

Parametrization of Torsion Units in $U_1(\mathbb{Z}D_5)$

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Abstract. In this study, a representation of $\mathbb{Z}D_5$ is obtained by using a faithful irreducible representation of 2^{nd} degree of D_5 . In the group ring $\mathbb{Z}D_5$, all torsion units are expressed in terms of parameters by means of this representation. It is shown that any torsion unit in $U_1(\mathbb{Z}D_5)$ can be expressed in terms of four parameters, two of which are free and the others are dependent on free parameters by a quadratic equation.

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1. INTRODUCTION

Let $U_1(\mathbb{Z}D_5)$ be normalized unit group of an integral group ring of D_5 . In mid-sixties, Zassenhaus made seemingly very strong conjectures[5]. The first one is

$$(ZC1) \quad \gamma \in U_1(\mathbb{Z}G), |\gamma| < \infty \Rightarrow \exists g \in G, \alpha \in \mathbb{Q}G \text{ such that } \gamma = \alpha g \alpha^{-1}.$$

This conjecture has been proved by Bhandari and Luthar [1] for metacyclic groups of order pq , where p and q are different primes.

Theorem 1.1. *Let G be a split extension of a cyclic p -group by a cyclic p' -group with faithful action. Let $u \in U_1(\mathbb{Z}G)$ be an element of finite order. Then there exists a $\gamma \in \mathbb{Q}G$ such that $u = \gamma^{-1}g\gamma$.*

For a wider class of metacyclic groups, it was given an affirmative answer by Milies and Sehgal [4]. **ZC1** was also verified for some metabelian groups by Sehgal and Wiess [6]. Dokuchaev studied on torsion units in integral group ring of nilpotent metabelian groups[2]. In this study, we have given a characterization of torsion units in terms of parameters in the integral group ring of the dihedral group D_5 .

Now, let us write the representation of $\mathbb{Z}D_5$ obtained by using faithful irreducible representation of $D_5 = \langle a, b : a^5 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ (James and Liebeck [3]).

$$\rho : D_5 \rightarrow GL(2, \mathbb{Q}(\omega))$$

$$a \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$$

$$b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $\omega = e^{\frac{2\pi i}{5}}$ is the first quintic root of unity and $\bar{\omega}$ is its complex conjugate. By extending this representation ρ of D_5 over Z , we can get the representation $\bar{\rho}$ of $\mathbb{Z}D_5$ as follows:

$$\begin{aligned} \bar{\rho} : \mathbb{Z}D_5 &\rightarrow \mathbb{Q}(\omega)^{2 \times 2} \\ \sum \gamma_g g &\rightarrow \sum \gamma_g \rho(g) \end{aligned}$$

For $\gamma = \alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \alpha_4 a^4 + \beta_0 b + \beta_1 ba + \beta_2 ba^2 + \beta_3 ba^3 + \beta_4 ba^4 \in \mathbb{Z}D_5$, Let us denote $\alpha = \sum \alpha_i \omega^i$ and $\beta = \sum \beta_i \omega^i$ then the representation of γ can be shortened as follows:

$$(1.1) \quad \bar{\rho}(\gamma) = \begin{bmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}$$

Now, let us recall some basic results which are used to characterize the torsion units of $U_1(\mathbb{Z}D_5)$.

Lemma 1.2. *Let G be a metacyclic group and $N = G'$, its commutator subgroup. Consider the following natural ring homomorphism:*

$$\begin{aligned} \varphi : \mathbb{Q}G &\rightarrow \mathbb{Q}(G/N) \\ \sum \gamma_g g &\rightarrow \sum \gamma_g gN. \end{aligned}$$

If $\gamma \in U_1(\mathbb{Z}G)$ is a rational conjugate to for some $g \in G$, then $\varphi(\gamma) = gN$.

Proof. Let $\gamma \in U_1(\mathbb{Z}G)$ be a torsion unit. If γ is a rational conjugate to g for some $g \in G$, then we write, $\gamma = \alpha g \alpha^{-1}$ for some $\alpha \in \mathbb{Q}G$. Since G/N is abelian and φ is the natural ring homomorphism, we have

$$\varphi(\gamma) = \varphi(\alpha g \alpha^{-1}) = \varphi(\alpha) \varphi(g) \varphi(\alpha)^{-1} = \varphi(g) = gN.$$

□

Lemma 1.3. *Let $\bar{\rho}$ be the representation of $\mathbb{Z}G$ obtained by linear extension of a representation ρ of finite degree of a group G . If $\gamma \in U_1(\mathbb{Z}G)$ is a rational conjugate to g , for some $g \in G$ then $|\bar{\rho}(\gamma)| = |\rho(g)|$.*

Lemma 1.4. *Let $\bar{\rho}$ be the representation of $\mathbb{Z}G$ obtained by linear extension of a representation ρ of finite degree of a group G . If $\gamma \in U_1(\mathbb{Z}G)$ is a rational conjugate to g , for some $g \in G$ then $\text{tr}\bar{\rho}(\gamma) = \text{tr}\rho(g)$.*

Proposition 1.5 (Berman-Higman[5]). *Let G be a finite group. Suppose $\gamma \in U(\mathbb{Z}G)$ is a torsion unit, namely, $\gamma^n = 1$ for some natural number n . If $\gamma_e \neq 0$ then $\gamma = \pm 1$.*

Corollary 1.6. *If $\gamma \in U_1(\mathbb{Z}G)$ is a rational conjugate to $g \in \mathcal{Z}(G)$, then $\gamma = g$. That is, any torsion unit which is rational conjugate to an element in the center of a group is trivial.*

2. PARAMETRIZATION OF TORSION UNITS

By Theorem 1.1, we can say that any torsion unit of $U_1(\mathbb{Z}D_5)$ must be conjugate to one of the conjugate classes of $D_5 = \{e\} \cup \{a, a^4\} \cup \{a^2, a^3\} \cup \{b, ba, ba^2, ba^3, ba^4\}$. By Corollary 1.6, any torsion unit which is conjugate to e is trivial. As a result, a torsion unit of $U_1(\mathbb{Z}D_5)$ of order 2 is conjugate to b and a torsion unit of $U_1(\mathbb{Z}D_5)$ of order 5 is a conjugate to either a or a^2 . Let us characterize these three different types of torsion units together.

Theorem 2.1. *If $\gamma = \alpha_0 + \alpha_1a + \alpha_2a^2 + \alpha_3a^3 + \alpha_4a^4 + \beta_0b + \beta_1ba + \beta_2ba^2 + \beta_3ba^3 + \beta_4ba^4 \in U_1(\mathbb{Z}D_5)$ is a torsion unit and $\gamma \neq 1$. Then among the coefficients of γ following relations hold :*

- i) $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 = \beta_0\beta_1 + \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_4 + \beta_4\beta_0$
- ii) $\alpha_1\alpha_3 + \alpha_2\alpha_4 + \alpha_4\alpha_1 = \beta_0\beta_2 + \beta_1\beta_3 + \beta_2\beta_4 + \beta_3\beta_0 + \beta_4\beta_1$
- iii) $\alpha_0 = 0$ and one of the following relations holds. Either $\alpha_4 = 1 - \alpha_1, \alpha_3 = -\alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$ or, $\alpha_4 = -\alpha_1, \alpha_3 = 1 - \alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$ or, $\alpha_4 = -\alpha_1, \alpha_3 = -\alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$.

Proof. Let $\gamma \in U_1(\mathbb{Z}D_5)$ be a torsion unit different from 1. By Lemma 1.2, Since $N = D'_5 = \langle a \rangle$, we have

$$(2.1) \quad \varphi(\gamma) = N \text{ or } bN \Rightarrow \begin{cases} \sum_{i=0}^4 \alpha_i = 0, & \sum_{i=0}^4 \beta_i = 1 \text{ or} \\ \sum_{i=0}^4 \alpha_i = 1, & \sum_{i=0}^4 \beta_i = 0. \end{cases}$$

If we choose $\delta = \sum_{i=0}^4 \alpha_i$, then we write $1 - \delta = \sum_{i=0}^4 \beta_i$. since δ and $1 - \delta$ are idempotents, we have

$$(2.2) \quad \delta = \left(\sum \alpha_i\right)^2 = \sum \alpha_i^2 + 2 \sum \alpha_j \alpha_k \quad (0 \leq i \leq 4, 0 \leq j < k \leq 4).$$

By taking $\alpha_5 = \alpha_0$ and $\alpha_6 = \alpha_1$, we get the square of the norm of α as follows :

$$\|\alpha\|^2 = \left(\sum_{i=0}^4 \alpha_i \omega^i\right) \left(\sum_{j=0}^4 \alpha_j \bar{\omega}^j\right) = \sum_{i=0}^4 \alpha_i^2 + (\omega + \bar{\omega}) \sum_{i=0}^4 \alpha_i \alpha_{i+1} + (\omega^2 + \bar{\omega}^2) \sum_{i=0}^4 \alpha_i \alpha_{i+2}$$

or equivalently, for $0 \leq i \leq 4$ and $0 \leq j < k \leq 4$, we write

$$(2.3) \quad 2\|\alpha\|^2 = 2 \sum \alpha_i^2 - \sum \alpha_j \alpha_k - \sqrt{5} \sum (-1)^{j-k} \alpha_j \alpha_k.$$

Similarly, let us compute the square of the norm of β :

$$(2.4) \quad (1 - \delta) = \left(\sum \beta_i\right)^2 = \sum \beta_i^2 + 2 \sum \beta_j \beta_k \quad (0 \leq i \leq 4, 0 \leq j < k \leq 4).$$

By assigning $\beta_5 = \beta_0$ and $\beta_6 = \beta_1$, we get the square of the norm of β as follows:

$$\|\beta\|^2 = \sum_{i=0}^4 \beta_i^2 + (\omega + \bar{\omega}) \sum_{i=0}^4 \beta_i \beta_{i+1} + (\omega^2 + \bar{\omega}^2) \sum_{i=0}^4 \beta_i \beta_{i+2}$$

or equivalently, for $0 \leq i \leq 4$ and $0 \leq j < k \leq 4$, we write

$$(2.5) \quad 2\|\beta\|^2 = 2 \sum \beta_i^2 - \sum \beta_j \beta_k - \sqrt{5} \sum (-1)^{j-k} \beta_j \beta_k.$$

Now let us consider Lemma 1.3

$$|\bar{\rho}(\gamma)| = \|\alpha\|^2 - \|\beta\|^2 = \left(\sum_{i=0}^4 \alpha_i \omega^i\right) \left(\sum_{j=0}^4 \alpha_j \bar{\omega}^j\right) - \left(\sum_{i=0}^4 \beta_i \omega^i\right) \left(\sum_{j=0}^4 \beta_j \bar{\omega}^j\right).$$

By using identities in (2.3) and (2.5), for $0 \leq i \leq 4$ and $0 \leq j < k \leq 4$, we get

$$(2.6) \quad 2|\bar{\rho}(\gamma)| = 2 \sum (\alpha_i^2 - \beta_i^2) - \sum (\alpha_j \alpha_k - \beta_j \beta_k) - \sqrt{5} \sum (-1)^{j-k} (\alpha_j \alpha_k - \beta_j \beta_k).$$

On the other hand, by identities in (2.2) and (2.4), for $0 \leq i \leq 4$ and $0 \leq j < k \leq 4$, we obtain

$$(2.7) \quad 2(|\bar{\rho}(\gamma)| + 1 - 2\delta) = -5 \sum (\alpha_j \alpha_k - \beta_j \beta_k) - \sqrt{5} \sum (-1)^{j-k} (\alpha_j \alpha_k - \beta_j \beta_k).$$

Now let us check all possible values of $2(|\bar{\rho}(\gamma)| + 1 - 2\delta)$. If γ is a rational conjugate to a or a^2 then by Lemma 1.3 and Lemma 1.4, we have $|\bar{\rho}(\gamma)| = 1 = \delta$. If γ is a rational conjugate to b then by the same reason, we have $|\bar{\rho}(\gamma)| = -1$ and $\delta = 0$. In both cases, $|\bar{\rho}(\gamma)| + 1 - 2\delta = 0$. Hence, for $0 \leq i \leq 4$ and $0 \leq j < k \leq 4$ we get

$$\sum \alpha_j \alpha_k = \sum \beta_j \beta_k \quad \text{and} \quad \sum (-1)^{j-k} \alpha_j \alpha_k = \sum (-1)^{j-k} \beta_j \beta_k,$$

or equivalently by taking $\alpha_5 = \alpha_0, \alpha_6 = \alpha_1, \beta_5 = \beta_0$ and $\beta_6 = \beta_1$, we write

$$(2.8) \quad \sum_{i=0}^4 \alpha_i \alpha_{i+1} = \sum_{i=0}^4 \beta_i \beta_{i+1} \text{ and } \sum_{i=0}^4 \alpha_i \alpha_{i+2} = \sum_{i=0}^4 \beta_i \beta_{i+2}.$$

Since γ is a torsion unit different from 1, by Proposition 1.5 we have $\alpha_0 = 0$. Therefore, identities in (2.8) can be written as follows:

$$\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_4 = \beta_0 \beta_1 + \beta_1 \beta_2 + \beta_2 \beta_3 + \beta_3 \beta_4 + \beta_4 \beta_0$$

$$\alpha_1 \alpha_3 + \alpha_2 \alpha_4 + \alpha_4 \alpha_1 = \beta_0 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_4 + \beta_3 \beta_0 + \beta_4 \beta_1.$$

On the other hand by Lemma 1.3 and Lemma 1.4, If γ is a rational conjugate to a then

$$\alpha_4 = 1 - \alpha_1, \alpha_3 = -\alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$$

If γ is a rational conjugate to a^2 then

$$\alpha_4 = -\alpha_1, \alpha_3 = 1 - \alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0$$

If γ is a rational conjugate to b then

$$\alpha_4 = -\alpha_1, \alpha_3 = -\alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1.$$

□

Remark 2.2. Let $\gamma \in U_1(\mathbb{Z}D_5)$ be a torsion unit different from 1. Then we write \mathfrak{J}^d relation in Theorem(2.1) as follows:

$$\alpha_4 = \delta_1 - \alpha_1, \alpha_3 = \delta_2 - \alpha_2, \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1 - (\delta_1 + \delta_2).$$

where δ_1 and δ_2 determined by $\text{tr}\bar{\rho}(\gamma) = \delta_1(\omega + \varpi) + \delta_2(\omega^2 + \varpi^2)$.

Proposition 2.3. If the equation $v^2 - uv - u^2 = \pm 1$ holds for (v, u) integer pairs then, it is the same as to say that the equation $v^2 - uv - u^2 = 1$ holds for (v, u) integer pairs.

Proof. If (v, u) integer pairs hold $v^2 - uv - u^2 = 1$ then we have

$$-1 = -v^2 + uv + u^2 = (-u^2) - (-u)v - v^2.$$

So, $(-u, v)$ pairs hold the negative sign of the same equation. □

Theorem 2.4. A torsion unit in $U_1(\mathbb{Z}D_5)$ different from 1 can be written as 4 parameters, two of which are free and the others depend on free parameters by a quadratic equation.

Proof. Let $\gamma = \sum_{i=0}^4 \alpha_i a^i + \beta_i b a^i \in U_1(\mathbb{Z}D_5)$ be a torsion unit different from 1.

By substituting $\beta_1 - \alpha_1 = p, \beta_2 - \alpha_2 = q, \beta_3 - \alpha_3 = r, \beta_4 - \alpha_4 = s$, the equations in Theorem 2.1 we get

$$1 - 2(\delta_1 + \delta_2) - p - q - r - s = \beta_0 \text{ and}$$

$$\beta_0(p + s + \delta_1) + r\delta_1 + (q + s)\delta_2 + pq + qr + rs = \alpha_1(r - q) + \alpha_2(-p + q - r + s),$$

$$\beta_0(p + s + \delta_2) + (p + q)\delta_1 + p\delta_2 + pr + qs + ps = \alpha_1(-r + q + p - s) + \alpha_2(p - s).$$

Now let us substitute once more as $r - q = u$ and $s - p = v$ then we write

$$\begin{bmatrix} u & v - u \\ -u - v & -v \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \beta_0(p + s + \delta_1) + r\delta_1 + (q + s)\delta_2 + pq + qr + rs \\ \beta_0(p + s + \delta_2) + (p + q)\delta_1 + p\delta_2 + pr + qs + ps \end{bmatrix}$$

Since the square matrix above is an unimodular matrix $v^2 - uv - u^2 = \pm 1$. But by Remark 2.2 we can write $v^2 - uv - u^2 = 1$ for integer pairs (v, u) . then we can get the following solution :

$$\alpha_1 = -v[\beta_0 - \beta_0^2 + (p + q + r)\delta_1 + (p + q + s)\delta_2 + pq + pr + ps + qr + qs + rs]$$

$$+ u[\beta_0(q + r + \delta_2) + (p + q)\delta_1 + s\delta_2 + pr + qs + rs],$$

$$\alpha_2 = u[\beta_0 - \beta_0^2 + (p + q + r)\delta_1 + (p + q + s)\delta_2 + pq + pr + ps + qr + qs + rs]$$

$$+ v[\beta_0(p + s + \delta_1) + r\delta_1 + (q + s)\delta_2 + pq + qr + rs].$$

Consequently, all coefficient of a torsion unit different from 1 can be written in terms of four parameters, where $p, q \in \mathbb{Z}$ free parameters and $u, v \in \mathbb{Z}$ parameters depend on each other by a quadratic equation $v^2 - uv - u^2 = \pm 1$. If we assign $\lambda = \text{sgn}(v^2 - uv - u^2)$ then the other coefficient are as follows :

$$\alpha_0 = 0, \quad \alpha_3 = \delta_2 - \lambda\alpha_2, \quad \alpha_4 = \delta_1 - \lambda\alpha_1, \quad \beta_0 = 1 - 2(\delta_1 + \delta_2) - p - q - r - s,$$

$$\beta_1 = p + \lambda\alpha_1, \quad \beta_2 = q + \lambda\alpha_2, \quad \beta_3 = r - \alpha_2 + \delta_2, \quad \beta_4 = s - \alpha_1 + \delta_1.$$

□

Corollary 2.5. *any torsion unit in $U_1(\mathbb{Z}D_5)$ different from 1 can be expressed in terms of four parameters, two of which are free and the others are dependent on free parameters by a quadratic equation.*

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