Radicals of an Ordered Semigroup in Terms of Type of Ordered Semigroups

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Abstract
A type \( \mathcal{F} \) of ordered semigroups is a class of ordered semigroups such that (i) if \( S \) belongs to \( \mathcal{F} \) and \( S' \) is isomorphic to \( S \), then \( S' \) belongs to \( \mathcal{F} \), and (ii) any one-element (ordered) semigroup belongs to \( \mathcal{F} \). Given a type \( \mathcal{F} \) of ordered semigroups and an ordered semigroup \( S \), the \( \mathcal{F} \)-rad\( S \) is the intersection of all pseudoorders of \( S \) having type \( \mathcal{F} \) (a pseudoorder \( \sigma \) on \( S \) has type \( \mathcal{F} \) if the quotient semigroup of \( S \) by the congruence \( \overline{\sigma} = \sigma \cap \sigma^{-1} \) has also type \( \mathcal{F} \) - we consider the quotient semigroup as an ordered semigroup under the induced order relation by \( \sigma \)). The derived type \( \mathcal{F}' \) of \( \mathcal{F} \) is the class of all ordered semigroups \( S \) such that \( \mathcal{F}' \)-rad\( S \) is the order relation of \( S \). An \( \mathcal{F} \) maximal homomorphic image of an ordered semigroup of an ordered semigroup \( S \) is an ordered semigroup \( S' \) in \( \mathcal{F} \) for which there exists a homomorphism \( \eta \) of \( S \) onto \( S' \) with the factorization property: if \( \varphi \) is a homomorphism of \( S \) onto an ordered semigroup of type \( \mathcal{F} \), then there exists a homomorphism \( \theta \) of \( S' \) onto \( T \) such that \( \theta \circ \eta = \varphi \). We give sufficient and necessary condition under which an ordered semigroup admits an \( \mathcal{F} \) maximal homomorphic image. We show that every ordered semigroup has an \( \mathcal{F}' \) maximal homomorphic image and finally for a type \( \mathcal{F} \) of ordered semigroups we prove that every ordered semigroup has an \( \mathcal{F} \) maximal homomorphic image if and only if \( \mathcal{F}' = \mathcal{F} \).

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### 1. Introduction

In this paper, given a type \( \mathfrak{F} \) of ordered semigroups, we study the notions of the \( \mathfrak{F} \)-radical of an ordered semigroup, the derived type \( \mathfrak{F}' \) of a type \( \mathfrak{F} \) of ordered semigroups, \( \mathfrak{F} \) maximal homomorphic image of an ordered semigroup, pseudoorder on an ordered semigroup having type \( \mathfrak{F} \), quotient semigroup as an ordered semigroup.

- \( \mathfrak{F} \)-radS denotes the intersection of all pseudoorders of S having type \( \mathfrak{F} \). A pseudoorder \( \sigma \) on S has type \( \mathfrak{F} \) if the quotient semigroup of S by the congruence \( \overline{\sigma} := \sigma \cap \sigma^{-1} \) has also type \( \mathfrak{F} \). Note that we consider the quotient semigroup as an ordered semigroup under the order relation defined in \[3\] (see also \[10\],\[1\], \[4\], \[5\], \[6\]).
- the derived type \( \mathfrak{F}' \) of \( \mathfrak{F} \) is the class of all ordered semigroups S such that \( \mathfrak{F} \)-radS is the order relation of S.
- an \( \mathfrak{F} \) maximal homomorphic image of an ordered semigroup of an ordered semigroup S is an ordered semigroup \( S' \) in \( \mathfrak{F} \) for which there exists a homomorphism \( \eta \) of S onto \( S' \) with the factorization property, that is, if \( \varphi \) is a homomorphism of S onto an ordered semigroup of type \( \mathfrak{F} \), then there exists a homomorphism \( \theta \) of \( S' \) onto T such that \( \theta \circ \eta = \varphi \).

The above notions have been studied in the case of plain semigroups \[9\] (see also \[8; Proposition 1.7\]). In the paper we are following the “steps” of A. H. Clifford and G. B. Preston \[9\] in the case of ordered semigroups using the concept of pseudoorder on an ordered semigroup. We see once again (e.g. we refer to \[10\], \[1\], \[3\], \[4\], \[5\], \[6\], \[7\], \[2\]) the important role that the concept of pseudoorder plays in the theory of ordered semigroups (if \( \sigma \) is a pseudoorder on an ordered semigroup \( (S, \cdot, \leq) \), then we define \[3, 5\] the congruence \( \overline{\sigma} := \sigma \cap \sigma^{-1} \) on S and...
an order relation “≤_σ” on \( \frac{S}{\sigma} \) such that the quotient semigroup \( \frac{S}{\sigma} \) becomes an ordered semigroup with the property that the mapping

\[ \sigma^*: S \to \frac{S}{\sigma}, \quad \sigma^*(\alpha) := (\alpha)_\sigma \]

is a homomorphism between ordered semigroups – the converse also holds [5]: if \( \rho \) is a congruence on \( S \) and “\( \prec \)” is an order relation on \( \frac{S}{\rho} \) such that \( (\frac{S}{\rho}, \ast, \prec) \) is an order semigroup with respect to the order of \( S \), then \( \rho = \sigma \) for some \( \sigma \) pseudoorder of \( S \).

2. Preliminary notes

We recall (see [10], [1], [3], [4], [5], [6], [2]) some basic definitions and results concerning ordered semigroups that will be used below in the paper.

An ordered semigroup \( (S, \cdot, \leq) \) is a semigroup \( (S, \cdot) \) endowed with an order relation “\( \leq \)” which is compatible with the operation “\( \cdot \)” (i.e. if \( \alpha, b, c \in S \) such that \( \alpha \leq b \), then \( \alpha \cdot c \leq b \cdot c \) and \( c \cdot \alpha \leq c \cdot b \)). According to [3] by a congruence on \( S \) we mean an equivalence relation \( \sigma \) on \( S \) such that for \( \alpha, b, c \in S \) with \( (\alpha, b) \in \sigma \) we have \( (\alpha c, bc) \in \sigma \) and \( (c \alpha, cb) \in \sigma \). If \( \sigma \) is a congruence on \( S \) then it is well known that the quotient semigroup \( \frac{S}{\sigma} := \{(\alpha)_\sigma : \alpha \in S\} \) (where \( (\alpha)_\sigma \) is the \( \sigma \)-class of \( S \) containing \( \alpha \in S \)) is a semigroup with operation “\( \ast \)” defined via the operation of \( S \) by the rule that

\[ (\alpha)_\sigma \ast (b)_\sigma := (\alpha b)_\sigma, \quad \alpha, b \in S \]

We always consider \( \frac{S}{\sigma} \) as a semigroup with the above operation. In general the quotient semigroup \( \left( \frac{S}{\sigma}, \ast \right) \) is not always an ordered semigroup with respect to the order of \( S \) (see [3]), that is, there does not always exist an order relation “\( \prec \)” on \( \frac{S}{\sigma} \) such that (i) \( \left( \frac{S}{\sigma}, \ast, \prec \right) \) is an ordered semigroup and (ii) for \( \alpha, b \in S \), \( \alpha \leq b \) implies \( (\alpha)_\sigma \prec (b)_\sigma \). To define an order relation “\( \prec \)” on \( \frac{S}{\sigma} \) so that \( \left( \frac{S}{\sigma}, \ast, \prec \right) \) becomes an ordered semigroup with respect to the order of \( S \), we consider the concept of pseudoorder on \( S \) [3]. A binary relation \( \rho \) on \( S \) is said to be a pseudoorder on \( S \) [3] if
i) \( \leq \subseteq \rho \)
ii) \( \rho \) is transitive
iii) for \( \alpha, b \in S \), \( (\alpha, b) \in \rho \) implies \( (\alpha c, bc) \in \rho \) and \( (\alpha, cb) \in \rho \).

Clearly the order relation “\( \leq \)" of \( S \) and the binary relation \( S \times S \) are pseudoorders on \( S \). The intersection of a (nonempty) family of pseudoorders on \( S \) is also a pseudoorder on \( S \).

If \( \rho \) is a pseudoorder on \( S \) then we define \( \rho^{-1} := \{(\alpha, b) \in S \times S : (b, \alpha) \in \rho\} \). It is straightforward to prove that \( \rho \) is a congruence on \( S \). On \( S \) we define a binary relation “\( \leq \rho \)” as follows:

\[ (\alpha)_\rho \leq (b)_\rho \iff (\alpha, b) \in \rho \]

Then [3] (see also [10], [5]) \( \left(S/\rho, \ast, \leq \rho\right) \) is an ordered semigroup with respect to the order of \( S \). Moreover [4] if \( \sigma \) is a congruence on \( S \) and “\( \prec \)" is an order relation on \( S/\sigma \) such that \( \left(S/\sigma, \ast, \prec\right) \) is an order semigroup with respect to the order of \( S \), then \( \sigma = \rho \) for some \( \rho \) pseudoorder of \( S \).

Now let \( \rho, \nu \) be pseudoorders on \( S \) such that \( \rho \subseteq \nu \). We define

\[ \nu/\rho := \left\{ ((\alpha)_\rho, (b)_\rho) \in S/\rho \times S/\rho : (\alpha, b) \in \nu \right\} \]

Then [5] \( \nu/\rho \) is a pseudoorder on \( S/\rho \). Additionally [10] it holds

**Theorem 1:** Let \( S \) be an ordered semigroup, \( \rho \) be a pseudoorder on \( S \) and \( \tau \) be a pseudoorder on \( S/\rho \). Then there exists a (unique) pseudoorder \( \nu \) on \( S \) with \( \rho \subseteq \nu \) such that \( \tau = \nu/\rho \).

**Proposition 2:** \( \nu/\rho \leq \rho \) if and only if \( \rho = \nu \).

**Proof:**

(\( \Rightarrow \)) Let \( \nu/\rho \leq \rho \) and \( \alpha, b \in S \). Then

\[ (\alpha, b) \in \nu \Rightarrow ((\alpha)_\rho, (b)_\rho) \in \nu/\rho \Rightarrow ((\alpha)_\rho, (b)_\rho) \leq \rho \Rightarrow (\alpha, b) \in \rho \]

(\( \Leftarrow \)) Let \( \rho = \nu \) and \( \alpha, b \in S \). Then

\[ ((\alpha)_\rho, (b)_\rho) \in \nu/\rho \Rightarrow (\alpha, b) \in \nu \Rightarrow (\alpha, b) \in \rho \Rightarrow (\alpha)_\rho \leq (b)_\rho \]

If \( (S, \cdot, \leq), (T, \cdot, \leq') \) are ordered semigroups, then a mapping \( f : S \to T \) is called
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i) **homomorphism** if
   i1) \( f(\alpha \cdot b) = f(\alpha) \cdot f(b) \) for each \( \alpha, b \in S \)
   i2) for \( \alpha, b \in S, \alpha \leq b \) implies \( f(\alpha) \leq f(b) \) (i.e. \( f \) is an isotone mapping).

ii) **reverse isotone** if for \( \alpha, b \in S, f(\alpha) \leq f(b) \) implies \( \alpha \leq b \) (every reverse isotone mapping is 1-1).

iii) **isomorphism** if \( f \) is homomorphism, reverse isotone and onto.

Let \( \rho \) be a pseudoorder on \( S \). We define
\[
\rho^* : S \rightarrow S/\rho
\]
\[
\alpha \rightarrow (\alpha)_\rho
\]

In [5] (see also [1]) it has been proved the following

**Theorem 3:** Let \( S \) be an ordered semigroup and \( \rho \) be a pseudoorder on \( S \). Then the mapping \( \rho^* \) is homomorphism and onto. \( \square \)

**Remark 4:** Take now \( \mu := \leq \). Because \( \mu \) is a pseudoorder on \( S \), we can consider the ordered semigroup \( (S/\mu^*, \leq_\mu) \). Since \( \mu^* = 1_S = \{(\alpha, \alpha) : \alpha \in S\} \) and so \( S/\mu^* = \{\{\alpha\} : \alpha \in S\} \), then it is straightforward to prove that \( \mu^* \) is reverse isotone and thus \( \mu^* \) is an isomorphism. Hence \( S = S/\mu^* \). \( \square \)

Let \( S, T \) be ordered semigroups and \( f \) be a homomorphism of \( S \) into \( T \). Then [3] (see also [5], [1]) the binary relation on \( S \) defined by
\[
f : \{(\alpha, b) \in S \times S : f(\alpha) \leq f(b)\}
\]
is a pseudoorder on \( S \) (usually denoted by \( \text{Ker} f \)) since for \( \alpha, b \in S, (\alpha, b) \in f \) if and only if \( f(\alpha) = f(b) \). Then [1] (see also [2], [5], [10]) we have the following

**Theorem 5:** Let \( S, T \) be ordered semigroups and \( f : S \rightarrow T \) be a homomorphism. Then \( S/\text{Im} f \) is a pseudoorder on \( S \) (usually denoted by \( \text{Ker} f \)). Then [1] (see also [2], [5], [10]) we have the following

**Theorem 5:** Let \( S, T \) be ordered semigroups and \( f : S \rightarrow T \) be a homomorphism. Then \( S/\text{Im} f = T \). \( \square \)

**Remark 6:** From the previous Theorem it is immediate that if \( f \) is onto then \( S/\text{Im} f = T \). \( \square \)

The next Theorem has been proved in [1].

**Theorem 7:** Let \( S, T \) be ordered semigroups, \( f : S \rightarrow T \) be a homomorphism and \( \rho \) be a pseudoorder on \( S \) such that \( \rho \subseteq f \). Then the mapping
\( \varphi: S/\rho \to T \)

\[ (\alpha)_\rho \mapsto f(\alpha) \]

is the unique homomorphism of \( S/\rho \) into \( T \) such that \( \varphi \circ \rho^s = f \) (the converse holds as well: if \( \rho \) is a pseudoorder on \( S \) for which there exists a homomorphism \( \varphi: S/\rho \to T \) such that \( \varphi \circ \rho^s = f \), then \( \rho \subseteq f \)).

The next Theorem has also been proved in [1] (see also [10]).

**Theorem 8:** Let \( S \) be an ordered semigroup and \( \rho, \nu \) be pseudoorders on \( S \) such that \( \rho \subseteq \nu \). Then \( S/\rho \nu \equiv S/\nu \).

\[ \square \]

### 3. Main Results

By a *type of ordered semigroups* we mean a class \( \mathcal{F} \) of ordered semigroups such that

i) if \( S \in \mathcal{F} \) and \( S \) is isomorphic to \( S' \), then \( S' \in \mathcal{F} \), and

ii) any one-element (ordered) semigroup belongs to \( \mathcal{F} \).

Let \( \mathcal{F} \) be a type of ordered semigroups and \((S,\cdot,\leq)\) be an ordered semigroup. A pseudoorder on \( S \) is said to be *of type* \( \mathcal{F} \) if \( S/\mathcal{F} \in \mathcal{F} \). We define \( \mathcal{F}_s \) as the set of all pseudoorders on \( S \) of type \( \mathcal{F} \). It is clear that \( \mathcal{F}_s \neq \emptyset \) since \( S \times S \in \mathcal{F}_s \) (if \( \sigma := S \times S \) then \( \overline{\sigma} = S \times S \) and so \( S/\overline{\sigma} \) is an one-element ordered semigroup – therefore \( S/\overline{\sigma} \in \mathcal{F} \)). The \( \mathcal{F} \)–radical of \( S \) is defined by

\[ \mathcal{F} - \text{rad} S := \bigcap_{\sigma \in \mathcal{F}_s} \sigma \]

Obviously \( \mathcal{F} - \text{rad} S \) is a pseudoorder on \( S \). The *derived type* \( \mathcal{F}' \) of \( \mathcal{F} \) is the class of all ordered semigroups \((T,\cdot,\leq')\) such that \( \mathcal{F} - \text{rad} T = \leq' \).

**Remark 9:** Let \( \rho := \mathcal{F} - \text{rad} S \). Then \( \rho \subseteq \sigma \) for all \( \sigma \in \mathcal{F}_s \) and hence we have

i) \[ \{ \sigma \subseteq S \times S : \sigma \in \mathcal{F}_s, \rho \subseteq \sigma \} = \{ \sigma \subseteq S \times S : \sigma \in \mathcal{F}_s \} \]

ii) \[ \rho = \bigcap \{ \sigma \subseteq S \times S : \sigma \in \mathcal{F}_s, \rho \subseteq \sigma \} \]

\[ \square \]
Proposition 10: Let \((S, \leq)\) be an ordered semigroup and \(\mathcal{F}\) be a type of ordered semigroups. The following are equivalent:

i) \(\leq \in \mathcal{F}_S\)

ii) \(S \in \mathcal{F}\).

Proof:

i) \(\Rightarrow\) ii) Let \(\leq \in \mathcal{F}_S\). Set \(\rho := \mathcal{F} - \text{rad} S\). Then clearly \(\leq \subseteq \rho\). Also, since \(\leq \in \mathcal{F}_S\), \(\rho \subseteq \mathcal{F}\). Therefore \(\rho = \leq\) and so (by Remark 4) \(\mathcal{F}/\rho \cong S\). In addition, since \(\rho \in \mathcal{F}_S\), \(\mathcal{F}/\rho \in \mathcal{F}\). Consequently \(S \in \mathcal{F}\).

ii) \(\Rightarrow\) iii) Let \(S \in \mathcal{F}\). Consider \(\mu := \leq\). Then (by Remark 4) \(S \cong \mathcal{F}/\mu\) and thus (since \(S \in \mathcal{F}\)) \(\mathcal{F}/\mu \in \mathcal{F}\) which means \(\mu \in \mathcal{F}_S\). Therefore \(\leq \in \mathcal{F}_S\). \(\Box\)

Remark 11: From the proof of i) \(\Rightarrow\) ii) of the previous Proposition, we immediately have that if \(\leq \in \mathcal{F}_S\) then \(\mathcal{F} - \text{rad} S = \leq\). \(\Box\)

Proposition 12: Let \(\mathcal{F}\) be a type of ordered semigroups. Then \(\mathcal{F} \subseteq \mathcal{F}'\).

Proof: Let \((S, \leq)\) be an ordered semigroup such that \(S \in \mathcal{F}\). Then, by Proposition 10, \(\leq \in \mathcal{F}_S\) and so, from Remark 11, we have \(\mathcal{F} - \text{rad} S = \leq\). Hence \(S \in \mathcal{F}'\). \(\Box\)

Proposition 13: Let \(\mathcal{F}\) be a type of ordered semigroups and \(S\) be an ordered semigroup. Then \(\mathcal{F}_S \subseteq \mathcal{F}'_S\).

Proof:

\[ \rho \in \mathcal{F}_S \iff \mathcal{F}/\rho \in \mathcal{F} \quad \Rightarrow \quad \mathcal{F}/\rho \in \mathcal{F}' \quad \Rightarrow \quad \rho \in \mathcal{F}'_S \] \(\Box\)

Proposition 14: Let \(\mathcal{F}\) be a type of ordered semigroups, \(S\) be an ordered semigroup and \(\rho \in \mathcal{F}'_S\). Then \(\mathcal{F} - \text{rad} \mathcal{F}/\rho = \leq\).

Proof: Since \(\rho \in \mathcal{F}'_S\) then we have \(\mathcal{F}/\rho \in \mathcal{F}'\) which means that \(\mathcal{F} - \text{rad} \mathcal{F}/\rho = \leq\). \(\Box\)

Proposition 15: Let \(\mathcal{F}\) be a type of ordered semigroups, \((S, \leq)\) be an ordered semigroup and \(\rho, \sigma\) be pseudoorders on \(S\) such that \(\rho \subseteq \sigma\). Then \(\mathcal{F}/\rho \in \mathcal{F}'_{S/\rho}\) if and only if \(\sigma \in \mathcal{F}_S\).
Proof:
\[
\sigma / \rho \in \hat{\sigma}_{F} / \rho \iff \frac{S}{\rho} / \sigma \in \hat{\sigma} \iff \frac{S}{\sigma} \in \hat{\sigma} \iff \sigma \in \hat{\sigma}_{S} \quad \square
\]

Lemma 16: Let \( \hat{\sigma} \) be a type of ordered semigroups, \((S, \cdot, \leq)\) be an ordered semigroup and \( \rho \) be a pseudoorder on \( S \). Then
\[
\underline{\hat{\sigma} - \text{rad} \, S / \rho} = \bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} / \rho
\]

Proof: From Proposition 12 and Theorem 1, it immediately follows that
\[
\underline{\hat{\sigma} - \text{rad} \, S / \rho} = \bigcap \{ \tau \subseteq \underline{S / \rho} \times S / \rho : (\exists \sigma \in \hat{\sigma}_{S}) \rho \subseteq \sigma \text{ and } \tau = \sigma / \rho \}
\]
Moreover
\[
\bigcap \{ \tau \subseteq \underline{S / \rho} \times S / \rho : (\exists \sigma \in \hat{\sigma}_{S}) \rho \subseteq \sigma \text{ and } \tau = \sigma / \rho \} = \bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} / \rho
\]
Therefore
\[
\underline{\hat{\sigma} - \text{rad} \, S / \rho} = \bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} / \rho \quad \square
\]

Proposition 17: Let \( \hat{\sigma} \) be a type of ordered semigroups, \((S, \cdot, \leq)\) be an ordered semigroup and \( \rho \in \hat{\sigma}_{S}' \). Then
\[
\bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} / \rho
\]
which means, by Lemma 16, \[
\bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} / \rho = \leq_{\rho}
\]
and thus (by Proposition 2) we have
\[
\bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} = \rho \quad \square
\]

Theorem 18: Let \((S, \cdot, \leq)\) be an ordered semigroup and \( \hat{\sigma} \) be a type of ordered semigroups. Then \( \underline{\hat{\sigma} - \text{rad} \, \underline{S / \rho}} \) is an ordered semigroup of type \( \hat{\sigma}' \).

Proof: Let \( \rho \models \hat{\sigma} - \text{rad} \, S \). Since \( \rho \) is a pseudoorder on \( S \), then, from Lemma 16, we have
\[
\underline{\hat{\sigma} - \text{rad} \, S / \rho} = \bigcap \{ \sigma \subseteq S \times S : \sigma \in \hat{\sigma}_{S}, \rho \subseteq \sigma \} / \rho
\]
Moreover
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\[
\bigcap\{\sigma \subseteq S \times S : \sigma \in \mathcal{S}_S, \rho \subseteq \sigma\} = \rho \leq \rho\quad (\text{Remark 9ii)} \bigcap = \rho\quad (\text{Proposition 2}) \leq \rho
\]

Consequently

\[
\mathcal{S} - \text{rad} S\big/ \rho = \rho
\]

and so \(S\big/ \rho = S\big/ (\mathcal{S} - \text{rad} S)\) is an ordered semigroup of type \(\mathcal{S}'\).

**Theorem 19:** Let \((S, \cdot, \leq)\) be an ordered semigroup and \(\mathcal{S}\) be a type of ordered semigroups. Then \(\mathcal{S}' - \text{rad} S = \mathcal{S} - \text{rad} S\).

**Proof:**

Let \(\alpha, \beta \in S\) such that \((\alpha, \beta) \in \mathcal{S}' - \text{rad} S\) and \(\gamma \in \mathcal{S}_S\). We shall prove that \((\alpha, \beta) \in \gamma\). By Proposition 13 we have \(\gamma \in \mathcal{S}_S\) and thus, since \((\alpha, \beta) \in \mathcal{S}' - \text{rad} S\), it follows immediately that \((\alpha, \beta) \in \gamma\). Because \(\gamma\) is an arbitrary element of \(\mathcal{S}_S\), we have \((\alpha, \beta) \in \mathcal{S}' - \text{rad} S\).

\[
\mathcal{S} - \text{rad} S \subseteq \mathcal{S}' - \text{rad} S
\]

Let \(\alpha, \beta \in S\) such that \((\alpha, \beta) \in \mathcal{S} - \text{rad} S\) and \(\mu \in \mathcal{S}_S\). We shall prove that \((\alpha, \beta) \in \mu\). By Proposition 17 we have

\[
\bigcap\{\sigma \subseteq S \times S : \sigma \in \mathcal{S}_S, \mu \subseteq \sigma\} = \mu
\]

Obviously

\[
\mathcal{S} - \text{rad} S \subseteq \bigcap\{\sigma \subseteq S \times S : \sigma \in \mathcal{S}_S, \mu \subseteq \sigma\} = \mu
\]

and so \(\mathcal{S} - \text{rad} S \subseteq \mu\). Therefore \((\alpha, \beta) \in \mu\) and since \(\mu\) is an arbitrary element in \(\mathcal{S}_S\), we have \((\alpha, \beta) \in \mathcal{S}' - \text{rad} S\).

**Theorem 20:** Let \(\mathcal{S}\) be a type of ordered semigroups. Then \(\mathcal{S}'' = \mathcal{S}'\).

**Proof:** From Proposition 12 it follows that \(\mathcal{S}' \subseteq \mathcal{S}''\). Hence we need only to prove \(\mathcal{S}'' \subseteq \mathcal{S}'\). Let \((S, \cdot, \leq)\) be an ordered semigroup such that \(S \in \mathcal{S}''\). Then \(\mathcal{S}' - \text{rad} S \leq \mathcal{S}''\) and thus (by Theorem 19) \(\mathcal{S} - \text{rad} S \leq \mathcal{S}''\) which means \(S \in \mathcal{S}'\).

**Proposition 21:** Let \(\mathcal{S}\) be a type of ordered semigroups and \(S\) be an ordered semigroup. Then \(\mathcal{S}_S'' = \mathcal{S}_S'\).
Proof:

\[ \rho \in \mathcal{F}_s'' \iff \frac{S}{\rho} \in \mathcal{F}'' \iff \frac{S}{\rho} \in \mathcal{F}' \iff \rho \in \mathcal{F}'_s \]

Now let \( \mathcal{F} \) be any type of ordered semigroups and \( S \) be any ordered semigroup. An ordered semigroup \( S' \) is called \( \mathcal{F} \) maximal homomorphic image of \( S \) if

- there exists a homomorphism \( \eta \) of \( S \) onto \( S' \) having the factorization property:
  "for each \( T \in \mathcal{F} \) and \( \varphi \) homomorphism of \( S \) onto \( T \), there exists a homomorphism \( \theta \) of \( S' \) onto \( T \) such that \( \theta \circ \eta = \varphi \)."

**Proposition 22:** Let \( S \) be an ordered semigroup, \( T \) be an ordered semigroup of type \( \mathcal{F} \) and \( \varphi \) be a homomorphism of \( S \) onto \( T \). Then \( \mathcal{F} - \text{rad} S \subseteq \varphi \).

**Proof:** Since \( \varphi \) is a homomorphism of \( S \) onto \( T \), we have (by Remark 6) that \( \frac{S}{\varphi} = T \). Thus (since \( T \in \mathcal{F} \)) \( \frac{S}{\varphi} \in \mathcal{F} \) which means that \( \varphi \in \mathcal{F}_S \). Therefore \( \mathcal{F} - \text{rad} S \subseteq \varphi \).

**Theorem 23:** Let \( \mathcal{F} \) be a type of ordered semigroups and \( S \) be an ordered semigroup. Then \( S \) admits an \( \mathcal{F} \) maximal homomorphic image if and only if \( \frac{S}{\mathcal{F} - \text{rad} S} \) has type \( \mathcal{F} \).

**Proof:**

(\( \Rightarrow \)) Since \( S \) admits an \( \mathcal{F} \) maximal homomorphic image, then there exist an ordered semigroup \( (S', \cdot, \leq') \) and a homomorphism of \( S \) onto \( S' \) satisfying the factorization property. Therefore (from Remark 6) \( \frac{S}{\eta} = S' \in \mathcal{F} \) and so \( \eta \in \mathcal{F}_S \).

It suffices to show that \( \eta = \mathcal{F} - \text{rad} S \). If we prove that \( \eta \subseteq \sigma \) for each \( \sigma \in \mathcal{F}_S \), then (since \( \eta \in \mathcal{F}_S \)) it immediately follows \( \eta = \mathcal{F} - \text{rad} S \). So let us consider \( \sigma \in \mathcal{F}_S \) and \( \alpha, \beta \in S \) such that \( (\alpha, \beta) \in \eta \). The mapping \( \sigma^* : S \to \frac{S}{\sigma} \) is (by Theorem 3) a homomorphism of \( S \) onto \( \frac{S}{\sigma} \). Since \( \sigma \in \mathcal{F}_S \), then \( \frac{S}{\sigma} \in \mathcal{F} \). Thus, from the factorization property of \( \eta \), there exists a homomorphism \( \theta \) of \( S' \) onto \( \frac{S}{\sigma} \) such that \( \theta \circ \eta = \sigma^* \). Then
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\[(\alpha, b) \in \eta \Rightarrow \eta(\alpha) \leq^* \eta(b) \quad (\text{is isomorphism}) \Rightarrow \theta(\eta(\alpha)) \leq_\sigma \theta(\eta(b)) \Leftrightarrow \sigma^*(\alpha) \leq_\sigma \sigma^*(b) \Leftrightarrow \]
\[(\alpha)_{\eta} \leq_{\sigma} (b)_{\eta} \Leftrightarrow (\alpha, b) \in \sigma\]

and hence \(\eta \subseteq \sigma\).

\((\Leftarrow)\) Let \(\rho := \tilde{\sigma} - \text{rad}S\). Then (by hypothesis) \(\frac{S}{\rho} \in \tilde{\sigma}\). We will prove that \(\rho^*\) has the factorization property. By Theorem 3 we have that \(\rho^*\) is a homomorphism of \(S\) onto \(\frac{S}{\rho}\). Let \(T\) be an ordered semigroup of type \(\tilde{\sigma}\) and \(\varphi\) a homomorphism of \(S\) onto \(T\). Then, from Proposition 22, \(\rho \subseteq \varphi\) and so if we consider

\[\theta : \frac{S}{\rho} \rightarrow T\]
\[(x)_\rho \rightarrow \varphi(x)\]

then we immediately have (Theorem 7) that \(\theta\) is a homomorphism such that \(\theta \circ \rho^* = \varphi\). Also \(\theta\) is clearly onto (since \(\varphi\) is onto). Consequently \(\frac{S}{\rho} (= \frac{S}{\tilde{\sigma} - \text{rad}S})\) is an \(\tilde{\sigma}\) maximal homomorphic image of \(S\). \(\square\)

**Remark 24:** Based on the proof of Theorem 23, if \(S\) admits an \(\tilde{\sigma}\) maximal homomorphic image, then \(\frac{S}{\tilde{\sigma} - \text{rad}S}\) is an \(\tilde{\sigma}\) maximal homomorphic image of \(S\). \(\square\)

According to Theorem 23, every ordered semigroup does not admit a maximal homomorphic image for every type of ordered semigroups. The next Theorem gives a type of ordered semigroups for which every ordered semigroup has a maximal homomorphic image. This type is the derived type of every type of ordered semigroups.

**Theorem 25:** Let \(S\) be an ordered semigroup and \(\tilde{\sigma}\) be a type of ordered semigroups. Then \(\frac{S}{\tilde{\sigma} - \text{rad}S}\) is an \(\tilde{\sigma}'\) maximal homomorphic image of \(S\).

**Proof:** From Theorem 18 we have that \(\frac{S}{\tilde{\sigma} - \text{rad}S}\) is an ordered semigroup of type \(\tilde{\sigma}'\). From Theorem 19 we have \(\tilde{\sigma}' - \text{rad}S = \tilde{\sigma} - \text{rad}S\) and thus \(\frac{S}{\tilde{\sigma} - \text{rad}S} = \frac{S}{\tilde{\sigma}' - \text{rad}S}\). Therefore \(\frac{S}{\tilde{\sigma}' - \text{rad}S}\) is an ordered semigroup of type \(\tilde{\sigma}'\). Then, by Theorem 23, \(S\) admits an \(\tilde{\sigma}'\) maximal homomorphic image.
and, from Remark 24, \( \frac{S}{\mathcal{F} - \text{rad}S} \) is an \( \mathcal{F}' \) maximal homomorphic image of \( S \).

From the above we immediately conclude that \( \frac{S}{\mathcal{F} - \text{rad}S} \) is an \( \mathcal{F}' \) maximal homomorphic image of \( S \).

\( \square \)

**Theorem 26:** Let \( \mathcal{F} \) be a type of ordered semigroups. The following are equivalent:

i) Every ordered semigroup has an \( \mathcal{F} \) maximal homomorphic image.

ii) \( \mathcal{F} = \mathcal{F}' \).

**Proof:** Since ii) \( \Rightarrow \) i) is immediate from Theorem 25, we need only to prove the implication i) \( \Rightarrow \) ii). So let us assume that i) holds. Due to Proposition 12, it suffices to prove \( \mathcal{F}' \subseteq \mathcal{F} \). For this consider an ordered semigroup \( (S, \cdot, \leq) \) of type \( \mathcal{F}' \). We are to show that \( S \in \mathcal{F} \). From our hypothesis we have that \( S \) admits an \( \mathcal{F} \) maximal homomorphic image. Then, by Theorem 23, \( \frac{S}{\mathcal{F} - \text{rad}S} \in \mathcal{F} \).

Moreover, \( S \in \mathcal{F}' \) implies \( \mathcal{F} - \text{rad}S = \leq \). Thus (by Remark 4) \( \frac{S}{\mathcal{F} - \text{rad}S} \simeq S \).

Therefore (since \( \frac{S}{\mathcal{F} - \text{rad}S} \in \mathcal{F} \) ) \( S \in \mathcal{F} \).  

\( \square \)

**References**


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