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On the Set of Gaps in Numerical Semigroups

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Abstract

In this paper, we give some results on the set of gaps in numerical semigroups. Also, we determine S by the set of gaps $H(S)$.

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1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \cdots n, \cdots\}$ and $S \subseteq \mathbb{N}$. S is called a $numerical semigroup$ if S is sub-semigroup of $(\mathbb{N}, +)$ with $0 \in S$.

It is known that every numerical semigroup is finitely generated, i.e. there exist elements of S, say $n_0, n_1, \cdots, n_p$ such that $n_0 < n_1 < \cdots < n_p$ and

$$S = \langle n_0, n_1, \cdots, n_p\rangle = \{\sum_{i=0}^{p} k_i n_i \quad : k_i \in \mathbb{N}\}$$

and

$$G.C.D.(n_0, n_1, \cdots, n_p) = 1 \Leftrightarrow Card(\mathbb{N}\backslash S) < \infty$$

by [1].

Let us give the following definitions known for numerical semigroup S.

$g(S) = max\{x \in \mathbb{Z} : x \notin S\}$ is called the $Frobenius\, number$ of S, where $\mathbb{Z}$ is the integer set. Thus, S numerical semigroup is $S = \{0, n_0, n_1, \cdots, g(S) + 1 \rightarrow \cdots\}$ (The arrow "$\rightarrow$" means that every integer which is greater then $g(S) + 1$ belongs to S).

We say that a numerical semigroup is *symmetric* if for every $x \in \mathbb{Z}\backslash S$, we have $g(S) - x \in S$ by [2].

For $n \in S\backslash\{0\}$, we define the Apéry set of the element n as the set

$$Ap(S,n) = \{s \in S : s - n \notin S\}.$$

It can easily be proved that $Ap(S,n)$ is formed by the smallest elements of S belonging to the different congruence classes $modn$. Thus, $\sharp(Ap(S,n)) = n$ and $g(S) = max(Ap(S,n)) - n$, where $\sharp(A)$ stands for $cardinality(A)$ by [3]. The elements of $\mathbb{N}\backslash S$, denote by $H(S)$,are called *gaps* of S. A gap x of a numerical semigroup S is $fundamental$ if $\{2x, 3x\} \subset S$. We denote by $FH(S)$ the set of fundamental gaps of S.
Let $D(x_i) = \{x \in \mathbb{N} : x \mid x_i, \exists i \in \{1, 2, ..., r\}\}$. Since S is a numerical semigroup and, $i \in \{1, 2, ..., r\}$, for $x_i \in FH(S)$ we write $S = \mathbb{N}\backslash D(x_i)$ by [4].
If $S = \langle s_1, s_2\rangle$, then

$$Ap(S, s_1) = \{(s_1 - r)s_2 : r = 1, 2, \cdots, s_1\}$$

and

$$Ap(S, s_2) = \{(s_2 - r)s_1 : r = 1, 2, \cdots, s_2\},$$

the set of gaps of S is

$$H(S) = \{0, 1, 2, ..., g(S)\}\backslash(Ap(S, s_1) \bigcup \; Ap(S, s_2) \bigcup \{s_1 + s_2\})$$

by [5].

2. Main Results

In this section, we give some results for the set of gaps of S.

Theorem 1. If S_1 and S_2 are two numerical semigroups, then we write

$$H(S_1 \bigcap S_2) = H(S_1) \bigcup H(S_2).$$

Proof. We have $a \in H(S_1 \bigcap S_2)$.It follows easily that $a \in H(S_1) \bigcup H(S_2)$. Since

$$a \in H(S_1 \bigcup S_2) \Longrightarrow a \notin (S_1 \bigcap S_2) \Longrightarrow \left\{ \begin{array}{c} (i) \;\; a \in S_1, a \notin S_2 \\ (ii) \;\; a \notin S_1, a \in S_2 \\ (iii) \;\; a \notin S_1, a \notin S_2 \end{array} \right\}.$$

Conversely,let us assume that $b \in H(S_1) \bigcup H(S_2)$. Since $b \in H(S_1) \bigcup H(S_2) \Longrightarrow b \in H(S_1) \vee H(S_2)$, we obtain the required result by definition of gaps.

Corollary. For $1 \leq i \leq n$,S_i are numerical semigroups, then we can write

$$H(\bigcap_{i=1}^{n} S_i) = \bigcup_{i=1}^{n} H(S_i).$$

Theorem 2. If S_1 and S_2 are two numerical semigroups such that $S_1 \nsubseteq S_2$ or $S_2 \nsubseteq S_1$, then we have

$$FH(S_1 \bigcap S_2) = FH(S_1) \bigcup FH(S_2).$$

Proof. If $x \in FH(S_1 \bigcap S_2) \Longleftrightarrow x \in H(S_1 \bigcap S_2) \wedge \{2x, 3x\} \subset (S_1 \bigcap S_2)$, then $x \in FH(S_1) \bigcup FH(S_2)$ by definition of fundamental gaps.

Corollary. If S_i are numerical semigroups such that $S_i \nsubseteq S_j$ and $S_j \nsubseteq S_i$, where $1 \leq i, j \leq n$, then we can write

$$FH(\bigcap_{i=1}^{n} S_i) = \bigcup_{i=1}^{n} FH(S_i).$$

Theorem 3. Let S_1 and S_2 are two numerical semigroups. $K = S_1 \bigcap S_2$ numerical semigroup is determined as follows

$$K = S_1 \bigcap S_2 = \mathbb{N} \backslash D(x_j) = \mathbb{N} \backslash (H(S_1) \bigcup H(S_2)),$$

where $x_j \in FH(S_1) \bigcap FH(S_2), j \in \{1, 2, ..., r\}$.

Proof. We find the required result by Theorem 1 and Theorem 2.

Example 1. Let

$$S_1 = \langle 3, 5, 7 \rangle = \{0, 3, 5, 6, 7, 8, \rightarrow, \cdots\}$$

and

$$S_2 = \langle 4, 5, 11 \rangle = \{0, 4, 5, 8, 9, 10, 11, 12, \rightarrow, \cdots\}.$$

be two numerical semigroups. Thus,we find that

$$S_1 \bigcap S_2 = \{0, 5, 8, 9, 10, \rightarrow, \cdots\},$$

$H(S_1) = \{1, 2, 4\}$,$H(S_2) = \{1, 2, 3, 6, 7\}$,$FH(S_1) = \{4\}$ and $FH(S_2) = \{6, 7\}$. Hence, we obtain that

$$H(S_1 \bigcap S_2) = H(S_1) \bigcup H(S_2) = \{1, 2, 3, 4, 6, 7\}$$

and

$$FH(S_1 \bigcap S_2) = FH(S_1) \bigcup FH(S_2) = \{4, 6, 7\}.$$

Moreover, we write $D(4, 6, 7) = \{1, 2, 3, 4, 6, 7\}$. Finally, we determine

$$K = S_1 \bigcap S_2 = \{0, 5, 8, 9, 10, \rightarrow, \cdots\}$$

numerical semigroup as follows

$$K = S_1 \bigcap S_2 = \{0, 5, 8, 9, 10, \rightarrow, \cdots\} = \mathbb{N} \backslash D(4, 6, 7) = \mathbb{N} \backslash (H(S_1) \bigcup H(S_2)).$$

Example 2. Let

$$S_1 = \langle 4, 5 \rangle = \{0, 4, 5, 8, 9, 10, 12, 13, \rightarrow, \cdots\}$$

and

$$S_2 = \langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, \rightarrow, \cdots\}.$$

be two numerical semigroups.In this case,

$$Ap(S_1,4)=\{5(4-r):r=1,2,3,4\}=\{0,5,10,15\},$$

$$Ap(S_1,5)=\{4(5-r):r=1,2,3,4,5\}=\{0,4,8,12,16\}$$

and $g(S_1)=11$. Hence, we write

$$H(S_1)=\{0,1,2,...,11\}\backslash Ap(S_1,4)\bigcup Ap(S_1,5)\bigcup\{9\})=\{1,2,3,6,7,11\}$$

and $FH(S_1)=\{6,7,11\}$. If we apply the same operations for S_2 above, then we find that

$$Ap(S_2,3)=\{5(3-r):r=1,2,3\}=\{0,5,10\}$$

and

$$Ap(S_2,5)=\{3(5-r):r=1,2,3,4,5\}=\{0,3,6,9,12\}$$

and $g(S_2)=7$.
Thus,we write

$$H(S_2)=\{0,1,2,...,7\}\backslash Ap(S_2,3)\bigcup Ap(S_2,5)\bigcup\{8\})=\{1,2,4,7\}$$

and $FH(S_2)=\{4,7\}$.In this case,for $K=S_1\bigcap S_2=\{0,5,8,9,10,12,13,\rightarrow,\cdots\}$, we obtain

$$H(S_1\bigcap S_2)=H(S_1)\bigcup H(S_2)=\{1,2,3,4,6,7,11\}.$$

and

$$FH(S_1\bigcap S_2)=FH(S_1)\bigcup FH(S_2)=\{4,6,7,11\}.$$

Moreover, we write $D(4,6,7,11)=\{1,2,3,4,6,7,11\}$ for $4,6,7,11\in FH(S_1\bigcap S_2)$. Thus,we determine numerical semigroup $K=S_1\bigcap S_2=\{0,5,8,9,10,12,13,\rightarrow,\cdots\}$ as follows:

$$K=S_1\bigcap S_2=\mathbb{N}\backslash D(4,6,7,11)=\mathbb{N}\backslash\{1,2,3,4,6,7,11\}=\mathbb{N}\backslash(H(S_1)\bigcup H(S_2)).$$

If we apply the same operations for numerical semigroups

$$S_1=\langle 4,5\rangle=\{0,4,5,8,9,10,12,13,\rightarrow,\cdots\}$$

and

$$S_3=\langle 2,5\rangle=\{0,2,4,5,6,7,8,9,\rightarrow,\cdots\}.$$

Then, we find

$$Ap(S_3, 2) = \{5(2-r) : r = 1, 2\} = \{0, 5\},$$

$$Ap(S_3, 5) = \{2(5-r) : r = 1, 2, 3, 4, 5\} = \{0, 2, 4, 6, 8\}$$

and $g(S_3) = 3$.Thus,we write

$$H(S_3) = \{0, 1, 2, 3\} \backslash Ap(S_3, 2) \bigcup Ap(S_3, 5) \bigcup \{7\}) = \{1, 3\}$$

and $FH(S_3) = \{3\}$. In this case, we find that

$$H(S_1 \bigcap S_3) = H(S_1) \bigcup H(S_3) = \{1, 2, 3, 6, 7, 11\},$$

but $FH(S_1 \bigcap S_3) = \{6, 7, 11\} \neq FH(S_1) \bigcup FH(S_3)$, for numerical semigroup

$$K = S_1 \bigcap S_3 = \{0, 4, 5, 8, 9, 10, 12, 13, \rightarrow, \cdots\} = S_1.$$

References

[1] R. Froberg, C. Gottlieb and R. Haggkvist,*On numerical semigroups*, Semigroup Forum, 35 (1987), 63-83.

[2] J.C.Rosales and J.I.Garcia- Garcia, *Hereditarily Finitely Generated Commutative Monoids* , Journal of Algebra, 221 (1999), 723-732.

[3] J.C. Rosales,*Numerical Semigroups with Apéry sets of Unique Expression* , Journal of Algebra 226 (2000), 479-487 .

[4] J.C. Rosales, P.A. Garcia-Sanchez, J.I.Garcia- Garcia, J.A. Jimenez Madrid,*Fundamental gaps in numerical semigroups*, Journal of pure and applied algebra, 189 (2004), 301-313.

[5] S. Ilhan, On Apéry sets of Symmetric Numerical Semigroups, International Mathematical Forum, 1, no.10, (2006), 481-484.

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