On the Growth Sequences of $\text{PSp}(2m, q)$

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Abstract

The aim of this paper is to give a lower bound for $h(2, \text{PSp}(2m, q))$, for all $2 \leq m \leq 5$, $m \geq 10$ and $q \geq 2$, where $h(2, G)$ is the maximum number such that $G^{h(2, G)}$ can be generated by 2 elements. Furthermore, we consider a problem which was conjectured by J.Wiegold and the first author in 1996, which says that $h(2, G)^2 > |G|$ for all finite non-abelian simple groups. We confirm the conjecture for the projective symplectic simple groups $\text{PSp}(2m, q)$ at the end.

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1 Introduction

Let $G$ be a finitely generated group, and $G^n$ the direct products of $n$ copies of $G$. The growth sequence of $G$ is the sequence $\{d(G^n)\}_{n \geq 1}$, where $d(G^n)$ is the

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minimum number of generators of \( G^n \). The growth sequence of finite groups are known with the accuracy in terms of various parameters [see 13, 14, 15, 16], and quite a number of results are known in the case of finitely generated infinite groups [17]. One of the main theoretical tools in the finite case is a result of P. Hall [6], showing that for a finite non-abelian simple group \( G \), and \( k \geq d(G) \)

\[
h(k, G) = \max\{ n : d(G^n) = k \} = \frac{1}{|\text{Aut}(G)|} \sum_{H \leq G} \mu(H, G)|H|,
\]

where \( \mu \) is the Möbius function of the subgroup lattice of \( G \). Of course, the Möbius function is hard to calculate, even for small groups such as the alternating group \( A_{10} \) or projective symplectic linear groups \( PSp(4, 4) \) or \( PSp(6, 2) \).

This article provides a lower bound for \( h(2, PSp(2m, q)) \), where \( 2 \leq m \leq 5 \), \( m \geq 10 \) and \( q \geq 2 \).

The method to find the lower bound given in the present article is to use a difficult theorem about the maximal subgroups of \( PSp(2m, q) \) by M. Aschbacher [1], and a list of maximal subgroups of \( PSp(2m, q) \) for some small values of \( m \) and \( q \), by P.B. Kliedman [9, 10, 11]. This idea comes from the probability of generating a finite classical group given by Kantor and Lubotzky [8] and a result of J. Thevenaz [12].

## 2 Some definitions and basic results

In this section, we give some definitions and known results, which are necessary for our purpose. We start with the following two concepts.

**Definition 2.1** We denote \( \phi_n(G) \) to be the total number of distinct \( n \) - tuples of elements of \( G \) which generate \( G \), and call \( \phi_n(G) \) the \( n \)\textsuperscript{th} Eulerian function. If \( G \) can not be generated by \( n \) elements, then \( \phi_n(G) = 0 \)

**Definition 2.2** For any fixed element \( x \) of a group \( G \), the size of the set

\[
\{ y \in G | < x, y > = G \}
\]

is denoted by \( cs(x) \), which is called the cospread of \( x \) in \( G \). Clearly \( cs(x) = 0 \) if \( G \) can not be generated by elements \( x \) and \( y \), for every \( y \in G \).
P. Hall [4] observed that $h(n, G) = \frac{\phi_n(G)}{|\text{Aut}(G)|}$ for non-abelian simple group $G$. Moreover, we may check that $h(2, G) = \frac{1}{|\text{Aut}(G)|} \sum_{x \in G} cs(x)$ (see [3]). So the lower bound for $\phi_2(PSp(2m, q))$ will provide a lower bound for $h(2, PSp(2m, q))$. Thus, we actually need to find a lower bound for $\phi_2(PSL(m, q))$. Now, we state the following inequality from [12], which gives a lower bound for the Eulerian function of $G$.

**Theorem 2.3** (J. Thevenaz [12]) Let $G$ be a non trivial finite group and $k$ a positive integer. Then

$$\phi_k(G) \geq |G|^k - \sum_i |M_i|^k,$$

where the sum is over all maximal subgroups $M_i$ of $G$.

The case $k = 2$ and $G = PSp(2m, q)$ is the important case for our investigation. On the other hand, it is not quite clear what the list of maximal subgroups of $PSp(2m, q)$ are for large $m$ and $q$. So, by the above theorem we need to find a suitable upper bound for $\sum_i |M_i|^2$. The best information on the maximal subgroups of $PSp(2m, q)$ is the following subgroup structure theorem of M. Aschbacher in [1].

**Theorem 2.4** (M. Aschbacher [1]) Let $M$ be a maximal subgroup of a finite simple classical group. Then either

(i) $M$ is a known geometric subgroup (stabilizer of a subspace, a direct sum decomposition, a tensor product decomposition, a subfield, an extension field, a form, etc), or

(ii) $M$ is almost simple, absolutely irreducible.

The geometric subgroups in (i) split naturally into 8 classes, which are denoted by $C_1, C_2, \ldots, C_8$. The set of the remaining maximal subgroups (i.e. class 9) is often denoted by $S$. We describe it briefly here; for full details refer to Kleidman and Liebeck [11].

Let $V$ be a vector space of dimension $n = 2m$ on a field $F_q$, $M$ be a maximal subgroup of $PSp(V, F) = PSp(2m, q)$ and $PSp(2m, q) \leq G \leq \text{Aut}(PSp(2m, q))$, where $n \geq 4$ and $q \geq 2$. Then $M$ falls into one of the following classes of subgroups:
C_1 : the stabilizer of a subspace of V or a pair of subspaces V_1, V_2 such that \( \dim V_1 + \dim V_2 = m \) and either \( V_1 \subseteq V_2 \) or \( V = V_1 \oplus V_2 \).

C_2 : the stabilizer of a direct sum decomposition \( V = V_1 \oplus V_2 \), where \( V_i \)'s are of the same dimensions.

C_3 : the stabilizer of a field extension \( F_q \), whose degree is a prime dividing \( m \).

C_4 : the stabilizer of a tensor product decomposition \( V = V_1 \otimes V_2 \).

C_5 : the centralizer of a field automorphism.

C_6 : the normalizer of a symplectic - type of groups, for a prime \( r \neq p \) (in an irreducible representation).

C_7 : the stabilizer of a tensor product decomposition \( V = V_1 \otimes V_2 \), where \( V_i \)'s are of the same dimensions.

C_8 : a classical subgroup embedded as usual.

S : \( M = N_G(T) \), where \( T \) is a non - abelian simple subgroup of \( PSp(V) \) such that \( T \leq M \leq Aut(T) \), and the universal cover \( \tilde{T} \) of \( T \) acts absolutely irreducible on \( V \) in a representation defined over no proper subfield of \( F_q \).

What necessary to provide here is to find an upper bound for \( \sum M |M| \) in each of 9 cases; of course, the number of conjugacy classes of maximal subgroups of each type is important for us. By W. M. Kantor and A. Lubotzky [8], the number of conjugacy classes of maximal subgroups bounded above by \( 2n \) classes in case \( C_1 \), \( n \) (as an upper bounded on the number of divisors of \( n \)) classes for each of the cases \( C_2, C_3 \) and \( C_4 \), \( \log_2 q \) in case \( C_5 \), one class in case \( C_6 \), \( \log_2 n \) in case \( C_7 \), and 4 classes in case \( C_8 \). It is not known the number of conjugacy classes of maximal subgroups in case \( S \) for all values \( n \) and \( q \), but there is an upper bound for \( \sum_{M \in S} |M| \) by [8], where the sum runs over all non - conjugate maximal subgroup \( M \) of \( PSp(2m, q) \) in \( S \) with \( n \geq 10 \) and \( q \geq 3 \). Now, we state the main results of this paper as the following:
**Theorem A** Let $G = \text{PSp}(2m, q)$ and simple, then

$$h(2, G) > \frac{88}{100} \frac{|G|}{|\text{Out}(G)|}$$

where $2 \leq m \leq 5$, $q \geq 2$; $m \geq 10$, $q \geq 2$ or $m \geq 9$, $q \geq 4$.

**Theorem B** Let $G = \text{PSp}(2m, q)$ as in Theorem A. Then

$$h(2, G)^2 > q^{m^2} |G|.$$

Comparing to the order of $G$ in Theorem A, one observes that $|\text{Out}G|$ is very small.

### 3 Proof of Theorem A

In [8], Kantor and Lubotzky gave the upper bound $3[5(n + 1) + \log_2(nq)]$ for the number of conjugacy classes of maximal subgroups of type $C_1 - C_8$ of $\text{PSp}(2m, q)$, where $n = 2m$. They have also proved that for every maximal subgroup $M \in C_1 - C_8$, $\frac{|M|}{|G|} \leq \frac{2}{q^{n-1}}$. Thus, we see that

$$\sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} \leq \frac{6[5(n + 1) + \log_2(nq)]}{q^{n-1}}.$$

The right hand side of the above inequality gives a value less than 1 for all $n = 2m \geq 4$, except $(n, q) = (4, 2)$, which the group $G$ is not simple in this case.

The estimate of $\sum_{M_i \in S} \frac{|M_i|}{|G|}$ is more problematic. Let $S' = S - E$ where $E$ consists of all maximal subgroups $M$ not satisfying the inequality $|M| \leq q^{3n}$. Kantor and Lubotzky [8] shown that

$$\sum_{M_i \in E} \frac{|M_i|}{|G|} \leq \frac{1}{q^m} = \frac{1}{q^{n/2}}$$

which is less than one for all $(n, q)$.

They also gave $18n^2q^{3n}(\log_2 q)^2$ as an upper bound for $\sum_{M_i \in S'} |M_i|$. So, we have

$$\sum_{M_i \in S'} \frac{|M_i|}{|G|} = \frac{1}{|G|} \sum_{M_i \in S'} |M_i| \leq \frac{18n^2q^{3n}(\log_2 q)^2}{|G|}.$$
which is less than 1, for all values \( n = 2m \geq 20, q \geq 2 \) and \( n = 2m \geq 18 \) if \( q \geq 4 \). Thus we may to consider the following three cases.

**Case 1:** \( n = 2m \geq 20 \) and \( q \geq 2 \).

Assume \( G = PSp(2m, q) \) with \( m \geq 10 \) and \( q \geq 2 \). Then we have

\[
\phi_2(G) \geq (1 - \sum_i \frac{|M_i|}{|G|})|G|^2
\]

\[
= [1 - \left( \sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} + \sum_{M_i \in E} \frac{|M_i|}{|G|} + \sum_{M_i \in S'} \frac{|M_i|}{|G|} \right)]|G|^2
\]

\[
\geq [1 - \left( \frac{6(5(n + 1) + \log_2(nq))}{q^{n-1}} + \frac{1}{q^2} + \frac{18(n^2q^{6n}(\log_2 q))^2}{|G|} \right)]|G|^2
\]

\[
\geq [1 - \left( \frac{6(105 + \log_2(40))}{2^{19}} + \frac{1}{2^{10}} + \frac{18(20)^22^{180}(\log_2 2)^2}{|PSp(20, 2)|} \right)]|G|^2
\]

\[
> \frac{99}{100}|G|^2.
\]

**Case 2:** \( n = 2m \geq 18 \) and \( q \geq 4 \).

By the similar method as above we can easily find that \( \phi_2(G) \geq \frac{99}{100}|G|^2 \).

**Case 3:** \( 4 \leq n = 2m \leq 10 \) and \( q \geq 2 \).

We may rely on the estimates derived from [8] for \( M_i \in C_1 - C_8 \), namely

\[
\sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} \leq \frac{6[5(n + 1) + \log_2(nq)]}{q^{n-1}},
\]

for \( 4 \leq n \leq 10 \) and \( q \geq 2 \). For \( M \in S \), we have to use some maximal subgroups structure from Kleidman’s Ph.D thesis [9]. Of course, sometimes we should compute \( \phi_2(G) \) directly for some small values \((n, q)\).

Now, we start with \( n = 10 \). If \( G = PSp(10, q) \) and \( q \geq 3 \) then we have

\[
\sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} \leq \frac{6[55 + \log_2 10q]}{q^9} \leq \frac{6[55 + \log_2 30]}{3^9} = 0.0182615.
\]

From Kleidman’s list of maximal subgroups given in [9], we have that if \( M \in S \), then \( |M| \leq |PSU(5, 2)| \). Also there are at most 12 conjugacy classes of maximal subgroups of type S. It follows that

\[
\sum_{M_i \in S} \frac{|M_i|}{|G|} \leq 12 \frac{|PSU(5, 2)|}{|PSp(10, q)|} \leq 12 \frac{|PSU(5, 2)|}{|PSp(10, 3)|} = 2.14797 \times 10^{-18}
\]

So, for \( G = PSp(10, q) \) and \( q \geq 3 \), we have
\[ \phi_2(G) \geq (1 - \sum_i \frac{|M_i|}{|G|})|G|^2 \geq (1 - (\sum_{M_i \in C_1-C_8} \frac{|M_i|}{|G|} + \sum_{M_i \in S} \frac{|M_i|}{|G|}))|G|^2 \]
\[ \geq (1 - (0.0182615 + 2.14797 \times 10^{-18})) \geq (1 - 0.0182615)|G|^2 > \frac{98}{100}|G|^2. \]

If \( q = 2 \) then by using the complete list of maximal subgroups of \( PSp(10,2) \) given in [9] we have
\[ \phi_2(G) \geq (1 - \sum_i \frac{|M_i|}{|G|})|G|^2 \geq (1 - 0.004926)|G|^2 > \frac{99}{100}|G|^2. \]

Thus, if \( G = PSp(10,q) \) and \( q \geq 2 \) then \( \phi_2(G) > \frac{98}{100}|G|^2. \)

If \( n = 8 \) and \( G = PSp(8,q) \) then for \( q \geq 4 \) we can similarly show that
\[ \sum_{M_i \in C_1-C_8} \frac{|M_i|}{|G|} \leq \frac{6[45 + \log_2(8q)]}{q^7} \leq \frac{6[45 + \log_2(32)]}{4^7} = 0.0183105. \]

Using Kleidman’s list, if \( M \in S \) then for \( 4 \leq q \leq 17 \) we have that \( |M| \leq |PSL(2,17)| \) and there are at most 7 conjugacy classes of maximal subgroups of type \( S \). Therefore
\[ \sum_{M_i \in S} \frac{|M_i|}{|G|} \leq 7 \frac{|M|}{|G|} \leq 7 \frac{|PSL(2,17)|}{|PSp(8,q)|} \leq 7 \frac{|PSL(2,17)|}{|PSp(8,4)|} = 3.88678 \times 10^{-18} \]

If \( q > 17 \) then \( |M| \leq |PSL(2,q)| \) and there are at most 7 conjugacy classes of maximal subgroups of type \( S \). So
\[ \sum_{M_i \in S} \frac{|M_i|}{|G|} \leq 7 \frac{|M|}{|G|} \leq 7 \frac{|PSL(2,q)|}{|PSp(8,q)|} \leq 7 \frac{q^{15}}{(17)^{15}} = 2.44548 \times 10^{-18} \]

Hence
\[ \phi_2(G) \geq (1 - \sum_i \frac{|M_i|}{|G|})|G|^2 \geq (1 - 0.0183105)|G|^2 > \frac{98}{100}|G|^2. \]

From Kleidman’s list one can compute precisely that
\[ \phi_2(PSp(8,3)) > \frac{99}{100}|G|^2, \quad \phi_2(PSp(8,2)) > \frac{97}{100}|G|^2. \]

Thus, if \( G = PSp(8,q) \) and \( q \geq 2 \) then \( \phi_2(G) > \frac{97}{100}|G|^2. \)
Assume that $G = PSp(6, q)$, then for $q \geq 5$ we have
\[
\sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} \leq \frac{6[35 + \log_2(6q)]}{q^5} \leq \frac{6[35 + \log_2(30)]}{5^5} = 0.076621
\]

If $M \in S$ then $|M| < |G_2(q)|$ and there are at most 17 conjugacy classes of $M$ by [9]. So, it follows that
\[
\sum_{M_i \in S} \frac{|M_i|}{|G|} \leq 17 \frac{|G_2(q)|}{|PSp(6, q)|} \leq 17 \frac{q^6(q^6 - 1)(q^2 - 1)}{(2q - 1)q^6(q^4 - 1)(q^4 - 1)(q^2 - 1)} \leq \frac{34}{q^3(q^4 - 1)} \leq \frac{34}{53(5^4 - 1)} = 4.358974 \times 10^{-4}.
\]

Therefore
\[
\phi_2(G) \geq (1 - \sum_i \frac{|M_i|}{|G|})|G|^2 \geq (1 - 0.076621)|G|^2 > \frac{92}{100} |G|^2.
\]

We have checked by the complete list of maximal subgroups of the groups $PSp(6, 2)$, $PSp(6, 3)$ and $PSp(6, 4)$ in [9] that
\[
\phi_2(PSp(6, 4)) > \frac{99}{100} |G|^2, \quad \phi_2(PSp(6, 3)) > \frac{96}{100} |G|^2, \quad \phi_2(PSp(6, 2)) > \frac{89}{100} |G|^2.
\]

Thus if $G = PSp(6, q)$ and $q \geq 2$ then $\phi_2(G) > \frac{89}{100} |G|^2$.

If $G = PSp(4, q)$, then for $q \geq 13$ we have
\[
\sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} \leq \frac{6[25 + \log_2(4q)]}{q^3} \leq \frac{6[25 + \log_2(52)]}{13^3} = 0.083842
\]

In the case $M \in S$, we can see that $|M| \leq q^2(q^2 + 1)(q - 1)$ and there are at most 5 conjugacy classes of maximal subgroups of type $S$ by [9]. So we have
\[
\sum_{M_i \in S} \frac{|M_i|}{|G|} \leq 5 \frac{q^2(q^2 + 1)(q - 1)}{(2q - 1)q^4(q^4 - 1)(q^2 - 1)} \leq \frac{10}{q^2(q^2 - 1)(q + 1)} \leq \frac{10}{13^2(13^2 - 1)(13 + 1)} = 2.515799 \times 10^{-5}.
\]

Putting the estimates together we have
\[
\phi_2(G) \geq (1 - \sum_i \frac{|M_i|}{|G|})|G|^2 \geq (1 - 0.083842)|G|^2 > \frac{91}{100} |G|^2.
\]
Using Kleidman’s list as in [9], if $7 \leq q \leq 11$, then for any $M \in C_1 - C_8$ we have $|M| \leq q^4(q^2 - 1)(q - 1)$ and so

$$\frac{|M|}{|G|} \leq \frac{q^4(q^2 - 1)(q - 1)}{(2q^2 - 1)(q^4 - 1)(q^2 - 1)} \leq \frac{2}{(q^2 + 1)(q + 1)}.$$

We know that there are at most 13 conjugacy classes of maximal subgroups of type $S$. Thus

$$\sum_{M_i \in C_1 - C_8} \frac{|M_i|}{|G|} \leq \frac{26}{(q^2 + 1)(q + 1)} \leq \frac{26}{(7^2 + 1)(7 + 1)} = 0.065$$

Similarly, if $M \in S$ then we have $|M| \leq q^2(q^2 + 1)(q - 1)$ and there are at most 5 conjugacy classes of maximal subgroups of type $S$. Hence

$$\sum_{M_i \in S} \frac{|M_i|}{|G|} \leq \frac{10}{q^2(q^2 - 1)(q + 1)} \leq \frac{10}{7^2(7^2 - 1)(7 + 1)} = 0.000531$$

Therefore

$$\phi_2(G) \leq (1 - \sum_i |M_i|/|G|)|G|^2 \geq (1 - 0.065)|G|^2 > \frac{93}{100}|G|^2,$$

when $7 \leq q \leq 11$.

Again by direct computation we have,

$$\phi_2(PSp(4, 5)) \geq \frac{98}{100}|G|^2, \quad \phi_2(PSp(4, 4)) \geq \frac{94}{100}|G|^2, \quad \phi_2(PSp(4, 3)) \geq \frac{88}{100}|G|^2.$$  

Thus, if $G = PSp(4, q)$ and $q \geq 3$ then $\phi_2(G) > \frac{88}{100}|G|^2$.

Finally, by putting all the above cases we can state that $\phi_2(G) > \frac{88}{100}|G|^2$, when $G = PSp(2m, q)$ for all $(m, q)$ in which $2 \leq m \leq 5$, $q \geq 2$; $m \geq 10$, $q \geq 2$ or $m \geq 9$, $q \geq 4$. Thus

$$h(2, G) = \frac{\phi_2(G)}{|\text{Aut}(G)|} > \frac{88}{100} \frac{|G|}{|\text{Out}(G)|}$$

and the proof of Theorem A is completed.

## 4 Proof of Theorem B

To proof Theorem B, it is enough to show that

$$\left(\frac{88}{100} \frac{|G|}{|\text{Out}(G)|}\right)^2 > q^{m^2}|G|$$
or equivalently
\[
\left( \frac{88}{100} \right)^2 \frac{|G|}{|Out(G)|^2 q^{m^2}} > 1.
\]
by Theorem A.

We know that if \( G = PSp(2m, q) \) then
\[
|Out(G)| = \begin{cases} 
  de & m \geq 3 \\
  (2, p)de & m = 2 
\end{cases},
\]
where \( d = (2, q - 1) \) and \( q = p^e \). Therefore \( |Out(G)| \leq 2e \) in each cases. Thus we have
\[
\left( \frac{88}{100} \right)^2 \frac{|G|}{|Out(G)|^2 q^{m^2}} \geq \left( \frac{88}{100} \right)^2 \frac{q^{m^2}(q^{2m} - 1)(q^{2m - 2} - 1) \cdots (q^2 - 1)}{4de^2 q^{m^2}}
\]
\[
\geq \left( \frac{88}{100} \right)^2 \frac{(q^{2m} - 1)(q^{2m - 2} - 1) \cdots (q^2 - 1)}{8e^2}
\]
\[
\geq \left( \frac{88}{100} \right)^2 \frac{q^4 - 1}{8}
\]
\[
\geq \left( \frac{88}{100} \right)^2 \frac{2^4 - 1}{8} > 1
\]
and the proof of Theorem B is completed.

One may note that for all \( m \geq 3 \), we can find the better lower bound
\[
h(2, PSp(2m, q))^2 > q^{m^2}(q^{2m} - 1)(q^{2m - 2} - 1) \cdots (q^6 - 1)|G|.
\]
The lower bound given in Theorem B conforms the conjecture \( h(2, G)^2 > |G| \); for all groups \( PSp(2m, q) \) and all values \( (n, q) \) as given in Theorem A. Of course, in spite of the fact that we believe the conjecture is true for the remaining values of \( (n, q) \) but we have not been able to proof it yet.

References


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