Abstract

In [9], we have proven that if $A$ is a four-dimensional absolute valued algebra having two different subalgebras isomorphic to $\mathbb{C}$, then $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$. Here we complete the study of $A$. Indeed, we show if $A$ has two different subalgebras isomorphic to $\mathbb{C}$. Then $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$. Furthermore, we classify all four-dimensional absolute valued algebras containing a nonzero idempotent commuting with all idempotents, such an algebra $A$ contains at least two different subalgebras isomorphic to $\mathbb{C}$. Which means that $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$.

Keywords: Absolute valued algebra, pre-Hilbert algebra, omnipresent idempotent.

1 Introduction

Let $A$ be a non necessarily associative real algebra which is normed as real vector space. We say that a real algebra $A$ is an absolute valued algebra, if it’s
satisfy the equality \( \|ab\| = \|a\|\|b\| \), for all \( a, b \in A \). We recall that every absolute valued algebra is a normed algebra, note that, the norm \( \|\cdot\| \) of any finite dimensional absolute valued algebra comes from an inner product \((\cdot,\cdot)\) \[4\]. In 1947 Albert proved that the finite dimensional unital absolute valued algebras are classified by \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) and that every finite dimensional absolute valued algebra is isotopic to one of the algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) and so has dimension \( 1, 2, 4 \) or \( 8 \) \[1\]. Urbanik and Wright proved in 1960 that all unital absolute valued algebras are classified by \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) \[14\]. It is easily seen that the one-dimensional absolute valued algebras are classified by \( \mathbb{R} \), and it is well-known that the two-dimensional absolute valued algebras are isomorphic to \( \mathbb{C}, \mathbb{C}^*, \mathbb{C}^* \) or \( \mathbb{C} \) \[11\]. Moreover, the four-dimensional absolute valued algebras have been described by M.I. Ramírez Álvarez in \[10\] and she gave an example of four-dimensional absolute valued algebra containing no subalgebra of dimension two. Which means that, the problem of classifying all four-dimensional absolute valued algebras seems still to be open.

Motivated by these facts, we became interested in the study of four-dimensional absolute valued algebras having two different commutative subalgebras of dimension two. According to Rodríguez’s theorem \[11\], \( \mathbb{C} \) and \( \mathbb{C}^* \) are the only commutative absolute valued algebra of dimension two. Furthermore, we know that \( \mathbb{H} \) and \( \mathbb{H}^* \) are examples of four-dimensional absolute valued algebras containing at least two different commutative subalgebras of dimension two. This reinforces our construction, by algebraic method, of this class. Note that these new algebras contain a nonzero omnipresent idempotent (proposition 3.1). We recall that, the classification of four-dimensional absolute valued algebras containing a unique two-dimensional commutative subalgebra is still an open problem.

In 2016 \[7\], we classified all four-dimensional absolute valued algebras containing a nonzero central idempotent and we also proved such an algebra contains a commutative subalgebra of dimension two. Here (theorem 4.3) we extend this result to more general situation. Indeed, we prove that, if \( A \) is a four-dimensional absolute valued algebra with a nonzero idempotent commuting with all idempotents, then \( A \) is isomorphic to a new absolute valued algebras of dimension four. We also show, in proposition 2.6, that a central idempotent is an omnipresent idempotent, the reciprocal does not hold in general, and the counter example is given (remark 3.3. (3)). In the same context, M. L. El-Mallah, proved that every absolute valued algebra \( A \) with a nonzero central idempotent and satisfying \( x^2x = xx^2 \) is isomorphic to \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{C}^*, \mathbb{H}^* \) and \( \mathbb{O} \) (see \[5\] and \[6\]). The same result was found by J. A. Cuenca by replacing the condition (central idempotent) by another weaker one (nonzero idempotent commuting with the remaining idempotents) \[2\]. We note that, the classification of four-dimensional absolute algebras containing a unique idempotent or
two different idempotents is still an open problem. From now on, we denote by \( I(A) \) the set of all nonzero idempotents of \( A \) (according to Segre’s theorem [13], \( I(A) \) is not empty) and we assume that \( I(A) \) contains at least three elements linearly independent. From the comments below, it arises in a natural way the following question: What is the classification of four-dimensional absolute valued algebras with a nonzero idempotent commuting with all idempotents? This paper is devoted to shed some light on this problem.

In section 2 we introduce the basic tools for the study of four-dimensional absolute valued algebras. We also give some properties related to idempotents satisfying some restrictions on commutativity (lemmas 2.4, 2.5 and 2.8). Moreover, the section 3 is devoted to construct, by algebraic method, some new class of the four-dimensional absolute valued algebras having two different subalgebras isomorphic to \( \mathcal{C} \) (respectively, containing a nonzero idempotent commuting with all idempotents), namely \( \mathcal{M}_1, \mathcal{M}_2 \) and \( \mathcal{M}_3 \).

The paper ends, in section 4, with the following main results.

**Theorem 1.1** Let \( A \) be a four dimensional absolute valued algebra having two different subalgebras isomorphic to \( \mathcal{C} \). Then \( A \) is isomorphic to \( \mathcal{H}, \mathcal{M}_1, \mathcal{M}_2 \) or \( \mathcal{M}_3 \).

**Theorem 1.2** Let \( A \) be a four dimensional absolute valued algebra with a nonzero idempotent \( e \) commuting with all idempotents of \( A \). If \( I(A) \) contains at least three elements \( e, f \) and \( g \) linearly independent, then \( A \) is isomorphic to \( \mathcal{H}, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \).

## 2 Preliminary Notes

Throughout this paper, the word algebra refers to a non-necessarily associative algebra over the field of real numbers \( \mathbb{R} \).

**Definition 2.1** Let \( B \) be an arbitrary algebra.

i) \( B \) is called a normed algebra (resp, absolute valued algebra) if it is endowed with a space norm: \( \| \cdot \| \) such that \( \| xy \| \leq \| x \| \| y \| \) (resp, \( \| xy \| = \| x \| \| y \| \)), for all \( x, y \in B \).

ii) \( B \) is called a pre-Hilbert algebra if it is endowed with a space norm comes from an inner product \((\cdot/\cdot)\) such that

\[
(\cdot/\cdot) : B \times B \to \mathbb{R} \quad (x, y) \mapsto (x/y) = \frac{1}{4}((\|x + y\|^2 - \|x - y\|^2))
\]
ii) $B$ is called a division algebra if for all nonzero $a \in B$, the operators $L_a(x) = ax$ and $R_a(x) = xa$ (for all $x \in B$) of left and right multiplication by $a$ are bijective. Note that every finite-dimensional absolute valued algebra is a division algebra.

iii) We mean by a nonzero omnipresent idempotent, an idempotent which is contained in all two-dimensional subalgebras of $B$.

In 2017, we showed that if $A$ is a real commutative algebraic algebra without divisors of zero, then $A$ is finite dimensional ($\dim A \leq 2$) [8]. Moreover, we proved by a simple algebraic method, there is no four-dimensional real commutative division algebra, which means that the dimension of commutative absolute valued algebras is less than or equal to two.

On the other hand, the most natural examples of absolute valued algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the algebra of Hamilton quaternion) and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values [3] and [12]. The algebras $\mathbb{C}^*$, $\mathbb{C}^*$, and $\mathbb{C}^*$ obtained by endowing the space $\mathbb{C}$ with the products defined by $x \ast y = \bar{xy}$, $x \ast y = x\bar{y}$ and $x \ast y = \bar{x}\bar{y}$ respectively where $x \rightarrow \bar{x}$ is the standard conjugation of $\mathbb{C}$. Note that the algebras $\mathbb{C}$ and $\mathbb{C}^*$ are the only two-dimensional commutative absolute valued algebras. Also by $\mathbb{H}$ we shall denote the real algebras obtained by endowing the space $\mathbb{H}$ with the products defined by $x \ast y = \bar{xy}$, where $x \rightarrow \bar{x}$ is the standard conjugation of $\mathbb{H}$.

We need the following relevant results:

**Theorem 2.2** [4] The norm of any finite dimensional absolute valued algebra comes from an inner product.

**Theorem 2.3** [11] If $A$ is a two-dimensional absolute valued algebra, then $A$ is isomorphic to $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^*$ and $\mathbb{C}^*$.

**Lemma 2.4** [7] Let $A$ be a finite-dimensional absolute valued algebra containing a nonzero central idempotent $f$, then the following statements hold:

i) $A$ contains a subalgebra of dimension two,

ii) $x^2 = -\|x\|^2 f$, for all $x \in \{f\}^\perp$.

**Lemma 2.5** [9] Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central idempotent $e$ and a subalgebra $B$ of dimension two. If $x, y \in B$, then $xy \in B$.

**Proposition 2.6** [9] Let $A$ be a four-dimensional absolute valued algebra containing a nonzero central idempotent $f$, then $f$ is an omnipresent idempotent.
Lemma 2.7 [13] every algebra in which $x^2 = 0$ only if $x = 0$ contains a nonzero idempotent.

Lemma 2.8 let $A$ be a four-dimensional absolute valued algebra having two different commutative subalgebras $B_1$ and $B_2$ of dimension two. Then $B_1 \cap B_2 \neq \{0\}$.

Proof. According to theorem 2.2, $A$ is an inner product space and by theorem 2.3, $B_1$ and $B_2$ are isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$. We can set $B_1 = A(e_1, i)$ and $B_2 = A(e_2, j)$, such that

$$i^2 = -e_1, \quad ie_1 = e_1i = \pm i$$

and

$$j^2 = -e_2, \quad je_2 = e_2j = \pm j$$

We know that

$$(e_1/i) = (e_2/j) = 0$$

We assume that $B_1 \cap B_2 = \{0\}$, so the family $\{e_1, i, e_2, j\}$ is a basis of $A$. According to lemma 2.5, we have the following equalities:

$$(e_1e_2 - e_2e_1/e_1) = (e_1e_2 - e_2e_1/i) = 0$$

Hence $(e_1e_2 - e_2e_1)^2 \in B_1$, also

$$(e_1e_2 - e_2e_1/e_2) = (e_1e_2 - e_2e_1/j) = 0$$

That is $(e_1e_2 - e_2e_1)^2 \in B_2$, this last imply $(e_1e_2 - e_2e_1)^2 = 0$ or $e_1e_2 = e_2e_1$. Similarly

$$(e_1j - je_1/e_1) = (e_1j - je_1/i) = 0$$

Which gives $(e_1j - je_1)^2 \in B_1$, also

$$(e_2i - ie_2/e_2) = (e_2i - ie_2/j) = 0$$

Thus $(e_2i - ie_2)^2 \in B_2$, we get $e_2i = ie_2$. Likewise

$$(ij - ji/e_1) = (ij - ji/i) = 0$$

Hence $(ij - ji)^2 \in B_1$, and

$$(ij - ji/e_2) = (ij - ji/j) = 0$$

Therefore $(ij - ji)^2 \in B_2$, that is, $ij = ji$. Consequently $A$ is a four-dimensional commutative division algebra, which is absurd according to [15]. Therefore $B_1 \cap B_2 \neq \{0\}$, that is $dim(B_1 \cap B_2) = 1$. 
Remark 2.9 Let $A$ be a four dimensional absolute valued algebra, then $(xy/yx) = -(x^2/y^2)$ for all $x, y \in A$ such that $(x/y) = 0$.

Proof. We get this result by a simple linearisation of the identity $\|x^2\| = \|x\|^2$.

Theorem 2.10 [9] Let $A$ be a four-dimensional absolute valued algebra having two different subalgebras $B_1$ and $B_2$ isomorphic to $\mathbb{C}$. Then $A$ is isomorphic to $\mathbb{H}, M_1, M_2$ or $M_3$ defined by:

$$
\begin{array}{c|c|c|c|c}
 & e & i & j & k \\
\hline
M_1 & e & i & j & -k \\
 i & -i & k & j & \\
 j & -k & -j & i & \\
 k & j & i & e & \\
\end{array}
\quad
\begin{array}{c|c|c|c|c}
 & e & i & j & k \\
\hline
M_2 & e & i & j & -k \\
 i & -i & k & j & \\
 j & -k & -j & i & \\
 k & j & i & e & \\
\end{array}
\quad
\begin{array}{c|c|c|c|c}
 & e & i & j & k \\
\hline
M_3 & e & i & j & -k \\
 i & -i & k & j & \\
 j & -k & -j & i & \\
 k & j & -i & e & \\
\end{array}
$$

3 New class of four-dimensional absolute valued algebras

Now we construct some new class of four-dimensional absolute valued algebras having two different subalgebras isomorphic to $\mathbb{C}$.

We set $M$ one of the principal absolute valued algebras $M_1, M_2$ or $M_3$ (defined in theorem 2.10) and $\overset{*}{M}$, the standard isotopes of $M$, other than $M$, that is the algebras having $M$ as vectorial space and products given by $x \ast y = \bar{x}\bar{y}$, where $x \rightarrow \bar{x}$ is the standard conjugation of $M$. We denote these new algebras by $\overset{*}{M}_1, \overset{*}{M}_2$ and $\overset{*}{M}_3$.

Proposition 3.1 The algebras $\overset{*}{M}_1, \overset{*}{M}_2$ and $\overset{*}{M}_3$ are an absolute valued algebras with a nonzero omnipresent idempotent $e$.

Proof. Let $M$ be one of principal absolute valued algebras $M_1, M_2, M_3$ or $M_4$, and $x, y \in M$ we have

$$
\|x \ast y\| = \|\bar{x}\bar{y}\| = \|\bar{x}\|\|\bar{y}\| = \|x\|\|y\|
$$
Therefore $\mathbb{M}_1^*$, $\mathbb{M}_2^*$ and $\mathbb{M}_3^*$ are an absolute valued algebras. Moreover, if $B$ is a two-dimensional subalgebra of $\mathbb{M}$, then $B$ is a two-dimensional subalgebra of $\mathbb{M}$. Which means that $\mathbb{M}$ and $\mathbb{M}$ have the same omnipresent idempotent $e$, hence The algebras $\mathbb{M}_1^*$, $\mathbb{M}_2^*$ and $\mathbb{M}_3^*$ are an absolute valued algebras with a nonzero omnipresent idempotent $e$.

Remark 3.2  
1. Let $F = \{e, i, j, k\}$ be the orthonormal basis of the algebra $\mathbb{M}$, where $e$ is a nonzero omnipresent idempotent of $\mathbb{M}$. The multiplication tables the elements of the base $F$ of $\mathbb{M}_1^*$, $\mathbb{M}_2^*$ and $\mathbb{M}_3^*$ are given by:

\[
\begin{array}{cccc}
\mathbb{M}_1^* & e & i & j & k \\
e & e & -i & -j & k \\
i & -i & -e & k & j \\
j & -j & -k & -e & -i \\
k & k & -j & i & -e \\
\end{array}
\quad
\begin{array}{cccc}
\mathbb{M}_2^* & e & i & j & k \\
e & e & -i & -j & -k \\
i & -i & -e & k & -j \\
j & -j & -k & -e & i \\
k & k & -j & -e & i \\
\end{array}
\quad
\begin{array}{cccc}
\mathbb{M}_3^* & e & i & j & k \\
e & e & -i & -j & k \\
i & -i & -e & k & j \\
j & -j & -k & -e & -i \\
k & -k & j & -i & e \\
\end{array}
\]

2. The algebras $\mathbb{M}_1^*$, $\mathbb{M}_2^*$ and $\mathbb{M}_3^*$ contain at least two different subalgebras isomorphic to $\mathbb{C}$.

3. $e$ is a nonzero omnipresent idempotent of $\mathbb{M}_2^*$ and $\mathbb{M}_3^*$ which is not a central idempotent.

4 Main Results

4.1 Commuting idempotents in four-dimensional absolute valued algebras

Theorem 4.1 Let $A$ be a four dimensional absolute valued algebra having two different subalgebras $B_1$ and $B_2$ isomorphic to $\mathbb{C}$. Then $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1^*$, $\mathbb{M}_2^*$ or $\mathbb{M}_3^*$.

Proof. If we define a new multiplication on $A$ by $x \ast y = \overline{xy}$, we obtain an algebra $\mathring{A}$ which contains two different subalgebras isomorphic to $\mathbb{C}$. Therefore,
applying theorem 2.10, \( A \) is isomorphic to \( \mathbb{H}, M_1^*, M_2^* \) or \( M_3^* \). Consequently, \( A \) is isomorphic to \( \mathbb{H}, M_1^*, M_2^* \) or \( M_3^* \).

In the rest of this subsection, we denote by \( I(A) \) the set of all nonzero idempotents of \( A \) (according to lemma 2.7, \( I(A) \) is not empty) and we assume that \( I(A) \) contains at least three different elements linearly independent. Let \( e \) be a nonzero idempotent commuting with all idempotents of \( A \), in this case, \( A \) contains at least two different commutative subalgebras isomorphic to \( \mathbb{C} \).

Lemma 4.2 Let \( A \) be a finite-dimensional absolute valued algebra containing a nonzero two idempotents \( f \) and \( e \) such that \( e \neq f \) and \( ef = fe \), then the subalgebra \( A(e, f) \) of \( A \) is isomorphic to \( \mathbb{C} \).

Proof. According to theorem 2.2, \( A \) is an inner product space. Since \( e \) and \( f \) are different nonzero idempotents. We have

\[
\|e - f\| = \|e^2 - f^2\| = \|e - f\|\|e + f\|
\]

That is \( \|e + f\| = 1 \), this imply \( (e/f) = -\frac{1}{2} \). Thus,

\[
e + f + ef = 0
\]

Therefore, the subalgebra \( A(e, f) \) is isomorphic to \( \mathbb{C} \).

Lemma 4.3 If \( I(A) \) contains at least three elements \( e, f \) and \( g \) linearly independent, then \( fg \neq gf \).

Proof. We assume that \( fg = gf \), by lemma 4.2 the subalgebra \( A(f, g) \) is isomorphic to \( \mathbb{C} \). That is,

\[
f + g + fg = 0
\]

Since \( ef = fe \) and \( eg = ge \), hence \( A(e, f) \) and \( A(e, g) \) are isomorphic to \( \mathbb{C} \). Which means that

\[
e + f + ef = 0
\]

and

\[
e + g + eg = 0
\]

We conclude that \( A(e, f, g) \) is a three-dimensional commutative absolute valued algebra, which is absurd \([\]\). Therefore \( fg \neq gf \).

Theorem 4.4 Let \( A \) be a four dimensional absolute valued algebra with a nonzero idempotent \( e \) commuting with all idempotents of \( A \). If \( I(A) \) contains at least three elements \( e, f \) and \( g \) linearly independent, then \( A \) is isomorphic to \( \mathbb{H}, M_1^*, M_2^*, M_3^* \).
Proof. According to theorem 2.2, A is an inner product space. Since $ef = fe$ and $eg = ge$, hence $A(e, f)$ and $A(e, g)$ are isomorphic to $\mathbb{C}$ and by lemma 4.3 $fg \neq gf$. Therefore A contains at least two different commutative subalgebras isomorphic $\mathbb{C}$ and the theorem 4.1 completes the proof.

4.2 Conclusion

In this section, we have the following main results.

Theorem 4.5 Let $A$ be a four dimensional absolute valued algebra having two different subalgebras isomorphic to $\mathbb{C}$. Then $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$ or $\mathbb{M}_3$.

Theorem 4.6 Let $A$ be a four dimensional absolute valued algebra with a nonzero idempotent $e$ commuting with all idempotents of $A$. If $I(A)$ contains at least three elements $e$, $f$ and $g$ linearly independent, then $A$ is isomorphic to $\mathbb{H}$, $\mathbb{M}_1$, $\mathbb{M}_2$, $\mathbb{M}_3$.

Acknowledgements. The authors express their deep gratitude to the referee for the carefully reading of the manuscript and the valuables comments that have improved the final version of the same.

References


*Received: April 15, 2023; Published: May 9, 2023*