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Segre Embedding and Desingularization

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Abstract

For a given noetherian affine algebraic variety $U_0 = Spec A_0$ with $A_0 = \frac{k[x_1,\ldots,x_m]}{<f_1,\ldots,f_l>}$, the blowing-up of U_0 along an ideal sheaf $\mathcal{I}_1$ is a quasi-projective variety U_1. The blowing-up of U_1 along another ideal sheaf $\mathcal{I}_2$ is a variety U_2 which is the result of gluing of some pieces and obeys a very complicated process. Our aim in this paper is giving a definition for the ideal family $Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$ and by using the Segre embedding and some relevant calculation try to find a simpler variety X' which is isomorphic to U_2.

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1 Introduction

The resolution of singularities of varieties and schemes is a basic concept in algebraic geometry. Over an algebraically closed field of characteristic zero, it is proved by Hironaka that a regular model exists. (cf. Main Theorem I on p. 132 of [7] and see also [9]). The successive blowing-up procedure for finding a desingularization for U_0 contains a lot of difficulties in the second step and also gluing of the result pieces is a very complicated task.
The main question in this paper is as follows:

Does there exist a method reducing the difficulties of finding a regular model?
In [1], Bodnar has introduced a unique ideal in such a way that the blowing-up along it is the same with the result of successive blowing-ups.
In this paper we present an ideal family $Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$ and after considering

its properties in §2 and §4 and its effect on Segre embedding we find some varieties which are less complicated than the result of successive blowing-ups but isomorphic to them. We show it for two consecutive step. The generalization of this method would be a nice conjecture as it is stated in the following question.

Question. *Can we use this method to construct* $Ang(U_0, \mathcal{I}_1, \ldots, \mathcal{I}_n)$ *for* $n \geq 3$*?*

This paper consists of five different sections. In the coming section we define $Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$. Later on we discuss about its properties in §3. In §4 we use Segre embedding and consider the result of two blowing-ups. In the last section we determine a specific element of $Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$ and use the Gröbner basis tool which simplifies the process. We also use the Gröbner basis and Singular software through an example. For more details see [3] and [4].
Note that by blowing-up along an ideal I, we mean the blowing-up with the center $Y = V(I)$. Our notations are similar to [5].

2 Preliminaries

Let $k = \bar{k}$ be a field of characteristic zero and $A_0 = k[x_1, \ldots, x_m]/< f_1, \ldots, f_l >$. Consider the noetherian scheme $U_0 = SpecA_0$. Suppose $\mathcal{I}_1$ is an ideal sheaf on U_0 such that $I_1 = \mathcal{I}_1(U_0) =< g_0, \ldots, g_n > A_0$. Then the blowing-up of U_0 along $\mathcal{I}_1$ is $U_1 = \tilde{U}_0 = ProjA_1$, where $A_1 = \tilde{A}_0 = \frac{A_0[y_0,\ldots,y_n]}{J_1}$ and J_1 is the kernel of $\phi_1 : A_0[y_0, \ldots, y_n] \to A_0[t]$, keeping the elements of A_0 and sending each y_i to $g_i t$.

Convention. By $I^{dH_{y_i}}$ we mean the dehomogenization of I, which means that we consider $y_i = 1$. For a polynomial f, $f^{dH_{y_i}}$ is the dehomogenization of f i.e. putting $y_i = 1$ in f. We also use $I^{H_{y_i}}$ for the homogenization of I with respect to y_i and use $f^{H_{y_i}}$ for the homogenization of f with respect to y_i. If it is not ambiguous we use I^{dH_i} rather than $I^{dH_{y_i}}$ and I^{H_i} rather than $I^{H_{y_i}}$ and also use similar abbreviation for f. We leave the proof of the following proposition to the reader:

Proposition 2.1 *By using the above notation the ith affine patch of* U_1 *is* $U_{1i} = SpecA_{1i}$ *where* $A_{1i} = \frac{A_0[y_0,\ldots,\hat{y_i},\ldots,y_n]}{J_1^{dH_i}}$.

Suppose we tend to find the blowing-up of the quasi-projective variety U_1 along the ideal sheaf $\mathcal{I}_2$ where the restriction of $\mathcal{I}_2$ on U_1 is the homogeneous ideal $I_2 =< h_0, \ldots, h_s >$. It is enough to find the blowing-up of the affine patches U_{1i} of U_1 along $I_2^{dH_i}$, $i = 0, 1, \ldots, n$, and then glue them together. Similar to the above discussion the result of blowing-up of U_{1i} along this ideal is a projective variety with the coordinate ring $\tilde{A}_{1i} = \frac{A_{1i}[z_0,\ldots,z_s]}{ker\phi_{2i}}$, where ϕ_{2i} is defined similar to the former step and corresponding to the ideal I^{dH_i} of A_{1i}. Let $J_{2i} = ker\phi_{2i}$. We can prove that

$$\tilde{U}_{1i} = Proj\frac{A_0[y_0,\ldots,\hat{y}_i,\ldots,y_n]}{J_1^{dH_i} + J_{2i}}[z_0,\ldots,z_s]$$

and $U_2 = \tilde{U}_1$ is the gluing of quasi-projective varieties $\tilde{U}_{1i}$, $i = 0,\ldots,n$. Furthermore according to the above proposition, for $j = 0,\ldots,s$, the jth affine patch of $\tilde{U}_{1i}$ is equal to

$$\tilde{U}_{ij} = Spec\frac{A_0[y_0,\ldots,\hat{y}_i,\ldots,y_n]}{J_1^{dH_i} + J_{2i}^{dH_{z_j}}}[z_0,\ldots,\hat{z}_j,\ldots,z_s].$$

So far our knowledge about the quasi- projective variety U_2 is very limited and we just know that $U_2 \subset U_0 \times \mathbb{P}_k^n \times \mathbb{P}_k^s$. In coming section we look for those ideals that by equating y_i and z_j to 1 we can get the ideal related to the coordinate ring of the ijth affine patch of U_2.

Definition 2.2 *A homogeneous ideal* $I \subset A_0[\underline{y_i}, \underline{z_j}]$ *is called a member of the ideal family* $Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$ *in the case that* $I^{dH_{y_i,z_j}}$, *(i.e. the dehomogenization of* I *with respect to* y_i *and* z_j*) be equal to the dehomogenization of* $J_1^{dH_i} + J_{2i}$ *with respect to* z_j, *in fact be equal to* $J_1^{dH_i} + J_{2i}^{dH_{z_j}}$.

In coming section by considering the properties of the ideals belonging to $Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$ we get some results which will enable us to get a better knowledge about U_2.

3 Calculating the ideal I

Let $i \in \mathbb{N} \bigcup \{0\}$ be fixed. As we mentioned in the previous section we consider the ring homomorphism ϕ_{2i} to find the blowing-up of U_{1i} along $\mathcal{I}_2$. Now suppose $f \in J_{2i}$ be one of its generators. According to the properties of ϕ_{2i}, the ideal J_{2i} is homogeneous with respect to z_j's. Hence we can assume that f is homogeneous. With out loss of generality, suppose that f is of degree one and has the form $f = \sum_{k=0}^{s} f_k z_k$ where f_k's are written in the indeterminates $y_0, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$. So we have

$$F = f_0 h_0^{dH_i} + \ldots + f_s h_s^{dH_i} \in J_1^{dH_i}. \tag{1}$$

Note that $F \in J_1^{dH_i}$ does not imply $F^{H_i} \in J_1$.

Let $F_1, \ldots, F_r$ be a generating set of the homogeneous ideal J_1. By (1) there exist polynomials $G_i \in A_0[\underline{y_k}]$, $i = 0, \ldots, r$ such that $F = G_1 F_1^{dH_i} + \ldots + G_r F_r^{dH_i}$.

If F' is the ith homogenization of F, then there exist non-negative integers $d_1, \ldots, d_r$ such that

$$F' = y_i^{d_1} G_1^{H_i} (F_1^{dH_i})^{H_i} + \ldots + y_i^{d_r} G_r^{H_i} (F_r^{dH_i})^{H_i}.$$

It is possible that for some k, $(F_k^{dH_i})^{H_i} \neq F_k$. This shows that F_k has a factor of the form $y_i^{m_k}$. Let $m^{(i)} = max_{k=1,\ldots,r} m_k$. We will multiply $y_i^{m^{(i)}}$ to $(F_k^{dH_i})^{H_i}$ and get a product of some powers of y_i to F_k's. Hence there exist integers $d'_1, \ldots, d'_r$ such that

$$F'' := y_i^{m^{(i)}} F' = \sum_{k=1}^{r} y_i^{d'_k} G_k^{H_i} F_k. \tag{2}$$

We observe that corresponding to each polynomial f there exists a polynomial $F'' \in J_1$. On the other hand by (1) there exist numbers $p_0, \ldots, p_r$ such that

$$F' = y_i^{p_0} f_0^{H_i} (h_0^{dH_i})^{H_i} + \ldots + y_i^{p_s} f_s^{H_i} (h_s^{dH_i})^{H_i}.$$

Similar to the former method it is possible that for some k, $(h_k^{dH_i})^{H_i} \neq h_k$. Hence there exists non-negative integers $n^{(i)}$ and p'_j, $j = 0, \ldots, s$, such that

$$y_i^{n^{(i)}} F' = y_i^{p'_0} f_0^{H_i} h_0 + \ldots + y_i^{p'_s} f_s^{H_i} h_s. \tag{3}$$

Considering (2) and (3), we define the polynomial G_f as $G_f := y_i^{m^{(i)}+n^{(i)}} F'$. By using the former relations we get

$$G_f = y_i^{m^{(i)}} [y_i^{p'_0} f_0^{H_i} h_0 + \ldots + y_i^{p'_s} f_s^{H_i} h_s]. \tag{4}$$

G_f is homogeneous in y_i's. Based on the definition of G_f and corresponding to f which we started with it, we define L_f as follows:

$$L_f := y_i^{m^{(i)}} [y_i^{p'_0} f_0^{H_i} z_0 + \ldots + y_i^{p'_s} f_s^{H_i} z_s] \in (A_0[y_0, \ldots, y_n])[z_0, \ldots, z_s]. \tag{5}$$

Theorem 3.1 *Let $L_f^{dH_k}$ be the dehomogenization of L_f with respect to y_k, then for each k, $k = 0, \ldots, n$, $L_f^{dH_k} \in J_{2k}$.*

Sketch of proof. Considering (1) we realize that the homogenization of L_f with respect to y_i is f and as a result $\phi_{2i}(L_f^{dH_i}) = \phi_{2i}(f) = 0$, so we have $L_f^{dH_i} \in J_{2i}$.

For $k \neq i$, by putting $y_k = 1$ in L_f and considering (5) we get $L_f^{dH_k}$. Now we substitute z_r by the polynomial h_r, r=0,...,s, and get the polynomial

$$y_i^{m^{(i)}} y_i^{n^{(i)}} [y_i^{p_0} (f_0^{H_i})^{dH_k} (h_0^{dH_i})^{H_i} + \ldots + y_i^{p_s} (f_s^{H_i})^{dH_k} (h_s^{dH_i})^{H_i}].$$

If we dehomogenize the above polynomial with respect to y_k we get the polynomial

$$y_i^{n^{(i)}}[y_i^{m^{(i)}}[y_i^{p_0}(f_0^{H_i})^{dH_k}\{(h_0^{dH_i})^{H_i}\}^{dH_k}+\ldots+y_i^{p_s}(f_s^{H_i})^{dH_k}\{(h_s^{dH_i})^{H_i}\}^{dH_k}]].$$

As we see the polynomial inside the inner brackets is F'^{dH_k} and hence the polynomial in the outside brackets is F''^{dH_k}. So the above polynomial is

$$y_i^{n^{(i)}}F''^{dH_k}:=y_i^{m^{(i)}}[y_i^{p'_0}(f_0^{H_i})^{dH_k}h_0^{dH_k}+\ldots+y_i^{p'_s}(f_s^{H_i})^{dH_k}h_s^{dH_k}]\in J_1^{dH_k}. \qquad (6)$$

Notice that the polynomial in (6) is the image of $L_f^{dH_k}$ under the homomorphism ϕ_{2k}, i.e. $L_f^{dH_k}\in J_{2k}$.

Remark. So far we have L_f corresponding to f in such a way that its kth dehomogenization, $k=0,1,\ldots,n$, is an element of J_{2k}. If J is the ideal generated by $\{L_f : f\in J_{2i}, i=0,\ldots,n\}$, then $J^{dH_i}=J_{2i},\ i=0,1,\ldots,n$. Now we put $I:=J+J_1$.
I has almost all properties of elements of $Ang(U_0,\mathcal{I}_1,\mathcal{I}_2)$. The only condition which is not possibly satisfied is being homogeneous with respect to y_i and z_j, $i=0,\ldots,n,\ j=0,\ldots,s$.

Definition 3.2 *Let* $S=\bigoplus_{d\geqslant 0}S_d$ *be a graded ring and* $I=\bigoplus_{d\geqslant 0}I_d$ *be a homogeneous ideal. Then* $I'=\bigoplus_{d\geqslant i}I_d$, $i\geqslant 0$ *is called the ith inverse saturation of* I.

Remark. (i) If $\{h_0,\ldots,h_s\}$ is a set of polynomials generating the homogeneous ideal I and $maxdegh_i=d,\ i=0,\ldots,s$, then the generating set of the dth inverse saturation of I consists of all elements which are the result of multiplication of monomials of degree $d-degh_i$ to $h_i,\ i=0,\ldots,s$. Note that if $I':=<h'_0,\ldots,h'_r>$, where $h'_i,\ i=0,\ldots,r$, is as above, then for each i, $degh'_i=d$. (ii) According to exercise 5.10(b) in [5], the subscheme corresponding to I' is the same with the subscheme corresponding to I and the result of blowing-up centered on each of them gives the same thing.

Now suppose that $d=maxdegh_i,\ i=0,\ldots,s$, and I'_2 be the dth inverse saturation of I_2 and $\{h'_0,\ldots,h'_r\}$ be its generating set. Now we use the above method for construction of L_f by using I'_2 rather than I_2. Since G_f is homogeneous and $degh'_i=d,\ i=0,\ldots,r$, the substitution of h'_i by z_i in G_f reduces the degree of G_f by $d-1$. Hence $I=J+J_1$ has the desired properties for being an element of $Ang(U_0,\mathcal{I}_1,\mathcal{I}_2)$.

Consider the quasi-projective variety $X=ProjA$ where $A=\frac{A_0[\underline{y_i},\underline{z_j}]}{I}$. The coordinate ring of the ijth affine patch of $U_2,\ i=0,\ldots,n,\ j=0,\ldots,s$ is isomorphic to the dehomogenization of A with respect to y_i and z_j i.e.

$$\frac{A_0[y_0,\ldots,\hat{y}_i,\ldots,y_n,z_0,\ldots,\hat{z}_j,\ldots,z_s]}{I^{dH_{y_i,z_j}}}.$$

4 Segre embedding

Recall that the Segre embedding is of the form

$$\phi : Proj A_0[y_0, \ldots, y_n] \times_{Spec A_0} Proj A_0[z_0, \ldots, z_s] \longrightarrow Proj A_0[v_{00}, \ldots, v_{ns}]$$

$$y_i z_j \longmapsto v_{ij}$$

for $i = 0, \ldots, n$, $j = 0, \ldots, s$. Now suppose that

$$\overline{\phi} : A_0[v_{00}, \ldots, v_{ns}] \longrightarrow A_0[\underline{y_i}, \underline{z_j}]$$

$$v_{ij} \longmapsto y_i z_j$$

then $image\phi = Proj \frac{A_0[\underline{v_{ij}}]}{ker\overline{\phi}}$. Consider the ring homomorphism

$$\psi : A_0[\underline{v_{ij}}] \longrightarrow \frac{A_0[\underline{y_i}, \underline{z_j}]}{I}$$

$$v_{ij} \longmapsto y_i z_j + I$$

and let $K = ker\psi$. We seek for a relation between $X' = Proj \frac{A_0[\underline{v_{ij}}]}{K}$ and X. We do it by comparing the corresponding affine patches of these two quasi-projective varieties.

Theorem 4.1 *With the above notation $U_2 \cong X'$.*

Proof. Without loss of generality consider the affine patch corresponding to $v_{00} = 1$. One can show that the ring homomorphism

$$\psi_{00} : \frac{A_0[v_{01}, \ldots, v_{ns}]}{K^{dH_{v_{00}}}} \longrightarrow \frac{A_0[y_0, \ldots, y_n, z_1, \ldots, z_s]}{I^{dH_{y_0, z_0}}}$$

$$v_{ab} \longmapsto y_a z_b \qquad (a \neq 0, b \neq 0)$$

$$v_{0b} \longmapsto z_b$$

$$v_{a0} \longmapsto y_a$$

is an isomorphism. Hence $Spec \frac{A_0[v_{01}, \ldots, v_{ns}]}{K^{dH_{v_{00}}}} \cong Spec \frac{A_0[y_1, \ldots, y_n, z_1, \ldots, z_s]}{I^{dH_{y_0, z_0}}}$. The left hand side of the above equation is the affine patch of X' corresponding to v_{00} and the right hand side is the 00th affine patch corresponding to $y_0 = 1$ and $z_0 = 1$ of U_2. Hence the gluing of the affine patches of U_2 is the same with the gluing of the affine patches of X'. In fact we have shown that

$$U_2 \cong X'_2 = Proj \frac{A_0[v_{00}, \ldots, v_{ns}]}{K}.$$

5 Construction of a better ideal

Let $\theta : A_0[\underline{y_i}, \underline{z_j}] \rightarrow \frac{A_0[\underline{y_i}]}{J_1}[t]$ be the ring homomorphism which fixes elements of $A_0[\underline{y_i}]$ and $z_i \mapsto h_i t$. Considering the properties of L_f of §2 we find out that $\theta(L_f) = 0$. Hence $I \subseteq ker\theta := H$. This means that $J_1^{dH_{y_i}} + {J_{2i}}^{dH_{z_j}} = I^{dH_{y_i,z_j}} \subseteq H^{dH_{y_i,z_j}}$. We can check that the inclusion $H^{dH_{y_i,z_j}} \subseteq {J_{2i}}^{dH_{z_j}} + {J_1}^{dH_{y_i}}$ is also satisfied. This implies $H \in Ang(U_0, \mathcal{I}_1, \mathcal{I}_2)$ and so it is a good candidate to substitute with I, which is less complicated than I.
To compute H we can use Gröbner basis to find $ker\theta$. In the following example we use the Singular software to do it.

Example. Let k be the algebraic closure of rationals and $U_0 = \mathbb{A}^2_k = SpecR_0$ where $R_0 = k[x_1, x_2]$. In the first step to find the blowing-up U_0 along the ideal $< x_1x_2, x_1^2 >$. Consider the ring homomorphism $\phi_1 : R_0[y_0, y_1] \rightarrow R_0[t]$ where ϕ_1 fixes the elements of R_0, $\phi_1(y_0) = x_1x_2t$ and $\phi_1(y_1) = x_1^2t$. By using the Gröbner basis as a tool and doing the relevant computation by the Singular software we find out that $J_1 := ker\phi_1 =< x_1y_0 - x_2y_1 >$. So we have $U_1 = \tilde{U}_0 = ProjR_1$, where $R_1 = \frac{R_0[y_0,y_1]}{J_1}$. Hence the affine patches of U_1 which are the result of substitution by $y_i = 1$ i.e. U_{10} and U_{11}, have the following coordinate rings respectively:

$$R_{10} = \frac{R_1}{<y_0-1>} \cong k[x_1, x_2, \tfrac{x_1}{x_2}] = k[x_2, \tfrac{x_1}{x_2}]$$

$$R_{11} = \frac{R_1}{<y_1-1>} \cong k[x_1, x_2, \tfrac{x_2}{x_1}] = k[x_1, \tfrac{x_2}{x_1}].$$

Now consider the ideal $I_2 =< x_1, x_2, y_0 >$. I_2 is a maximal ideal of R_{11}, hence it is a closed point of $SpecR_{11} = U_{11}$ which is singular. Furthermore $I_2 \notin U_{10}$. To find the blowing-up of U_1 along I_2 it is enough to find the blowing-up of U_{11} along I_2. But already we should find the first inverse saturation of I_2 to get $I'_2 \subset R_1$ and use I'_2 rather than I_2 in the process of blowing-up. In fact $I'_2 =< x_1y_1, x_2y_1, x_1y_0, x_2y_0, y_0 >$ and as an ideal of R_1 it will be simplified to get $I'_2 =< x_1y_1, x_2y_1, y_0 >$.

Now we compute $H = ker\theta$ as an element of $Ang(U_0, I_1, I_2)$. By using Singular software we get

$$H = ker\theta =< x_1z_2 - z_1, y_0z_0 - y_1z_1, x_2z_0 - x_1z_1, x_1y_0 - x_2y_1, x_2y_1z_2 - y_0z_1 > .$$

Now we go back to §4 and by using the ring homomorphism $\psi : R_0[v_{00}, \ldots, v_{12}] \rightarrow \frac{R_0[\underline{y_i}, \underline{z_j}]}{H}[t]$, $i = 0, 1$, $j = 0, 1, 2$, which fixes elements of R_0 and $\psi(v_{ij}) = y_iz_jt + H$ we get a new scheme isomorphic to U_2. Calculating $K = ker\psi$ by the Singular software we get:

$$K =< v_{00} - v_{11}, x_2v_{12} - v_{01}, x_1v_{12} - v_{11}, v_{02}v_{11} - v_{01}v_{12},$$
$$v_{02}v_{10} - v_{11}v_{12}, v_{01}v_{10} - v_{11}^2, x_2v_{10} - x_1v_{11}, x_1v_{02} - v_{01}, x_1v_{01} - x_2v_{11} >,$$

and hence we get $Proj\frac{R_0[v_{ij}]}{K}$ which is isomorphic to U_2.

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