On Totally Smooth Groups

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Abstract

A maximal chain in a finite lattice $L$ is called smooth if any two intervals of the same length are isomorphic. A group $G$ is called smooth if its subgroup lattice $L(G)$ has a smooth chain, and we call $G$ totally smooth if all maximal chains in $L(G)$ are smooth. In this paper we will determine all finite totally smooth groups and all finite groups all of whose proper subgroups are totally smooth.

Mathematics Subject Classification: 20D30, 20E15

Introduction. A maximal chain $0 = a_0 < a_1 < ... < a_n = I$ in a lattice $L$ with least element 0 and greatest element $I$ is called smooth if $[a_{i+j}/a_i] \cong [a_j/0]$ for all $i, j \in N$ such that $i + j \leq n$. A group $G$ is called smooth if its subgroup lattice $L(G)$ has a smooth chain. Finite smooth groups have been studied by Schmidt [3, 4]. We call a lattice $L$ totally smooth if all maximal chains of elements of $L$ are smooth. A group $G$ is called totally smooth if its subgroup lattice $L(G)$ is totally smooth.

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In this paper we will determine all finite totally smooth groups and also the finite groups all of whose proper subgroups are totally smooth.

We give the structure of totally smooth groups in the following Theorem.

**Theorem 1.** A finite group \( G \) is totally smooth if and only if one of the following holds:

(i) \( G \) is cyclic of prime power order.

(ii) \( G \) is a \( P \)-group.

(iii) \( G \) is cyclic of square free order.

**Proof.** It is clear that if \( G \) satisfies (i), (ii), or (iii), then every maximal chain of subgroups of \( G \) is smooth and hence \( G \) is totally smooth.

For the converse, suppose that the Theorem is false and let \( G \) be a counterexample of minimal order.

If \( L(G) \) has a maximal chain of length at most 2, then by [4; Lemma 1.3, p. 184], \( |G| = p^2 \) or \( |G| = pq \) or \( PQ \) where \( P \) is elementary abelian of order \( p^m \) and \( Q \) is cyclic of prime order operating irreducibly on \( P \) for \( p, q \in \pi(G) \) and \( m \geq 2 \).

Hence \( G \) is a cyclic group of prime power order or a \( P \)-group or cyclic of square free order which contradicts our choice of \( G \). So all maximal chains have length at least 3.

The minimality of \( G \) implies that every maximal subgroup of \( G \) satisfies (i), (ii), or (iii). So we have to handle the following three cases:

**Case 1.** There exists a cyclic maximal subgroup of prime power order \( p^k \) where \( k \geq 2 \). Then [4; Theorem A, p. 185] shows that \( G \) is a cyclic group of prime power order, a contradiction.

**Case 2.** There exists a maximal subgroup \( M \) which is a \( P \)-group.

If \( G \) is a \( p \)-group, then \( M \) is elementary abelian. Let \( H \) be any minimal subgroup of \( M \). Since \( G \) is totally smooth, it follows that \( [G/H] \cong L(M) \) and hence \( \Phi(G) \leq H \). Since \( |M| \geq p^2 \), there exist at least two minimal subgroups
of $M$, say, $H_1$ and $H_2$. Therefore, $\Phi(G) \leq H_1 \cap H_2 = 1$. Hence $G$ is elementary abelian, a contradiction.

Thus $G$ is not a $p$–group. Then $M$ is either elementary abelian or a semi-direct product of an elementary abelian group $L$, say, of order $p^{n-2}$ by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on $L$ (see [2], p. 49).

Suppose first that $M$ is elementary abelian and let $H$ be any minimal subgroup of $M$. Since $[G/H] \cong L(M)$, it follows from [4; Corollary 2.5, p.187] that $H \triangleleft G$ and that $G/H$ is a nonabelian $P$–group of order $p^{n-2}q, p > q$ for some $n$. So $M$ is the unique Sylow $p$–subgroup of $G$. As shown above, every subgroup of $M$ is normal in $G$ which implies that $Q$ induces a universal power automorphism in $M$. Then $G$ is a $P$–group, a contradiction.

Therefore $M$ is a nonabelian $P$–group of order $p^r q, p > q, r \in \mathbb{N}$. Let $K$ be a subgroup of $M$ such that $|K| = q$. Obviously, $K$ is not normal in $M$. Since $G$ is totally smooth, it follows that $[G/K] \cong L(M)$. Hence by [4; Proposition 2.4, p. 186], $K = N_G(K)$ and $G$ is a $P$–group, a contradiction.

**Case 3.** There is no maximal subgroup which is cyclic of prime power order or a $P$–group. Then all maximal subgroups of $G$ are cyclic of square free order. Therefore all Sylow subgroups of $G$ are of prime order. If $G$ would not be nilpotent, then $G$ is minimal non-nilpotent and by [1; Satz 5.2, p. 281], $|G| = pq$. It follows that the maximal chains of subgroups of $G$ have length 2, a contradiction. Thus $G$ would be nilpotent and hence all Sylow subgroups of $G$ are normal in $G$. Then $G$ is cyclic of square free order, a final contradiction.

We study now groups for which all maximal subgroups are totally smooth. The following theorem gives the structure of these groups. Recall that a $P$–group is a group lattice–isomorphic to an elementary abelian group (see [2], p. 49).

**Theorem 2.** All maximal subgroups of a finite group $G$ are totally smooth if and only if one of the following holds:
(i) $G$ is of order $p_1p_2p_3$; where $p_1, p_2$ and $p_3$ are not necessary distinct primes.

(ii) $G$ is a totally smooth group.

(iii) $G = PQ$, where $P$ is an elementary abelian normal Sylow $p$–subgroup and $Q$ is cyclic of order $q$ which operates irreducibly on $P$.

(iv) $G = PQ$, where $P$ is an elementary abelian normal Sylow $p$–subgroup of order $p^q$, $Q$ is cyclic of order $q^2$ which operates irreducibly on $P$ and $q|p - 1$.

**Proof.** Assume first that all maximal subgroups of $G$ are totally smooth. Applying Theorem 1, all maximal subgroups of $G$ are super-solvable and hence by Huppert’s Theorem (see[1; Satz 9.6, p. 718]) $G$ is solvable.

If the maximal length $n$ of a chain in $L(G)$ is at most 3, then $G$ is of order $p_1p_2p_3$ or $p_1p_2$ or $p_1$, where $p_1, p_2$ and $p_3$ are not necessarily distinct primes and hence we are done. Thus we may assume that $n \geq 4$.

Now we have the following cases:

**Case 1.** $G$ is a $p$–group for some prime $p$. By Theorem 1, every maximal subgroup of $G$ is cyclic or elementary abelian. Suppose that $M$ and $N$ are maximal subgroups of $G$, where $M$ is elementary abelian and $N$ is cyclic. Then $M \cap N \neq 1$ and since $M \cap N$ is cyclic and elementary abelian, it follows that $|M \cap N| = p$. Then $n = 3$, a contradiction since $n \geq 4$. Thus either all maximal subgroups of $G$ are elementary abelian or all maximal subgroups of $G$ are cyclic.

Assume first that all maximal subgroups of $G$ are cyclic. Then $G$ has exactly one subgroup of order $p$ and hence $G$ is cyclic since $n \geq 4$ (see[1; Satz 8.2, p. 310]).

Now assume that all maximal subgroups of $G$ are elementary abelian. Then $M_1 \cap M_2 \leq Z(G)$ where $M_i$ is a maximal subgroup of $G$ for $i = 1, 2$. Hence $G' \leq Z(G)$. We argue that $G$ is abelian. If not, then there exist $x, y \in G$ such
that \( xy \neq yx \). Then \( G = \langle x, y \rangle \) and hence \( G' = \langle [x, y] \rangle \) and \( G' \) is cyclic of order \( p \) since it is elementary abelian. Since \( G/G' = \langle x, y \rangle G'/G' \) has order \( p^2 \), it follows that \( |G| = p^3 \), a contradiction since \( n \geq 4 \). Thus \( G \) is abelian, and consequently \( G \) is elementary abelian. In particular, \( G \) is totally smooth.

**Case 2.** \( |G| \) is divisible by exactly two different primes \( p \) and \( q \). Since \( G \) is solvable, \( G \) has a minimal normal subgroup \( L \) of order \( p^r \), say. Let \( P \) be a Sylow \( p \)–subgroup and \( Q \) a Sylow \( q \)–subgroup of \( G \).

Suppose first that \( |L| = p \) and let \( Q_1 \) be a maximal subgroup of \( Q \). If \( L = P \), then \( LQ_1 \) is a proper subgroup of \( G \). It follows, by hypothesis, that \( LQ_1 \) is totally smooth. Hence Theorem 1 shows that \( LQ_1 \) is a nonabelian \( P \)–group or cyclic of square free order. Then \( |Q_1| = q \) and so \( |Q| = q^2 \), which implies that \( n = 3 \), a contradiction.

Thus \( L \) is a proper subgroup of \( P \). Hence \( LQ \) is a proper subgroup of \( G \). By hypothesis, \( LQ \) is totally smooth. Applying Theorem 1, \( LQ \) is a nonabelian \( P \)–group or cyclic of square free order and hence \( |Q| = q \).

Assume that \( Q < G \) and so \( HQ \) is a proper subgroup of \( G \), where \( H \) is a maximal subgroup of \( P \). Since \( HQ \) is totally smooth, Theorem 1 implies that \( |H| = p \) as \( Q < G \). Since \( H \) is a maximal subgroup of \( P \), we have that \( |P| = p^2 \) and hence \( |G| = p^2q \), and \( n = 3 \), a contradiction as \( n \geq 4 \). Thus \( Q \) is not normal in \( G \). By hypothesis, \( P \) is totally smooth. By Theorem 1, \( P \) is elementary abelian or cyclic. If \( P \) would not be normal in \( G \), it would follow that \( P = N_G(P) \) as \( P \) is a maximal subgroup of \( G \). By Burnside’s Theorem, \( Q < G \), a contradiction. Thus \( P < G \).

If \( P \) would be cyclic, \( H_1 < G \) where \( H_1 \) is a maximal subgroup of \( P \) and so \( H_1Q < G \) which implies, by using Theorem 1, that \( |H_1| = p \) since \( H_1 \) is cyclic. Then \( |G| = p^2q \), and \( n = 3 \), a contradiction as \( n \geq 4 \). Thus \( P \) is elementary abelian. By Maschke’s Theorem, \( P \) is completely reducible under \( Q \) and so \( L \) has a complement \( K \), say, in \( P \) which is normal in \( G \). Hence \( KQ \) is a proper
subgroup of $G$. Once again, by hypothesis and Theorem 1, $KQ$ is a nonabelian $P-$group or cyclic. Since $n \geq 4$ and $|L| = p$, it follows that $|K| > p$ and so $KQ$ is a nonabelian $P-$group. Then all subgroup of $K$ are normal in $KQ$ and hence normal in $G$.

Now let $P_1$ be any subgroup of $P$ of order $p$ such that $P_1 \neq L$. Since $n \geq 4$, $U = LP_1$ is a proper subgroup of $P$. By Dedekind’s rule $U = LK \cap U = L(K \cap U)$. Since $L$ and $K \cap U$ are $Q-$invariant, we have that $UQ$ is a proper subgroup of $G$ and so is totally smooth. Applying Theorem 1, $UQ$ is a nonabelian $P-$group as $|U| = p^2$. It follows that $Q$ dose not centralize $L$ and $P_1 \triangleleft UQ$. Since $P$ is elementary abelian, $P_1 \triangleleft G$. Therefore, every subgroup of $P$ is normal in $G$ and $Q$ induces a nontrivial power automorphism in $P$. Then $G$ is a nonabelian $P-$group. In particular, $G$ is totally smooth.

Now suppose that $|L| > p$. If $L$ is a proper subgroup of $P$, $LQ$ is a proper subgroup of $G$ and by hypothesis is totally smooth. By Theorem 1, $LQ$ is a nonabelian $P-$group. Then there exists a normal subgroup $L_1$, say, of $LQ$ of order $p$. Since $L_1 \triangleleft LQ$ and $L_1 \triangleleft P$, it follows that $L_1 \triangleleft G$ which contradicts the minimality of $L$. Thus $L = P$ and hence $P$ is a minimal normal subgroup of $G$.

If $|Q| = q$, then (iii) holds. So assume that $|Q| \geq q^2$ and let $Q_1$ be a maximal subgroup of $Q$. Then $PQ_1$ is a proper subgroup of $G$. By hypothesis, $PQ_1$ is totally smooth and by Theorem 1, $PQ_1$ is a $P-$group. Hence $|Q_1| = q$ and so $|G| = q^2$. If $Q$ would be elementary abelian, $PQ_i$ would be a nonabelian $P-$group for every maximal subgroup $Q_i$ of $Q$. Then $G$ would have a normal subgroup of order $p$ which would contradict the minimality of $P$. Thus $Q$ is cyclic, operates irreducibly on $P$ and $Q_1$ normalizes every subgroup of $P$. Since $q | p - 1$, it follows by [5; Lemma 3.1], that $|P| = p^q$. Thus (iv) holds.

Case 3. $|G|$ is divisible by $m \geq 3$ different primes $p_1, p_2, ..., p_m$. Suppose, for a contradiction, that $p_1^2 || G$.
Since $G$ is solvable, $G$ has a Sylow basis $\{P_1, P_2, ..., P_m\}$. Hence $P_1P_i$ is a proper subgroup of $G$ for $i = 2, 3, ..., m$. By our hypothesis, $P_1P_i$ is totally smooth and by Theorem 1, $P_1P_i$ is a nonabelian $P$–group since $|P_1| > p_1$.

Hence $|P_i| = p_i$ and $P_1$ has a normal subgroup $H$, say, of $G$ of order $p_1$. Consequently, $HP_2P_3...P_m$ is a proper subgroup of $G$. Then by Theorem 1, $HP_2P_3...P_m$ is cyclic of square free order which implies that $P_i$ centralizes $H$, a contradiction as $P_1P_i$ is a nonabelian $P$–group.

Thus every Sylow subgroup of $G$ is cyclic of prime order. Since $n \geq 4$, for every $i, j \in \{1, 2, ..., n\}$ with $i \neq j$, there exists $k \in \{1, 2, ..., n\}$ such that $P_iP_jP_k$ is a proper subgroup of $G$ where $i \neq k \neq j$. By Theorem 1, $P_iP_jP_k$ is cyclic and hence $[P_i, P_j] = 1$. It follows that $G$ is abelian and consequently $G$ is cyclic of square free order, in particular, $G$ is totally smooth.

Conversely, if $G$ satisfies (i), (ii), or (iii), then, clearly, all maximal subgroups of $G$ are totally smooth. So assume that $G = PQ$, where $P$ is an elementary abelian normal subgroup of order $p^q$, $Q$ is a cyclic subgroup of order $q^2$ operating irreducibly on $P$ and $q|p - 1$. Let $Q_1$ be the subgroup of $Q$ of order $q$. Since $q|p - 1$, $Q_1$ is not irreducible on $P$. It follows by [5; Lemma 3.1], that $Q_1$ normalizes every subgroup of $P$. So $PQ_1$ is a $P$–group and all maximal subgroups of $G$ are totally smooth.

Acknowledgements. I would like to express my exceedingly gratitude, appreciation and sincere thanks to my supervisor Prof. Dr. Roland Schmidt, for his excellent guidance, helpful and useful discussions.

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Received: July 21, 2006