Ore Extensions over Pseudo-valuation Rings

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Abstract. Let R be a commutative Noetherian $\mathbb{Q}$ -algebra ($\mathbb{Q}$ is the field of rational numbers). Let δ be a derivation of R and σ be an automorphism of R . Then we prove the following:

1. If R is a Pseudo-valuation ring, then $R[x, \delta]$ is also a Pseudo-valuation ring.
2. If R is a divided ring, then $R[x, \delta]$ is also a divided ring.
3. If R is a Pseudo-valuation ring, then $R[x, x^{-1}, \sigma]$ is also a Pseudo-valuation ring.
4. If R is a divided ring, then $R[x, x^{-1}, \sigma]$ is also a divided ring.

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1. INTRODUCTION

All rings are commutative with identity $1 \neq 0$. $N(R)$ denotes the set of all nilpotent elements of R . $Z(R)$ denotes the centre of R . $\mathbb{Q}$ denotes the field of rational numbers. We recall that as in Hedstrom [13], an integral domain R with quotient field F , is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). In Badawi, Anderson and Dobbs [4], the study of pseudo-valuation domains was generalized to arbitrary rings. In Badawi, Anderson and Dobbs [4], a prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. We note that a PVR is quasilocal by Lemma 1(b) of Badawi, Anderson and Dobbs [4]. An integral domain is a PVR if and only if it is a PVD by Proposition (3.1) of Anderson [1], Proposition (4.2) of Anderson [2] and Proposition (3) of Badawi [5]. We recall that a prime ideal P of R is said to be divided if it is

comparable (under inclusion) to every ideal of R . A ring R is called a divided ring if every prime ideal of R is divided. We denote the set of prime ideals of R by $\text{Spec}(R)$ and the set of strongly prime ideals of R by $\text{S.Spec}(R)$.

In Badawi [6], another generalization of PVDs is given. In Badawi [6], for a ring R with total quotient ring Q such that $N(R)$ is a divided prime ideal of R , let $\phi : Q \rightarrow R_{N(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus N(R)$. Then ϕ is a ring homomorphism from Q into $R_{N(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{N(R)}$ given by $\phi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by T . A prime ideal P of $\phi(R)$ is called a T -strongly prime ideal if $xy \in P$, $x \in T$, $y \in T$ implies that either $x \in P$ or $y \in P$. $\phi(R)$ is said to be a T -pseudo-valuation ring (T -PVR) if each prime ideal of $\phi(R)$ is T -strongly prime. A prime ideal S of R is called ϕ -strongly prime ideal if $\phi(S)$ is a T -strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ -strongly prime, then R is called a ϕ -pseudo-valuation ring (ϕ -PVR).

Also recall from Badawi [7], a ring R is called a ϕ -chained ring (ϕ -CR) if $N(R)$ is a divided prime ideal of R and for every $a \in T \setminus \phi(R)$, we have $a^{-1} \in \phi(R)$. In Proposition (2.6) of Badawi [8] it is shown that if $N(R)$ is a divided prime ideal of R and P is a regular ϕ -strongly prime ideal of R . Then the total quotient ring Q of R is ϕ -CR.

This article concerns the study of differential operator rings and skew Laurent polynomial rings over PVDs. Let R be a ring and δ be a derivation of R . We denote the differential operator ring $R[x, \delta]$ by $D(R)$. If I is an ideal of R such that $\delta(I) \subseteq I$, then we denote $I[x, \delta]$ by $D(I)$. We would like to mention that $D(R)$ is the usual polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = xa + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x, \delta]$ to be of the form $f(x) = \sum x^i a_i$, $0 \leq i \leq n$.

Let R be a ring and σ be an automorphism of R . We denote the skew Laurent polynomial ring $R[x, x^{-1}, \sigma]$ by $L(R)$. We would also like to mention that in $L(R)$ multiplication is subject to the relation $ax = x\sigma(a)$ for all $a \in R$. If I is an ideal of R such that $\sigma(I) = I$, then we denote $I[x, x^{-1}, \sigma]$ by $L(I)$.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [3, 9, 10, 11, 14, 15].

Let now R be a commutative Noetherian Q -algebra. Let δ be a derivation of R and σ be an automorphism of R . Then we prove the following:

1. If R is a Pseudo-valuation ring, then $R[x, \delta]$ is also a Pseudo-valuation ring.
2. If R is a divided ring, then $R[x, \delta]$ is also a divided ring.
3. If R is a Pseudo-valuation ring such that $\sigma(P) = P$ for all $P \in \text{Spec}(R)$, then $R[x, x^{-1}, \sigma]$ is also a Pseudo-valuation ring.
4. If R is a divided ring such that $\sigma(P) = P$ for all $P \in \text{Spec}(R)$, then $R[x, x^{-1}, \sigma]$ is also a divided ring.

These results are proved in Theorems 2.6, 2.7, 2.8 and 2.9 respectively.

2. POLYNOMIAL RINGS

We begin with the following known results:

Lemma 2.1. *Let R be a Noetherian Q -algebra. Let δ be a derivation of R . If P is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of R and $\delta(P \cap R) \subseteq P \cap R$.*

Proof. See Theorem (2.22) of Goodearl and Warfield [12]. $\square$

Lemma 2.2. *Let R be a Noetherian Q -algebra. Let δ be a derivation of R . If P is a prime ideal of R such that $\delta(P) \subseteq P$, then $D(P)$ is a prime ideal of $D(R)$ and $D(P) \cap R = P$.*

Proof. See Theorem (2.22) of Goodearl and Warfield [12]. $\square$

Lemma 2.3. *Let R be a Noetherian ring. Let σ be an automorphism of R . If I is a prime ideal of R such that $\sigma(I) \subseteq I$, then $L(I)$ is an ideal of $L(R)$ and if J is an ideal of $L(R)$, then $J \cap R$ is an ideal of R and $\sigma(J \cap R) \subseteq J \cap R$.*

Proof. See Ex. 2ZA of Goodearl and Warfield [12]. $\square$

Theorem 2.4. (Hilbert Basis Theorem): *Let R be a right/left Noetherian ring. Let α be an automorphism of R and ρ be an α -derivation of R . Then the ore extension $O(R) = R[x, \alpha, \rho]$ is right/left Noetherian. Also $R[x, x^{-1}, \alpha]$ is Noetherian.*

Proof. See Theorems (1.12) and (1.17) of Goodearl and Warfield [12]. $\square$

Proposition 2.5. *Let R be a ring. Let σ be an automorphism of R and δ be a σ -derivation of R . Then:*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.*
2. *For any completely prime ideal Q of $R[x, \sigma, \delta]$, $Q \cap R$ is a completely prime ideal of R .*

Proof. See Proposition (2.4) of Bhat, Raina and Prakash [9]. $\square$

Theorem 2.6. *Let R be a commutative Noetherian Q -algebra which is also a PVR. Then $D(R)$ is also a PVR.*

Proof. Let $J \in \text{Spec}(D(R))$. Then by Theorem (2.22) of Goodearl and Warfield [12] $J \cap R \in \text{Spec}(R)$ and $\delta(J \cap R) \subseteq J \cap R$. Now R is a PVR, therefore $J \cap R \in S.\text{Spec}(R)$. Now 2.5 implies that $D(J \cap R) \in S.\text{Spec}(D(R))$. But $D(J \cap R) = J$ by Theorem (2.22) of Goodearl and Warfield [12]. Therefore $J \in S.\text{Spec}(D(R))$. Hence $D(R)$ is a PVR. $\square$

Theorem 2.7. *Let R be a divided commutative Noetherian Q -algebra. Then $D(R)$ is also divided Noetherian.*

Proof. $D(R)$ is Noetherian by 2.4. Let $J \in \text{Spec}(D(R))$ and $0 \neq K$ be a proper ideal of $D(R)$. Now by Theorem (2.22) of Goodearl and Warfield [12]

$J \cap R \in \text{Spec}(R)$ and $\delta(J \cap R) \subseteq (J \cap R)$. Also $K \cap R$ is an ideal of R and $\delta(K \cap R) \subseteq (K \cap R)$ by Lemma (2.18) of Goodearl and Warfield [12]. Now R is divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $(J \cap R) \subseteq (K \cap R)$. Therefore $D(J \cap R) \subseteq D(K \cap R)$. Thus $J \subseteq K$. Hence $D(R)$ is divided Noetherian. $\square$

Theorem 2.8. *Let R be a Noetherian ring which is also a PVR and $\sigma(P) = P$ for all $P \in \text{Spec}(R)$. Then $L(R)$ is also a PVR.*

Proof. Let $J \in \text{Spec}(L(R))$. Then by Ex. (2ZA) of Goodearl and Warfield [12] there exists $K \in \text{Spec}(R)$ and an integer $m \geq 1$ such that $\sigma^m(K) = K$, and $(J \cap R) = K \cap \sigma(K) \cap \sigma^2(K) \dots \sigma^{m-1}(K)$. Now by hypothesis $\sigma^t(K) = K$ for all integers $t \geq 1$. Therefore $(J \cap R) = K$. Now $L(K) = L(J \cap R) = J$ by (2ZA) of Goodearl and Warfield [12]. Therefore it is enough to show that $L(K) \in S.\text{Spec}(L(R))$.

Now $K \in \text{Spec}(R)$, therefore $K \in S.\text{Spec}(R)$ as R is a PVR. Therefore by 2.5 $L(K) \in S.\text{Spec}(L(R))$. Thus $J \in S.\text{Spec}(L(R))$. Hence $L(R)$ is a PVR. $\square$

Theorem 2.9. *Let R be a divided Noetherian ring and $\sigma(P) = P$ for all $P \in \text{Spec}(R)$. Then $L(R)$ is also divided.*

Proof. $L(R)$ is Noetherian by 2.4. Let $J \in \text{Spec}(L(R))$. Then as in 2.8 $(J \cap R) \in \text{Spec}(R)$. Now let K be an ideal of $L(R)$. Then by Ex. (2ZA) of Goodearl and Warfield [12] $K \cap R$ is an ideal of R and $\sigma(K \cap R) = K \cap R$. Now R is divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $(J \cap R) \subseteq (K \cap R)$. Therefore $L(J \cap R) \subseteq L(K \cap R)$. Thus $J \subseteq K$. Hence $L(R)$ is divided Noetherian. $\square$

REFERENCES

- [1] D. F. Anderson, Comparability of ideals and valuation rings, Houston J. Math. 5(1979), 451-463.
- [2] D. F. Anderson, When the dual of an ideal is a ring, Houston J. Math. 9(1983), 325-332.
- [3] S. Annin, Associated primes over skew polynomial rings, Communications in Algebra 30 (2002) 2511-2528.
- [4] A. Badawi, D. F. Anderson, and D. E. Dobbs, Pseudo-valuation rings, Lecture Notes Pure Appl. Math., Vol. 185(1997), 57-67, Marcel Dekker, New York.
- [5] A. Badawi, On domains which have prime ideals that are linearly ordered, Comm. Algebra 23(1995), 4365-4373.
- [6] A. Badawi, On ϕ -pseudo-valuation rings, Lecture Notes Pure Appl. Math., Vol. 205(1999), 101-110, Marcel Dekker, New York.
- [7] A. Badawi, On ϕ -chained rings and ϕ -valuation rings, Houston J. Math. 27(2001), 725-736.
- [8] A. Badawi, On the complete integral closure of rings that admit ϕ -strongly prime ideals, Lecture Notes Pure Appl. Math., Vol. 231(2002), 15-22, Marcel Dekker, New York.
- [9] V. K. Bhat, Ravi Raina, and Om Prakash, Ore extensions over δ -rigid rings, Int. J. Contem. Math. Sci. 1(12) (2006), 591-595.

- [10] V. K. Bhat, A note on Krull dimension of skew polynomial rings, *Lobachevskii J. Math.* 22(2006), 3-6.
- [11] W. D. Blair and L. W. Small, Embedding differential and skew-polynomial rings into artinian rings, *Proc. Amer. Math. Soc.* 109(4) 1990, 881-886.
- [12] K. R. Goodearl and R. B. Warfield Jr, *An introduction to non-commutative Noetherian rings*, (Cambridge Uni. Press, 1989).
- [13] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, *Pacific J. Math.* 4(1978), 551-567.
- [14] C. Y. Hong, N. K. Kim and T. K. Kwak, Ore-extensions of baer and p.p.-rings. *J. Pure and Applied Algebra* 151(3) (2000), 215-226.
- [15] T. K. Kwak, Prime radicals of skew-polynomial rings, *Int. J. of Mathematical Sciences* 2(2) (2003), 219-227.

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