilpotent of class at most $c + 2$ (see [3] and [4]).

**Theorem 5.** Let $G$ be a non-periodic group, and let $c$ be any positive integer. If every aperiodic element of $G$ is contained in $B_c(G)$ then $G$ is nilpotent of class at most $c + 2$.

**Proof.** Suppose there exist periodic elements $a$ and $b$ in $G$ with $ab$ aperiodic. Thus $ab, ba \in B_c(G)$. Let consider the subgroup $H = \langle a, b \rangle$. Then $ab Z_{c+1}(H), ba Z_{c+1}(H) \in R_2(G/Z_{c+1}(H))$ by (3.1.3) of [2]. Hence, modulo $Z_{c+1}(H)$, we have $[b, a, a] = [ab, a, a] = 1$ and $[a, b, b] = [ba, b, b] = 1$. It follows that $H$ is nilpotent (of class at most $c + 3$). But this implies that $H$ is finite, and $ab$ is periodic, a contradiction. Therefore $T(G)$ is a subgroup of $G$. This implies that $G = T(G) \cup B_c(G)$ and $G = B_c(G)$ since $G$ is non-periodic. Hence the result follows.

Let $R_3(G) = \{a \in G : [a, x, x] = 1, \text{ for all } x \in G\}$ denote the set of all right 3-Engel elements of a group $G$. It is known that, in general, $R_3(G)$ need not be a subgroup of $G$ (see [5]). In [6], M.L. Newell has shown that if $a \in R_3(G)$ then $\langle a, b \rangle$ is nilpotent of class at most 5, for all $b \in G$. Moreover, $R_3(G)$ is contained in the Hirsch-Plotkin radical of $G$.

**Theorem 6.** Let $G$ be a non-periodic group. If every aperiodic element of $G$ is contained in $R_3(G)$ then $G$ is locally nilpotent. Moreover $G$ is generated by its right 3-Engel elements.

**Proof.** Suppose that there exist periodic elements $a$ and $b$ in $G$ with $ab$ aperiodic. Let consider the subgroup $H = \langle a, b \rangle$. Since $ab \in R_3(G)$, from [6] it follows that $H$ is nilpotent (of class at most 5). But this implies that $H$ is finite, and $ab$ is periodic, a contradiction. Therefore the periodic elements of $G$ form a subgroup $T(G)$. Of course $G = T(G) \cup \langle R_3(G) \rangle$ and $G = \langle R_3(G) \rangle$ since $G$ is non-periodic. Thus $G$ is contained in its Hirsch-Plotkin radical by [6], and the result follows.

**References**


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