Four Dimensional Absolute Valued Algebras
Having Two Different Subalgebras
Isomorphic to \( \mathbb{C} \)

Abdelhadi Moutassim

Centre Régional des Métiers de l’Education et de la Formation (CRMEF), Casablanca-Settat Annexe provinciale Settat, Morocco

Mohamed Louzari and Aziz Es.Sadiq

Laboratoire d’Algèbre et ses Applications (LAA) Faculté des Sciences Abdelmalek Essaâdi Tétouan, Morocco

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2023 Hikari Ltd.

Abstract

The aim of this paper is to construct, by algebraic methods, some new class of four-dimensional absolute valued algebras, namely \( M_1, M_2 \) and \( M_3 \) which are not isomorphic to the known algebra \( \mathbb{H} \). These new algebras contain a nonzero omnipresent idempotent. Furthermore, we classify all four-dimensional absolute valued algebras having two different subalgebras isomorphic to \( \mathbb{C} \). Note that there exists a four-dimensional absolute valued algebra containing no subalgebra of dimension two, which means that, the problem of classifying all four-dimensional absolute valued algebras seems still to be open.

Keywords: Absolute valued algebra, pre-Hilbert algebra, omnipresent idempotent.

1 Introduction

Let \( A \) be a non necessarily associative real algebra which is normed as real vector space. We say that a real algebra \( A \) is an absolute valued algebra, if it’s...
satisfy the equality \( \|ab\| = \|a\|\|b\| \), for all \( a, b \in A \). The reader is referred to [2] for basis facts and intrinsic characterizations of these classical absolute valued algebras. The last decades have known several works in the theme for algebras having finite-dimensional [4], [3] and [5]. We recall that every absolute valued algebra is a normed algebra, note that, the norm \( \| \cdot \| \) of any finite dimensional absolute valued algebras comes from an inner product \( \langle \cdot, \cdot \rangle \) [4] and [6]. In 1947 Albert proved that the finite dimensional unital absolute valued algebras are classified by \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) and that every finite dimensional absolute valued algebra is isotopic to one of the algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) and so has dimension 1, 2, 4 or 8 [1]. Urbanik and Wright proved in 1960 that all unital absolute valued algebras are classified by \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) [12]. It is easily seen that the one-dimensional absolute valued algebras are classified by \( \mathbb{R} \), and it is well-known that the two-dimensional absolute valued algebras are isomorphic to \( \mathbb{C}, \mathbb{C}^*, \mathbb{C}^* \) or \( \mathbb{C}^* \) [10]. Moreover, the four-dimensional absolute valued algebras have been described by M.I. Ramírez Álvarez in [9] and she gave an example of four-dimensional absolute valued algebra containing no subalgebra of dimension two. Which means that, the problem of classifying all four-dimensional absolute valued algebras seems still to be open.

Motivated by these facts, we became interested in the study of four-dimensional absolute valued algebras having two different subalgebras isomorphic to \( \mathbb{C} \). Note that, there is no four-dimensional real commutative division algebra, which means that the dimension of commutative absolute valued algebras is less than or equal to two [8]. Furthermore, we know that \( \mathbb{H} \) is an example of four-dimensional absolute valued algebras containing at least two different subalgebras isomorphic to \( \mathbb{C} \). This reinforces our construction, by algebraic method, of this class. Note that these new algebras contain a nonzero omnipresent idempotent (lemma 3.2). We recall that, the classification of four-dimensional absolute valued algebras containing a unique two-dimensional subalgebra isomorphic to \( \mathbb{C} \) is still an open problem.

In 2016 [7], we proved that if \( A \) is a four-dimensional absolute valued algebra containing a nonzero central idempotent, then \( A \) contains a commutative subalgebra of dimension two. Here we show, in proposition 2.6, that a central idempotent is an omnipresent idempotent, the reciprocal does not hold in general, and the counter example is given (remark 3.3). In section 2 we introduce the basic tools for the study of four-dimensional absolute valued algebras. We also give some properties related to subalgebras isomorphic to \( \mathbb{C} \) (lemmas 2.4, 2.5 and 2.7). Moreover, the section 3 is devoted to construct, by algebraic method, some new class of the four-dimensional absolute valued algebras having two different subalgebras isomorphic to \( \mathbb{C} \), namely \( M_1, M_2 \) and \( M_3 \).

The paper ends, in section 4, with the following main results.
Theorem 1.1  Let $A$ be a four dimensional absolute valued algebra having two different subalgebras $B_1$ and $B_2$ isomorphic to $\mathbb{C}$. Then $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$. 

2 Preliminary Notes

Throughout this paper, the word algebra refers to a non-necessarily associative algebra over the field of real numbers $\mathbb{R}$.

Definition 2.1  Let $B$ be an arbitrary algebra.

i) $B$ is called a normed algebra (resp, absolute valued algebra) if it is endowed with a space norm: $\|\|\|$ such that $\|xy\| \leq \|x\|\|y\|$ (resp, $\|xy\| = \|x\|\|y\|$), for all $x, y \in B$.

ii) $B$ is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product $(\cdot/\cdot)$ such that $$(\cdot/\cdot) : B \times B \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto (x/y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

ii) $B$ is called a division algebra if for all nonzero $a \in B$, the operators $L_a(x) = ax$ and $R_a(x) = xa$ (for all $x \in B$) of left and right multiplication by $a$ are bijective. Note that every finite-dimensional absolute valued algebra is a division algebra.

iii) We mean by a nonzero omnipresent idempotent, an idempotent which is contained in all two-dimensional subalgebras of $B$.

In 2017, we showed that if $A$ is a real commutative algebraic algebra without divisors of zero, then $A$ is finite dimensional ($\dim A \leq 2$) [8]. Moreover, we proved by a simple algebraic method, there is no four-dimensional real commutative division algebra, which means that the dimension of commutative absolute valued algebras is less than or equal two.

On the other hand, the most natural examples of absolute valued algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the algebra of Hamilton quaternion) and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values [2] and [11]. The algebras $*\mathbb{C}$, $\mathbb{C}^*$, and $\overline{\mathbb{C}}$ (obtained by endowing the space $\mathbb{C}$ with the products defined by $x \ast y = \bar{x}y$, $x \ast y = xy$ and $x \ast y = \bar{x}\bar{y}$ respectively) where $x \rightarrow \bar{x}$ is the standard conjugation of $\mathbb{C}$. 


We need the following relevant results:

**Theorem 2.2** [4] *The norm of any finite dimensional absolute valued algebra comes from an inner product.*

**Theorem 2.3** [11] *If $A$ is a two-dimensional absolute valued algebra, then $A$ is isomorphic to $\mathbb{C}$, $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{C}^*$.*

**Lemma 2.4** [7] *Let $A$ be a finite-dimensional absolute valued algebra containing a nonzero central idempotent $f$, then the following statements hold:

i) $A$ contains a subalgebra of dimension two,

ii) $x^2 = -\|x\|^2 f$, for all $x \in \{f\}^\perp$.*

**Lemma 2.5** [7] *Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central idempotent $e$ and a subalgebra $B$ of dimension two. If $x,y \in B$, then $xy \in B$.***

**Proposition 2.6** *Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central idempotent $f$, then $f$ is an omnipresent idempotent.*

**Proof.** The lemma 2.4.(i) proves the existence of a subalgebra $B := A(e,i)$ isomorphic to $\mathbb{C}$ or $\mathbb{C}$ ($i^2 = -e$ and $ei = ie = \pm i$). It suffices to show that $f \in B$. If $(f/i) = 0$ then $i^2 = -f \in B$ (lemma 2.4.(ii)). Otherwise and without loss of generality we may put $b = i - (i/f)f$. We have

\[
\begin{align*}
b^2 &= -e - 2(i/f)if + (i/f)^2 f \\
-\|b\|^2 f &= -e - 2(i/f)if + (i/f)^2 f \quad \text{(lemma 2.4.(ii))} \\
(-\|b\|^2 - (i/f)^2)f &= -e - 2(i/f)if \\
-f &= -e - 2(i/f)if \quad \left(\|b\|^2 = 1 - (i/f)^2\right) \\
-f &= f + ef - 2(i/f)if \quad \left(e + f + ef = 0\right) \\
f &= -f - ef + 2(i/f)if \\
f &= -(f + e) + 2(i/f)if \\
f &= -(f + e) + 2(i/f)i \\
2f &= -e + 2(i/f)i
\end{align*}
\]

We conclude that $f \in B$.

**Lemma 2.7** *Let $A$ be a four-dimensional absolute valued algebra having two different subalgebras $B_1$ and $B_2$ isomorphic to $\mathbb{C}$. Then $B_1 \cap B_2 \neq \{0\}$.*
**Proof.** Since $B_1$ and $B_2$ are isomorphic to $\mathbb{C}$, then we can set $B_1 = A(e_1, i)$ and $B_2 = A(e_2, j)$, such that

\[ i^2 = -e_1, \quad ie_1 = e_1i = i \]

and

\[ j^2 = -e_2, \quad je_2 = e_2j = j \]

We know that

\[ (e_1/i) = (e_2/j) = 0 \]

And we assume that $B_1 \cap B_2 = \{0\}$, so the family $\{e_1, i, e_2, j\}$ is a basis of $A$. According to lemma 2.5, we have the following equalities:

\[ (e_1e_2 - e_2e_1/e_1) = (e_1e_2 - e_2e_1/i) = 0 \]

Hence $(e_1e_2 - e_2e_1)^2 \in B_1$, also

\[ (e_1e_2 - e_2e_1/e_2) = (e_1e_2 - e_2e_1/j) = 0 \]

That is $(e_1e_2 - e_2e_1)^2 \in B_2$, this last imply $(e_1e_2 - e_2e_1)^2 = 0$ or $e_1e_2 = e_2e_1$. Similarly

\[ (e_1j - je_1/e_1) = (e_1j - je_1/i) = 0 \]

Which gives $(e_1j - je_1)^2 \in B_1$, also

\[ (e_2i - ie_2/e_2) = (e_2i - ie_2/j) = 0 \]

Thus $(e_2i - ie_2)^2 \in B_2$, we get $e_2i = ie_2$. Likewise

\[ (ij - ji/e_1) = (ij - ji/i) = 0 \]

Hence $(ij - ji)^2 \in B_1$, and

\[ (ij - ji/e_2) = (ij - ji/j) = 0 \]

Therefore $(ij - ji)^2 \in B_2$, that is, $ij = ji$. Consequently $A$ is a four-dimensional commutative division algebra, which is absurd according to [8]. Therefore $B_1 \cap B_2 \neq \{0\}$, that is $dim(B_1 \cap B_2) = 1$. Hence $e_1 = e_2$.

**Remark 2.8** Let $A$ be a four dimensional absolute valued algebra and $e$ be a nonzero idempotent in $A$, then $(xy/yx) = -(x^2/y^2)$ for all $x, y \in \{e\}^\perp$ such that $(x/y) = 0$.

**Proof.** We get this result by a simple linearisation of the identity $\|x^2\| = \|x\|^2$. 

3 New class of four-dimensional absolute valued algebras

Now we construct some new class of four-dimensional absolute valued algebras having two different subalgebras isomorphic to $\mathbb{C}$.

Let $\{e, i, j, k\}$ be the orthonormal basis of the algebra $\mathbb{H}$ of quaternions with the usual multiplication table:

\[
\begin{array}{cccc}
H & e & i & j & k \\
e & e & i & j & k \\
i & i & -e & k & -j \\
j & j & -k & -e & i \\
k & k & j & -i & -e \\
\end{array}
\]

We define a new multiplication of the space $\mathbb{H}$, we get new class of algebras with the multiplication tables defined respectively by:

\[
\begin{array}{cccc}
M_1 & e & i & j & k \\
e & e & i & j & -k \\
i & i & -e & k & j \\
j & j & -k & -e & -i \\
k & -k & -j & i & -e \\
\end{array}
\]

\[
\begin{array}{cccc}
M_2 & e & i & j & k \\
e & e & i & j & k \\
i & i & -e & k & -j \\
j & j & -k & -e & i \\
k & -k & -j & i & e \\
\end{array}
\]

\[
\begin{array}{cccc}
M_3 & e & i & j & k \\
e & e & i & j & -k \\
i & i & -e & k & j \\
j & j & -k & -e & -i \\
k & k & j & -i & e \\
\end{array}
\]

**Proposition 3.1** The algebras $M_1, M_2$ and $M_3$ are an absolute valued algebras having two different subalgebras isomorphic to $\mathbb{C}$.

**Proof.** It suffices to show that $M_1$ is an absolute valued algebra, and in the same way we prove the others. Let $x = \alpha_1 e + \beta_1 i + \gamma_1 j + \delta_1 k$ and $y = \alpha_2 e + \beta_2 i + \gamma_2 j + \delta_2 k$, we have

\[xy = (\alpha_1 \alpha_2 - \beta_1 \beta_2 - \gamma_1 \gamma_2 - \delta_1 \delta_2)e + (\alpha_1 \beta_2 + \beta_1 \alpha_2 - \gamma_1 \delta_2 + \delta_1 \gamma_2)i \]
\[+ (\alpha_1 \gamma_2 + \gamma_1 \alpha_2 + \beta_1 \delta_2 - \delta_1 \beta_2)j + (-\alpha_1 \delta_2 - \delta_1 \alpha_2 + \beta_1 \gamma_2 - \gamma_1 \beta_2)k \]

and

\[\|xy\|^2 = (\alpha_1 \alpha_2 - \beta_1 \beta_2 - \gamma_1 \gamma_2 - \delta_1 \delta_2)^2 + (\alpha_1 \beta_2 + \beta_1 \alpha_2 - \gamma_1 \delta_2 + \delta_1 \gamma_2)^2 \]
\[+ (\alpha_1 \gamma_2 + \gamma_1 \alpha_2 + \beta_1 \delta_2 - \delta_1 \beta_2)^2 + (-\alpha_1 \delta_2 - \delta_1 \alpha_2 + \beta_1 \gamma_2 - \gamma_1 \beta_2)^2 \]
\[= \alpha_1^2(\alpha_2^2 + \beta_2^2 + \gamma_2^2 + \delta_2^2) + \beta_1^2(\alpha_2^2 + \beta_2^2 + \delta_2^2 + \gamma_2^2) + \gamma_1^2(\beta_2^2 + \alpha_2^2 + \delta_2^2 + \gamma_2^2) + \delta_1^2(\alpha_2^2 + \beta_2^2 + \gamma_2^2 + \delta_2^2) \]
Four-dimensional AVA having two different subalgebras isomorphic to \(\mathbb{C}\)

\[
= (\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2)(\alpha_2^2 + \beta_2^2 + \gamma_2^2 + \delta_2^2)
= \|x\|^2\|y\|^2.
\]

By the same way we show that \(M_2\) and \(M_3\) are absolute valued algebras. Note that, the algebras \(M_1\), \(M_2\) and \(M_3\) contain two different subalgebras isomorphic to \(\mathbb{C}\).

**Lemma 3.2** \(e\) is a nonzero omnipresent idempotent of \(M_1\), \(M_2\) and \(M_3\).

**Proof.** Let \(A\) be a four-dimensional absolute valued algebra and \(B\) a two-dimensional subalgebra of \(A\), by theorem 2.3, \(B\) is isomorphic to \(\mathbb{C}, \mathbb{C}^*\) or \(\mathbb{C}^{*}\), that is, \(B = A(f, t)\) where \(f\) is a nonzero idempotent of \(B\), \((f/t) = 0\) and \(t^2 = \pm f\). Let \(\{e, i, j, k\}\) be an orthonormal basis of the algebra \(A\) such that \(e\) is a nonzero idempotent of \(A\). Then there exists \(\alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2 \in \mathbb{R}\) such that \(f = \alpha_1e + \beta_1i + \gamma_1j + \delta_1k\) and \(t = \alpha_2e + \beta_2i + \gamma_2j + \delta_2k\). We distinguish the following cases:

1. \(A\) is isomorphic to \(M_1\).
   We have \(e\) is a central idempotent of \(M_1\) and the proposition 2.6 completes the proof.

2. \(A\) is isomorphic to \(M_2\).
   We have
   \[i^2 = j^2 = -e, \quad k^2 = e, \quad ie = ei = i, \quad je = ej = j, \quad ek = k, \quad ke = -k\]
   And
   \[ik = ki = -j, \quad ij = -ji = k, \quad jk = kj = i\]
   Since \(f^2 = f\), then
   \[
   \alpha_1^2 - \beta_1^2 - \gamma_1^2 + \delta_1^2 = \alpha_1 \quad (1)
   
   2\alpha_1\beta_1 + 2\gamma_1\delta_1 = \beta_1 \quad (2)
   
   2\alpha_1\gamma_1 - 2\beta_1\delta_1 = \gamma_1 \quad (3)
   
   \delta_1 = 0 \quad (4)
   \]
   As \(\|f\| = 1\), then \(\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2 = 1\) also \(\alpha_1^2 - \beta_1^2 - \gamma_1^2 + \delta_1^2 = \alpha_1 \quad (1)\),
   we get
   \[2\alpha_1^2 - \alpha_1 - 1 = 0\]
   thus \(\alpha_1 = 1\) or \(\alpha_1 = -\frac{1}{2}\).
   (a) If \(\alpha_1 = 1\), therefore \(e = f \in B\)
   (b) If \(\alpha_1 = -\frac{1}{2}\), then the equalities (2), (3) and (4) give \(\beta_1 = \delta_1 = \gamma_1 = 0\). So \(f = -\frac{1}{2}e\) which is absurd \((\|f\| = \|e\| = 1)\). Consequently \(e\) is a nonzero omnipresent idempotent of \(M_2\).
3. $A$ is isomorphic to $M_3$.

We have

$$i^2 = j^2 = -e, \quad k^2 = e, \quad ie = ei = i, \quad je = ej = j, \quad ek = -k, \quad ke = k$$

And

$$ik = ki = j, \quad ij = -ji = k, \quad jk = kj = -i$$

Since $f^2 = f$, then

$$\alpha_1^2 - \beta_1^2 - \gamma_1^2 + \delta_1^2 = \alpha_1 \quad (5)$$

$$2\alpha_1\beta_1 - 2\gamma_1\delta_1 = \beta_1 \quad (6)$$

$$2\alpha_1\gamma_1 + 2\beta_1\delta_1 = \gamma_1 \quad (7)$$

$$\delta_1 = 0 \quad (8)$$

As $\|f\| = 1$, then $\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2 = 1$ also $\alpha_1^2 - \beta_1^2 - \gamma_1^2 + \delta_1^2 = \alpha_1$ (5), we get

$$2\alpha_1^2 - \alpha_1 - 1 = 0$$

that is, $\alpha_1 = 1$ or $\alpha_1 = -\frac{1}{2}$.

(a) If $\alpha_1 = 1$, therefore $e = f \in B$

(b) If $\alpha_1 = -\frac{1}{2}$, then the equalities (6), (7) and (8) give $\beta_1 = \delta_1 = \gamma_1 = 0$. So $f = -\frac{1}{2}e$ which is absurd ($\|f\| = \|e\| = 1$). Consequently $e$ is a nonzero omnipresent idempotent of $M_3$.

**Remark 3.3** $e$ is a nonzero omnipresent idempotent of $M_2$, and $M_3$ which is not a central idempotent.

## 4 Main Results

In this section, we classify all four-dimensional absolute valued algebras having two different subalgebras isomorphic to $\mathbb{C}$.

We state now the following important result:

**Theorem 4.1** Let $A$ be a four-dimensional absolute valued algebra having two different subalgebras $B_1$ and $B_2$ isomorphic to $\mathbb{C}$. Then $A$ is isomorphic to $\mathbb{H}$, $M_1$, $M_2$ or $M_3$.

**Proof.** According to theorem 2.2, $A$ is an inner product space, and by lemma 2.7, $B_1 \cap B_2 = \{e\}$ ($e$ is an idempotent of $A$). We put $B_1 = A(e,i)$ and $B_2 = A(e,j)$ where

$$i^2 = j^2 = -e, \quad ie = ei = i \quad and \quad je = ej = j$$
We know that \((e/i) = (e/j) = 0\), and without loss of generality we may assume that \((i/j) = 0\). Indeed, if \((i/j) \neq 0\) then \(t = \frac{j - (i/j)i}{\|j - (i/j)i\|}\) is orthogonal to \(e\). Since \(te = et = t\) and \(\|e\| = \|t\| = 1\), we get \(t^2 = -e\). Which implies that \(A(e, t)\) is isomorphic to \(\mathbb{C}\).

Now in \(A\) there exists an orthonormal subset \(\{e, i, j\}\) which can be extended to an orthonormal basis \(\{e, i, j, k\}\) for \(A\). Since \(k \in \{e, i, j\}\perp\), then \(k^2 \in A(e, i) \cap A(e, j) = \{e\}\). We get \(k^2 = \pm e\). But since
\[
(ek/e) = (ek/e^2) = (e/k) = 0 \quad \text{and} \quad (ke/e) = (ke/e^2) = (k/e) = 0
\]
\[
(ek/i) = (ek/ei) = (k/i) = 0 \quad \text{and} \quad (ke/i) = (ke/ie) = (k/i) = 0
\]
Also
\[
(ek/j) = (ek/ej) = (k/j) = 0 \quad \text{and} \quad (ke/j) = (ke/je) = (k/j) = 0
\]
we obtain \(ek = \varepsilon k\) and \(ke = \zeta k\), where \(|\varepsilon| = |\zeta| = 1\).

We conclude that \(A(e, k)\) is two-dimensional subalgebra of \(A\), that is \(A(e, k)\) is isomorphic to \(\mathbb{C}, \ast\mathbb{C}, \mathbb{C}^*\) or \(\ast\mathbb{C}\) (theorem 2.3). We distinguish the following cases:

1. \(A(e, k)\) is isomorphic to \(\mathbb{C}\).
   
   Then \(e\) will be the unit element of \(A\) and, therefore \(A\) is isomorphic to \(\mathbb{H}\).

2. \(A(e, k)\) is isomorphic to \(\ast\mathbb{C}\).
   
   So \(ke = ek = -k\) and \(k^2 = -e\), since
   
   \[
   (ij/e) = (ij/ - i^2) = -(i/j) = 0
   \]
   \[
   (ij/i) = (ij/ie) = (i/j) = 0
   \]
   and
   
   \[
   (ij/j) = (ij/j) = (i/j) = 0
   \]
   Hence \(ij = k\) or \(ij = -k\). In a similar manner, we can show that
   
   \[ ik = j \quad \text{or} \quad ik = -j \]
   
   and
   
   \[ jk = i \quad \text{or} \quad jk = -i \]
   Assume that \(ij = k\), in this case we have \(ik = j\) and \(jk = -i\). Indeed, if \(ik = -j\), then
   
   \[ i(j + k) = k - j = -ek - ej = -e(k + j) \]
Which gives $i = -e$ ($A$ has no zero divisors), contradiction. Also if $jk = i$, then

$$(i + j)k = j + i = (j + i)e$$

which implies $k = e$, absurd. Moreover, by remark 2.8, we have

$$(ij/ji) = -(i^2/j^2) = -1$$

which means that

$$||ij + ji||^2 = 0$$

So $ij = -ji$, and by the same way we have $ik = -ki$, and $jk = -kj$. Therefore, we get the multiplication table of $M_1$.

3. $A(e, k)$ is isomorphic to $\mathbb{C}^\ast$.

We have $ek = k$, $ke = -k$ and $k^2 = e$. Using remark 2.8, we have

$$(ik/ki) = -(i^2/k^2) = 1$$

which means that

$$||ik - ki||^2 = 0$$

So $ik = ki$, similarly, we get

$$jk = kj \text{ and } ij = -ji$$

By simple calculations, we show that

$$ij = k \text{ or } ij = -k,$$

$$ik = j \text{ or } ik = -j$$

and

$$jk = i \text{ or } jk = -i$$

Assume that $ij = k$, in this case we have $ik = -j$ and $jk = i$. Indeed, if $ik = j$, then

$$i(j + k) = k + j = ek + ej = e(k + j)$$

Which gives $i = e$ ($A$ has no zero divisors), contradiction. Also if $jk = -i$, then

$$(i + j)k = -j - i = -je - ie = -(j + i)e$$

which implies $k = -e$, absurd. So $A$ is isomorphic to $M_2$.

4. $A(e, k)$ is isomorphic to $\mathbb{C}^\ast$.

We have $ek = -k$, $ke = k$ and $k^2 = e$. By remark 2.8, we get

$$ik = ki, \quad jk = kj \text{ and } ij = -ji$$
And by simple calculations, we show that
\[ ij = k \quad \text{or} \quad ij = -k, \]
\[ ik = j \quad \text{or} \quad ik = -j \]
and
\[ jk = i \quad \text{or} \quad jk = -i \]
Assume that \( ij = k \), in this case we have \( ik = j \) and \( jk = -i \). Indeed, if \( ik = -j \), then
\[ i(j + k) = k - j = -ek - ej = -e(k + j) \]
This implies that \( i = -e \) (\( A \) has no zero divisors), contradiction. Also if \( jk = i \), then
\[ (i + j)k = j + i = je + ie = (j + i)e \]
which implies \( k = e \), absurd. So \( A \) is isomorphic to \( \mathbb{M}_3 \).

**Remark 4.2** Assume that \( ij = -k \), if we substitute \(-k = t\) we obtain \( ij = t \), that is we again get the same multiplication tables previously.

**Acknowledgements.** The authors express their deep gratitude to the referee for the carefully reading of the manuscript and the valuables comments that have improved the final version of the same.

**References**


[5] A. Moutassim, On Absolute Valued Algebras Satisfying \( (x,x,x) = 0 \), \( (x^2,x,x) = 0 \), \( (x,x,x^2) = 0 \), \( (x^2,x^2,x) = 0 \) or \( (x,x^2,x^2) = 0 \), International Journal of Algebra, 3 (2009), no. 11, 511- 524.


Received: March 1, 2023; Published: March 25, 2023