Relative Finitistic Projective and Injective Dimensions

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Abstract

Let $R$ be an associative ring with identity and $RC$ a weakly Wakamatsu tilting module with $S \cong \text{End}_R(C)$. We study the finitistic projective dimension of $R$ with respect to $C$. Under some conditions, it is shown that $\text{FP}_C D(R) = \text{FP}_D(S) = \sup \{G_{C-pd_R}(M) \mid M \in B_C(R) \text{ and } G_{C-pd_R}(M) < \infty\}$. Also we give the dual conclusion.

Keywords: Finitistic projective(injective) dimension; Weakly Wakamatsu tilting module; Weakly cotilting module

Mathematics Subject Classification: 18G15, 18E10, 18G35

1 Introduction

Semidualizing modules can be defined over different rings. Over a commutative Noetherian ring, they were introduced by Foxby [6], Golod [7] and Vasconcelos [12] (Foxby called them PG-modules of rank one, Golod called them suitable modules and Vasconcelos called them spherical modules). Araya et al.[1] furthered this definition to a non-commutative, but noetherian ring, while White

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extended the definition to the non-noetherian, but commutative, setting. The theory has developed quickly since Holm and White [9] initiated the study of semidualizing modules over arbitrary associative rings. Semidualizing modules have been a cornerstone of relative homological algebra. Several important results appeared in relevant literature. For instance, Holm [8, Theorem 2.28] proved that the classical finitistic projective dimension is equal to the related finitistic Gorenstein projective dimension. Over a commutative Noetherian ring $R$, Tavasoil [10] defined the finitistic projective dimension with respect to a semidualizing module $C$ as follows:

$$FP_C^D(R) = \sup \{ P_C^\mu_R(M) \mid M \in \text{Mod}(R) \text{ and } P_C^\mu_R(M) < \infty \},$$

and it is shown in [10, Theorem 3.1] that

$$FP_C^D(R) = \sup \{ G_C^\nu_R(M) \mid M \in B_C(R) \text{ and } G_C^\nu_R(M) < \infty \}.$$

The aim of this paper is to extend this result to a non-commutative ring. Throughout this article, $R$ will be an associative (not necessarily commutative) ring with identity. $\text{Mod}(R)$ denotes the category of left $R$-modules. We use $\text{Proj}(R)$ (resp., $\text{Inj}(R)$) to denote the class of all projective (resp., injective) left $R$-modules. For $C \in \text{Mod}(R)$, we use $\text{Add}_R(C)$ (resp., $\text{Prod}_R(C)$) to denote the class of all left $R$-modules which are isomorphic to direct summands of direct sums (resp., direct products) of copies of $C$. We also use $\text{add}_R(C)$ to denote the class of all left $R$-modules which are isomorphic to direct summands of finite direct sums of copies of $C$. This paper is organized as follows.

In Section 3, we define the finistic projective dimension of $R$ with respect to a left $R$-module $C$ as follows:

$$FP_C^D(R) = \sup \{ P_C^\mu_R(M) \mid M \in \text{Mod}(R) \text{ and } P_C^\mu_R(M) < \infty \}.$$  

In Theorem 3.4, we prove that

$$FP_C^D(R) = \sup \{ G_C^\nu_R(M) \mid M \in X \text{ and } G_C^\nu_R(M) < \infty \},$$

where $RC$ is a self-small weakly Wakamatsu tilting module, $X$ is a subcategory of $\text{Mod}(R)$ such that $\text{Add}_R(C) \subseteq X$ and $X$ is closed under extensions and cokernels of monomorphisms. Indeed, the theorem above is a generalized version of [10, Theorem 3.1].

In Section 4, we prove some dual results in Section 3. We define the finistic injective dimension of $S$ with respect to a left $R$-module $C$ as follows:

$$FI_C^D(S) = \sup \{ I_C^\mu_R(M) \mid M \in \text{Mod}(S) \text{ and } I_C^\mu_R(M) < \infty \}.$$

Suppose that $CS$ has a degreewise finite projective resolution with $R \cong \text{End}_S(C)$, $\text{Ext}^1_R(C,C) = 0$ and that there is an injective cogenerator $D$ which admits an exact left $\text{Prod}_S(C^\nu)$-resolution. Moreover, assume that $E$ is an injective cogenerator in $\text{Mod}(R)$ and $U = C^\nu = \text{Hom}_R(C,E)$. It is shown that

$$FI_C^D(S) = \sup \{ G_U^\mu_S(M) \mid M \in Y \text{ and } G_U^\mu_S(M) < \infty \},$$

where $Y$ denotes a subcategory of $\text{Mod}(S)$ such that $\text{Prod}_S(C^\nu) \subseteq Y$ and $Y$ is closed under extensions and kernels of epimorphisms (see Theorem 4.3).
2 Preliminaries

Definition 2.1. ([9, Definition 2.1]) A semidualizing module is a \((R, S)\)-bimodule \(C\) satisfying the following properties:
\begin{enumerate}
\item \(RC\) and \(CS\) both admit a degreewise finite projective resolution in the corresponding module categories.
\item \(\text{Ext}^1_R(C, C) = \text{Ext}^1_S(C, C) = 0\).
\item The natural homothety maps \(R \to \text{Hom}_S(C, C)\) and \(S \to \text{Hom}_R(C, C)\) are ring isomorphisms.
\end{enumerate}

Definition 2.2. ([13, Section 3]) A Wakamatsu tilting module is a left \(R\)-module \(RC\) satisfying the following properties:
\begin{enumerate}
\item \(RC\) admits a degreewise finite projective resolution.
\item \(\text{Ext}^1_R(C, C) = 0\).
\item There exists a \(\text{Hom}_R(\cdot, C)\)-exact exact sequence of \(R\)-modules
\[ X : 0 \to R \to C^0 \to C^1 \to \cdots, \]
where \(C^i \in \text{add}_R(C)\) for every \(i \in \mathbb{N}\).
\end{enumerate}

By [13, Corollary 3.2], \(RC_S\) is semidualizing if and only if \(RC\) is a Wakamatsu tilting module with \(S \cong \text{End}_R(C)\) if and only if \(CS\) is a Wakamatsu tilting module with \(R \cong \text{End}_S(C)\).

Definition 2.3. ([2, Definitions 2.1 and 4.2]) A left \(R\)-module \(C\) is weakly Wakamatsu tilting if it has the following two properties:
\begin{enumerate}
\item \(\text{Ext}^1_R(C, C^{(I)}) = 0\) for every set \(I\).
\item There exists an exact sequence of left \(R\)-modules
\[ X : 0 \to R \to C_0 \to C_1 \to C_2 \to \cdots, \]
where \(C_i \in \text{Add}_R(C)\) for every \(i \in \mathbb{N}\) and such that \(\text{Hom}_R(\cdot, X)\) leaves the sequence \(X\) exact whenever \(X \in \text{Add}_R(C)\).
\end{enumerate}

Dually, a left \(R\)-module \(U\) is said to be weakly cotilting if it has the following two properties:
\begin{enumerate}
\item \(\text{Ext}^1_R(U^{(I)}, U) = 0\) for every set \(I\).
\item There is an injective congenerator \(D\) in \(R\)-modules which admits an exact sequence left \(R\)-modules:
\[ Y : \ldots \to A_2 \to A_1 \to A_0 \to D \to 0, \]
where \(A_i \in \text{Prod}_R(U)\) for every \(i \in \mathbb{N}\) and such that \(\text{Hom}_R(Y, \cdot)\) leaves the sequence \(Y\) exact whenever \(Y \in \text{Prod}_R(U)\).
Definition 2.4. ([2, Definitions 2.2 and 4.1] Let $C, U, M \in \text{Mod}(R)$. $M$ is said to be $G_C$-projective if there exist an exact sequence of left $R$-modules

$$X = \ldots \to P_1 \to P_0 \to A^0 \to A^1 \to \cdots$$

where $P_i \in \text{Proj}(R)$, $A^i \in \text{Add}_R(C)$ for every $i \in \mathbb{N}$, $M \cong \text{Im}(P_0 \to A^0)$, and such that $\text{Hom}_R(X, Q)$ is exact whenever $Q \in \text{Add}_R(C)$.

Dually, $M$ is said to be $G_U$-injective if there exists an exact sequence of left $R$-modules

$$Y = \ldots \to Y_1 \to Y_0 \to I^0 \to I^1 \to \cdots$$

where the $I^i \in \text{Inj}(R)$, $Y_i \in \text{Prod}_R(U)$ for every $i \in \mathbb{N}$, $M \cong \text{Im}(Y_0 \to I^0)$ and such that $\text{Hom}_R(E, Y)$ is exact whenever $E \in \text{Prod}_R(U)$.

We use $G_C P(R)$ (resp., $G_U I(R)$) to denote the class of all $G_C$-projective (resp., $G_U$-injective) left $R$-modules.

Definition 2.5. ([5, Definition 4.1] Let $C \in \text{Mod}(R)$ and $S = \text{End}_R(C)$. A left $R$-module is said to be $\mathcal{P}_C$-projective if it is isomorphic to $C \otimes_S P$ for some $P \in \text{Proj}(S)$. A left $S$-module is said to be $\mathcal{I}_C$-injective if it is isomorphic to $\text{Hom}_R(C, I)$ for some $I \in \text{Inj}(R)$. We use $\mathcal{P}_C(R)$ (resp., $\mathcal{I}_C(S)$) to denote the categories of $\mathcal{P}_C$-projective (resp., $\mathcal{I}_C$-injective) left $R$ (resp., $S$)-modules.

Definition 2.6. ([11]) Let $\mathcal{X}$ be a subclass of $\text{Mod}(R)$. A left $R$-module $M$ is said to have $\mathcal{X}$-projective dimension less than or equal to $n$, $\mathcal{X}$-$\text{pd}_R(M) \leq n$, if there exists an exact sequence

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

with $X_i \in \mathcal{X}$ for every $i \in \{0, \ldots, n\}$. If $n$ is the least nonnegative integer for which a sequence exists then $\mathcal{X}$-$\text{pd}_R(M) = n$, and if there is no such $n$ then $\mathcal{X}$-$\text{pd}_R(M) = \infty$. Dually, we have the definition of $\mathcal{X}$-injective dimension.

In particular, When $\mathcal{X} = \mathcal{P}_C(R)$ (resp., $\mathcal{I}_C(R)$), we use $\mathcal{P}_C$-$\text{pd}_R(M)$ (resp., $\mathcal{I}_C$-$\text{id}_R(M)$) to denote the $\mathcal{P}_C$-projective (resp., $\mathcal{I}_C$-injective) dimension of $M$ (see [5, Definition 4.1]). When $\mathcal{X} = G_C P(R)$ (resp., $G_U I(R)$), we use $G_C$-$\text{pd}_R(M)$ (resp., $G_U$-$\text{id}_R(M)$) to denote the $G_C$-projective (resp., $G_U$-injective) dimension of $M$ (see [2, Definitions 3.1 and 4.10]).

Next we recall the definitions of Auslander and Bass classes.

Definition 2.7. ([4]) Let $C \in \text{Mod}(R)$ and $S = \text{End}_R(C)$, the Auslander class associated to $C$, $\mathcal{A}_C(S)$, is the class of all left $S$-module $M$ satisfying

(A1) $\text{Tor}^S_{\geq 1}(C, M) = 0$.

(A2) $\text{Ext}^1_R(C, C \otimes_S M) = 0$.

(A3) the canonical map $\mu_M : M \to \text{Hom}_R(C, C \otimes_S M)$ is an isomorphism of
left $S$-modules.
On the other hand, the Bass class associated to $C$, $\mathcal{B}_C(R)$, consists of all left $R$-modules $N$ satisfying
(B1) $\operatorname{Ext}^{\geq 1}_R(C, N) = 0$.
(B2) $\operatorname{Tor}^{\geq 1}_S(C, \operatorname{Hom}_R(C, N)) = 0$.
(B3) the canonical map $\nu_N : C \otimes_S \operatorname{Hom}_R(C, N) \to N$ is an isomorphism of left $R$-modules.

3 Relative finitistic projective dimension

In [8, Theorem 2.28], it is proved that there is an equality between the classical finitistic projective dimension:
$$\operatorname{FPD}(R) = \sup \{ \operatorname{pd}_R(M) \mid M \in \operatorname{Mod}(R) \text{ and } \operatorname{pd}_R(M) < \infty \}$$
and the related finitistic projective dimension:
$$\operatorname{FPD}(R) = \sup \{ \operatorname{Gpd}_R(M) \mid M \in \operatorname{Mod}(R) \text{ and } \operatorname{Gpd}_R(M) < \infty \},$$
where $\operatorname{Gpd}_R(M)$ denotes the Gorenstein projective dimension of $M$.

For $C \in \operatorname{Mod}(R)$, we define the finitistic projective dimension with respect to $C$ as follows:
$$\operatorname{FP}_C\operatorname{D}(R) = \sup \{ \operatorname{Pd}_C(M) \mid M \in \operatorname{Mod}(R) \text{ and } \operatorname{Pd}_C(M) < \infty \}.$$ 

In particular, when $C = R$, it is exactly the classical finitistic projective dimension FPD($R$).

**Definition 3.1.** ([2, Definition 5.1]) A left $R$-module $C$ is said to be self-small, if for every set $I$,
$$\operatorname{Hom}_R(C, C^{(I)}) \cong \operatorname{Hom}_R(C, C)^{(I)}.$$

A left $R$-module $M$ is said to be Hom-faithful, if $\operatorname{Hom}_R(M, N) = 0$, then $N = 0$.

**Proposition 3.2.** ([5, proposition 3.1]) If $RC$ is self-small and $S \cong \operatorname{End}_R(C)$, then $\operatorname{Add}_R(C) = C \otimes_S \operatorname{Proj}(S)$.

**Proposition 3.3.** Let $C$ be a weakly Wakamatsu tilting module. If $0 \to K \to G \to M \to 0$ is an exact sequence with $G \in G_C\operatorname{P}(R)$.
(1) If $M \in G_C\operatorname{P}(R)$, then $K \in G_C\operatorname{P}(R)$.
(2) If $G_C\operatorname{pd}_R(M) > 0$, then $G_C\operatorname{pd}_R(K) = G_C\operatorname{pd}_R(M) - 1$.

**Proof.** (1) If $M \in G_C\operatorname{P}(R)$, by [2, proposition 2.9], $G_C\operatorname{P}(R)$ is closed under kernels of epimorphisms. So we have $K \in G_C\operatorname{P}(R)$.
(2) Assume $G_C\operatorname{pd}_R(M) = m > 0$, by [2, Proposition 3.11], $G_C\operatorname{pd}_R(K) = \sup \{ G_C\operatorname{pd}_R(G), G_C\operatorname{pd}_R(M) - 1 \} = m - 1$ (note that $0 = G_C\operatorname{pd}_R(G) \neq G_C\operatorname{pd}_R(M) = m$). □
Theorem 3.4. Suppose \( rC \) is a self-small weakly Wakamatsu tilting module. If \( \text{Add}_R(C) \subseteq \mathcal{X} \), and \( \mathcal{X} \) is closed under extensions and cokernels of monomorphisms, then

\[
\text{FP}_C D(R) = \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \}.
\]

Proof. By assumptions, we know that \( \mathcal{X} \) contains modules with finite \( \mathcal{P}_C \)-projective dimensions. [3, Theorem 3.6] implies that

\[
\text{FP}_C D(R) \leq \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \}.
\]

If \( M \in \mathcal{X} \) with \( 0 < \text{Gpd}_R(M) = m < \infty \), then [2, Theorem 3.5] implies that there is a short exact sequence \( 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0 \) with \( \mathcal{P}_C - \text{pd}_R(K) < \infty \), \( \mathcal{P}_C - \text{pd}_R(K) = \text{Gpd}_R(K) = m - 1 \), then \( \mathcal{P}_C - \text{pd}_R(K) = \text{Gpd}_R(M) - 1 \). Therefore,

\[
\sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \} \leq \text{FP}_C D(R) + 1.
\]

Next we will prove that \( \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \} \leq \text{FP}_C D(R) \).

Let \( 0 < \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \} = m < \infty \) and let \( M \in \mathcal{X} \) with \( \text{Gpd}_R(M) = m \), we have a short exact sequence \( 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0 \) with \( \mathcal{P}_C - \text{pd}_R(K) = \text{Gpd}_R(K) = m - 1 \).

Since \( G \) is \( \text{Gpd}_R \)-projective, there exists a projective right \( S \)-module \( P \) such that \( G \subseteq C \otimes_S P \). Also we can consider the left \( R \)-module \( L = C \otimes_S P/K \), since \( K \subseteq G \), we get \( M \cong G/K \) with \( M \subseteq L \), therefore we can get an exact sequence \( 0 \rightarrow M \rightarrow L \rightarrow L/M \rightarrow 0 \), besides we can also get an exact sequence \( 0 \rightarrow G \rightarrow C \otimes_S P \rightarrow L/M \rightarrow 0 \) (note that \( L/M \cong C \otimes_S P/G \)). Because \( \mathcal{P}_C - \text{pd}_R(K) < \infty \), \( K \subseteq \mathcal{X} \). Thus \( G \subseteq \mathcal{X} \). Since \( C \otimes_S P \subseteq \mathcal{X} \), \( \mathcal{X} \) is closed under cokernels of monomorphisms, we get \( L/M \in \mathcal{X} \). We claim that \( L \notin \text{Gpd}_R(P) \).

If \( L \in \text{Gpd}_R(P) \), according to Proposition 3.3, \( \text{Gpd}_R(L/M) = m + 1 > m \), which contradicts \( \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \} = m \), hence \( L \notin \text{Gpd}_R(P) \). So we have an exact sequence \( 0 \rightarrow K \rightarrow C \otimes_S P \rightarrow L \rightarrow 0 \) with \( \text{Gpd}_R(L) = \text{Gpd}_R(K) + 1 = m \), \( \mathcal{P}_C - \text{pd}_R(K) < \infty \) and \( C \otimes_S P \) is \( \text{Add}_R(C) \), so \( \mathcal{P}_C - \text{pd}_R(L) < \infty \). Therefore \( \mathcal{P}_C - \text{pd}_R(L) = \text{Gpd}_R(L) = m \).

Corollary 3.5. If \( C \) is a self-small weakly Wakamatsu tilting module, \( S \cong \text{End}_R(C) \), \( rC \) is Hom-faithful, \( \text{Add}_R(C) \subseteq \mathcal{X} \), \( \mathcal{X} \) is closed under extensions and cokernels of monomorphisms, then

\[
\text{FP}_C D(R) = \text{FPD}(S) = \sup \{ \text{Gpd}_S(M) \mid M \in \text{Mod}(S) \text{ and } \text{Gpd}_S(M) < \infty \} = \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{X} \text{ and } \text{Gpd}_R(M) < \infty \}.
\]

Proof. The first equality follows from [3, Proposition 4.2] and the second one follows from [8, Theorem 2.28]. Finally, the third one is immediate by Theorem 3.4.

Corollary 3.6. If \( C \) is a self-small weakly Wakamatsu tilting module and \( rC \) is Hom-faithful, then

\[
\text{FP}_C D(R) = \sup \{ \text{Gpd}_R(M) \mid M \in \mathcal{B}_C(R) \text{ and } \text{Gpd}_R(M) < \infty \}.
\]
Proof. Let $X = B_C(R)$. According to [2, Proposition 5.6], $\text{Add}_R(C) \subseteq B_C(R)$ and $B_C(R)$ is closed under extensions and cokernels of monomorphisms. So the result follows from Corollary 3.5.

4 Relative finitistic injective dimension

In [8, Theorem 2.29], it is shown that there is an equality between the classical finitistic injective dimension:

$$\text{FID}(R) = \sup \{ \text{id}_R(M) \mid M \in \text{Mod}(R) \text{ and } \text{id}_R(M) < \infty \}$$

and the related finistic Gorenstein injective dimension:

$$\text{FID}(R) = \sup \{ \text{Gid}_R(M) \mid M \in \text{Mod}(R) \text{ and } \text{Gid}_R(M) < \infty \},$$

where $\text{Gid}_R(M)$ denotes the Gorenstein injective dimension of $M$.

For $C \in \text{Mod}(R)$, we define finistic injective dimension with respect to $C$ as follows:

$$\text{FI}_C D(S) = \sup \{ \mathcal{I}_C-\text{id}_S(M) \mid M \in \text{Mod}(S) \text{ and } \mathcal{I}_C-\text{id}_S(M) < \infty \}.$$ 

In particular, when $C = R$, it is exactly the classical finitistic injective dimension $\text{FID}(R)$.

Proposition 4.1. [5, Proposition 3.2] Assume that $C_S$ is self-orthogonal, it admits a degreewise finite projective resolution and $R = \text{End}_S(C)$. Then $\text{Prod}_S(C^\vee) = \text{Hom}_R(C, I(R)) = \mathcal{I}_C(S)$ ($I$ is injective in $\text{Mod}(R)$).

Proposition 4.2. Let $U$ be a weakly cotilting module. Suppose that $0 \rightarrow M \rightarrow G \rightarrow K \rightarrow 0$ is a short exact sequence with $G$ $G_U$-injective. Then the following statements hold.

(1) If $M \in G_U I(S)$, then $K \in G_U I(S)$.

(2) If $G_U-\text{id}_S(M) > 0$, then $G_U-\text{id}_S(K) = G_U-\text{id}_S(M) - 1$.

Proof. It is dual to the proof of Proposition 3.3.

Theorem 4.3. Suppose that $C_S$ has a degreewise finite projective resolution with $R \cong \text{End}_S(C)$, $\text{Ext}_{S_{op}}^1(C, C) = 0$ and that there is an injective cogenerator $D$ which admits an exact left $\text{Prod}_S(C^\vee)$-resolution. Let $U = C^\vee$. If $\text{Prod}_S(U) \subseteq \mathcal{Y}$, $\mathcal{Y}$ is closed under extensions and kernels of epimorphisms. Then

$$\text{FI}_C D(S) = \sup \{ G_U-\text{id}_S(M) \mid M \in \mathcal{Y} \text{ and } G_U-\text{id}_S(M) < \infty \}.$$ 

Proof. Note that $C^\vee$ is weakly cotilting by [4, Lemma 3.6]. By assumption, we get that $\mathcal{Y}$ contains modules with finite $\mathcal{I}_C$-injective dimensions. Using [3, Theorem 3.6] and Proposition 4.1, we can get that $G_U-\text{id}_S(M) = \mathcal{I}_C-\text{id}_S(M)$ when $\mathcal{I}_C-\text{id}_S(M) < \infty$. So we conclude $\text{FI}_C D(S) \leq \sup \{ G_U-\text{id}_S(M) \mid M \in \mathcal{Y} \text{ and } G_U-\text{id}_S(M) < \infty \}$. If $M \in \mathcal{Y}$ with $0 < G_U-\text{id}_S(M) = m < \infty$, by [4,
Lemma 3.6], \( U \) is weakly cotilting. Then the dual of [2, Theorem 3.5] implies that there is a short exact sequence \( 0 \to M \to G \to K \to 0 \) with \( \mathcal{I}_C\text{id}_S(K) = \text{G}_{U}\text{id}_S(M) - 1 \). Therefore,

\[
\sup \{ \text{G}_{U}\text{id}_S(M) \mid M \in \mathcal{Y} \text{ and } \text{G}_{U}\text{id}_S(M) < \infty \} \leq \text{FI}_C D(S) + 1.
\]

Next we will show that \( \sup \{ \text{G}_{U}\text{id}_S(M) \mid M \in \mathcal{Y} \text{ and } \text{G}_{U}\text{id}_S(M) < \infty \} \leq \text{FI}_C D(S) \). Let \( 0 < \sup \{ \text{G}_{U}\text{id}_S(M) \mid M \in \mathcal{Y} \text{ and } \text{G}_{U}\text{id}_S(M) < \infty \} = m < \infty \) and let \( M \in \mathcal{Y} \) with \( \text{G}_{U}\text{id}_S(M) = m \), from above, we have a short exact sequence \( 0 \to M \to G \to K \to 0 \) with \( \mathcal{I}_C\text{id}_S(K) = m - 1 \). Since \( G \) is \( U \)-injective, there exists an injective left \( R \)-module \( E \) such that \( \text{Hom}_R(C, E) \to G \) is epic, so \( \text{Hom}_R(C, E) \to G \) is also epic. We can consider \( H \cong \text{Ker}(\text{Hom}_R(C, E)) \to K \), we get an exact sequence \( 0 \to F \to H \to M \to 0 \) with \( F = \text{Ker}(H \to M) \), besides we also have an exact sequence: \( 0 \to F \to \text{Hom}_R(C, E) \to G \to 0 \). Similarly, we can get \( F \in \mathcal{Y} \). We claim \( H \notin \text{G}\text{id}_S(C,E) \), the proof is dual to that of Theorem 3.4. Finally we consider a short exact sequence \( 0 \to H \to \text{Hom}_R(C, E) \to K \to 0 \), and \( \text{G}_{U}\text{id}_S(H) = \text{G}_{U}\text{id}_S(K) + 1 = m \), and \( \mathcal{I}_C\text{id}(H) < \infty \), and \( H \in \mathcal{Y} \), we get \( \mathcal{I}_C\text{id}_S(H) = \text{G}_{U}\text{id}_S(H) = m \).

**Corollary 4.4.** Suppose \( C_S \) has a degreewise finite projective resolution with \( R \cong \text{End}_S(C) \), \( rC \) is Hom-faithful, \( \text{Ext}_{S^{op}}^{2}(C,C) = 0 \) and that there is an injective cogenerator \( D \) which admits an exact left \( \text{Prod}_S(C^\vee)\)-resolution. Let \( U = C^\vee \). Then

\[
\text{FI}_C D(S) = \sup \{ \text{G}_{U}\text{id}_S(M) \mid M \in \mathcal{A}_C(S) \text{ and } \text{G}_{U}\text{id}_S(M) < \infty \}.
\]

**Proof.** Let \( \mathcal{Y} = \mathcal{A}_C(S) \). According to [4, Lemma 3.2 (2)] and [2, Proposition 5.4], \( \text{Prod}_S(C^\vee) \subseteq \mathcal{A}_C(S) \) and \( \mathcal{A}_C(S) \) is closed under extensions and kernels of epimorphisms. So the result follows from Theorem 4.3. \( \square \)

**Corollary 4.5.** If \( \text{Inj} R \subseteq \mathcal{Y} \), \( \mathcal{Y} \) is closed under extensions and kernels of epimorphisms. Then

\[
\text{FI}_R D(S) = \text{FID}(R)
\]

\[
= \sup \{ \text{Gid}_R(M) \mid M \in \text{Mod}(R) \text{ and } \text{Gid}_R(M) < \infty \}
\]

\[
= \sup \{ \text{G}_{R^\vee}\text{id}_R(M) \mid M \in \mathcal{Y} \text{ and } \text{G}_{R^\vee}\text{id}_R(M) < \infty \}
\]

**Proof.** The second equality follows from [8, Theorem 2.29] and the third one is immediate by Theorem 4.3. \( \square \)

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References


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