Composition of Primes in Elliptic Curves

$y^2 = x^3 - 2px$

Shin-Wook Kim

Deokjin-gu, Songcheon 54823
101-703, I-Park Apt
Jeonju, Jeonbuk, Korea

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Abstract

We confront to generalized rank 1 often in elliptic curve $E_{-2p}: y^2 = x^3 - 2px$ with prime $p$. In this article, we shall enumerate rank of $E_{-2p}$ where prime $p$ is composed of many variables.

Mathematics Subject Classification: 11A41, 11G05

Keywords: Odd prime, elliptic curve

1 Introduction

In elliptic curve $E_{-2p}: y^2 = x^3 - 2px$, there deduced many results of generalized rank 1. For taking rank 1 prime should be the forms $p \equiv 3, 11, 5, 13 (mod 16)$. In $p \equiv 3, 11 (mod 16)$, we must search the solution of relating equations 4) $N^2 = -2M^4 + pe^4$ for $\Gamma$ and if $p$ is $p \equiv 5, 13 (mod 16)$ then, it is necessary to find the solution of equation 2) $N^2 = -M^4 + 2pe^4$ for $\Gamma$. Many forms of $p$ that derives rank 1 in $E_{-2p}$ are given as $p = Hu^4 + Iv^2v^2 + Kv^8 \ldots \ldots (A)$. There also exist the form $p = Hu^4 + Iv^4v^8 + Kv^8 \ldots \ldots (A)$. But that is not all thing. Other forms of prime $p$ also educes rank 1 in $E_{-2p}$. It can be the form $p$ that is consist of 3 variables and 6 terms. In this case, all terms are square forms. It is sufficient that only treating the form (A) in $E_{-2p}$ to attain rank 1. But by managing various forms of $p$ the treatment of rank 1 in $E_{-2p}$ became varied and in eventually it makes generalization of rank in elliptic curve progressive. In this article, we shall invest-
gate rank of curve $E_{-2p}$ where $p$ is comprised of 14 variables.

To start with, it is necessary to consider some notations in [5] and [6].

Take $E$ as an elliptic curve $y^2 = x^3 + ax^2 + bx$ and $\Gamma$ as the set of rational points on $E$. Due to Mordell’s Theorem, $\Gamma$ is a finitely generated abelian group. Furthermore, it is isomorphic to $E(Q)_{\text{tors}} \oplus Z^r$ with torsion subgroup $E(Q)_{\text{tors}}$ and Mordell-Weil rank $r$. Besides, we appoint that $Q^\times$ is multiplicative group that is comprised of non-zero rational numbers. Denote $Q^{\times 2}$ as subgroup of squares of elements of $Q^\times$.

Define $\alpha$ as a homomorphism in section 5 of Chapter III in [5] and $\bar{\alpha}$ as a homomorphism in [6].

For $\Gamma$, we designate

$$N^2 = b_1 M^4 + a M^2 e^2 + b_2 e^4$$

as relating equation that satisfies several conditions in section 6 of Chapter III in [5].

Take $\bar{E}$ as the curve $y^2 = x(x^2 - 2ax + a^2 - 4b)$ and $\bar{\Gamma}$ as the set of rational points on $\bar{E}$.

Assign $\bar{N}^2 = b_1 M^4 - 2a M^2 e^2 + b_2 e^4$ as relating equation for $\bar{\Gamma}$ where the conditions in [6] are hold.

Besides, there derived $2^r = \frac{\#\alpha(\Gamma)\#\bar{\alpha}(\bar{\Gamma})}{4}$ where $r$ denotes the rank of $E$.

2 Form $E_{-2p}$

In $E_{-2p}: y^2 = x^3 - 2px$, generally there are two kinds of forms $p$ that induces rank 1. The first one is composed of only square. The representative thing is (A). Most primes $p$ (duces rank 1 in $E_{-2p}$) are correlated to this first case. The second one is the form whose components are not all square. What the difference of these two kinds of primes $p$ is the value of $M$. In first form, $M$ is gotten as one value whereas in second form the value $M$ is given as polynomial with several variables. In both cases, it is fixed that $e = 1$. Now, we numerate the rank of $E_{-2p}: y^2 = x^3 - 2px$ where $p$ is composed of 14 variables. In the followings, we use the notations $r.4.2$ that was used in [3].

Theorem 2.1. (1). Indicate an odd prime $p$ as $p = 16h^4 + 3t^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^4 + t^4 + \delta^4 + \lambda^4 - 8h^2 t^2 + 8h^2 j^2 + 8h^2 k^2 + 8h^2 \cdot l^2 + 8h^2 s^2 + 8h^2 t^2 + 8h^2 u^2 + 8h^2 v^2 + 8h^2 w^2 + 8h^2 \eta^2 + 8h^2 \delta^2 + 8h^2 \lambda^2 + 8 \cdot h^2 \lambda^2 - 2i^2 j^2 - 2i^2 k^2 - 2i^2 l^2 - 2i^2 s^2 - 2i^2 t^2 - 2i^2 u^2 - 2i^2 v^2 - 2i^2 w^2 - 2i^2 \eta^2 - 2i^2 \delta^2 - 2i^2 \lambda^2 + 2j^2 k^2 + 2j^2 l^2 + 2j^2 s^2 + 2j^2 t^2 + 2j^2 u^2 + 2j^2 v^2 + 2j^2 w^2 + 2j^2 \eta^2 + 2j^2 \delta^2 + 2j^2 \lambda^2 + 2k^2 l^2 + 2k^2 s^2 + 2k^2 t^2 + 2k^2 u^2 + 2k^2 v^2 + 2k^2 w^2 + 2k^2 \eta^2 + 2k^2 \delta^2 + 2k^2 \lambda^2 + 2l^2 s^2 + 2l^2 t^2 + 2l^2 u^2 + 2l^2 v^2 + 2l^2 w^2 + 2l^2 \eta^2 + 2l^2 \delta^2 + 2l^2 \lambda^2 + 2s^2 t^2 + 2s^2 u^2 + 2s^2 v^2 + 2s^2 w^2 + 2s^2 \eta^2 + 2s^2 \delta^2 + 2s^2 \lambda^2 + 2t^2 u^2 + 2t^2 v^2 + 2t^2 w^2 + 2t^2 \eta^2 + 2t^2 \delta^2 + 2t^2 \lambda^2 + 2u^2 v^2 + 2u^2 w^2 + 2u^2 \eta^2 + 2u^2 \delta^2 + 2u^2 \lambda^2 + 2v^2 w^2 + 2v^2 \eta^2 + 2v^2 \delta^2 + 2v^2 \lambda^2 + 2w^2 \eta^2 + 2w^2 \delta^2 + 2w^2 \lambda^2$.
\[ +2t^2\eta^2 + 2t^2\nu^2 + 2t^2\delta^2 + 2t^2\lambda^2 + 2u^2\nu^2 + 2u^2w^2 + 2u^2\eta^2 + 2u^2t^2 + 2u^2\delta^2 \\
+ 2u^2\lambda^2 + 2v^2w^2 + 2v^2\eta^2 + 2v^2t^2 + 2v^2\delta^2 + 2v^2\lambda^2 + 2w^2\eta^2 + 2w^2t^2 + 2w^2\delta^2 \\
\cdot \delta^2 + 2w^2\lambda^2 + 2\nu^2t^2 + 2\eta^2\delta^2 + 2\nu^2\lambda^2 + 2t^2\delta^2 + 2t^2\lambda^2 + 2\delta^2 \lambda^2 \text{ with integers} \\
h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda \text{ and } (h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda) = 1 \text{ and } p \equiv 3(\text{mod} 16) \text{ in } E_{-2p} \text{ then, the consequence} \\
\text{rank}(E_{-2(16h^4+3i^4+j^4+k^4+l^4+s^4+t^4+\ldots+2\eta^2\delta^2+2\eta^2\lambda^2+2t^2\delta^2+2t^2\lambda^2+2\delta^2\lambda^2)}(Q)) = 1 \\
\text{is derived.}
\]

(2). If prime \( p = 6h^4+i^4+j^4+k^4+l^4+s^4+t^4+u^4+v^4+w^4+\eta^4+\iota^4+\delta^4+\lambda^4+60h^2i^2-20h^2 \)
\cdot h^2\nu^2 + 4h^2w^2 + 4h^2\eta^2 + 4h^2\iota^2 + 4h^2\delta^2 + 4h^2\lambda^2 - 2i^2j^2 - 2i^2k^2 - 2i^2t^2 - 2i^2\nu^2 - 2i^2w^2 - 2i^2\eta^2 - 2i^2\iota^2 - 2i^2\delta^2 - 2i^2\lambda^2 + 2j^2k^2 + 2j^2i^2 - 2j^2s^2 - 2j^2t^2 + 2j^2\nu^2 + 2j^2w^2 - 2j^2\eta^2 - 2j^2\iota^2 - 2j^2\delta^2 - 2j^2\lambda^2 + 2k^2l^2 + 2k^2s^2 + 2k^2t^2 + 2k^2\nu^2 + 2k^2w^2 + 2k^2\eta^2 + 2k^2\iota^2 + 2k^2\delta^2 + 2k^2\lambda^2 + 2l^2s^2 + 2l^2t^2 + 2l^2\nu^2 + 2l^2w^2 + 2l^2\eta^2 + 2l^2\iota^2 + 2l^2\delta^2 + 2l^2\lambda^2 + 2m^2n^2 + 2m^2o^2 + 2m^2p^2 + 2m^2\eta^2 + 2m^2\iota^2 + 2m^2\delta^2 + 2m^2\lambda^2 + 2m^2\nu^2 + 2m^2\omega^2 + 2m^2\eta^2 + 2m^2\iota^2 + 2m^2\delta^2 + 2m^2\lambda^2 + 2n^2o^2 + 2n^2p^2 + 2n^2\eta^2 + 2n^2\iota^2 + 2n^2\delta^2 + 2n^2\lambda^2 + 2o^2p^2 + 2o^2\eta^2 + 2o^2\iota^2 + 2o^2\delta^2 + 2o^2\lambda^2 + 2\eta^2\delta^2 + 2\eta^2\lambda^2 + 2\iota^2\delta^2 + 2\iota^2\lambda^2 + 2\delta^2\lambda^2 \text{ with integers} h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda \text{ and } (h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda) = 1 \text{ and } p \equiv 11(\text{mod} 16) \text{ in elliptic curve } E_{-2p} \text{ then, the consequence} \\
\text{rank}(E_{-2(6h^4+i^4+j^4+k^4+l^4+s^4+t^4+\ldots+2\eta^2\delta^2+2\eta^2\lambda^2+2t^2\delta^2+2t^2\lambda^2+2\delta^2\lambda^2)}(Q)) = 1 \\
is derived.

(3). We appoint that an odd prime \( p \) is gotten as the form \( p = 100h^4 + 11i^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^4 + \iota^4 + \delta^4 + \lambda^4 + 60h^2i^2 - 20h^2 \)
\cdot j^2 + 20h^2k^2 + 20h^2l^2 + 20h^2s^2 + 20h^2t^2 + 20h^2u^2 + 20h^2w^2 + 20h^2\nu^2 + 20h^2\iota^2 + 20h^2\delta^2 + 20h^2\lambda^2 - 6i^2j^2 + 6i^2k^2 + 6i^2l^2 + 6i^2s^2 + 6j^2i^2 + 6j^2k^2 + 6j^2l^2 + 6j^2s^2 + 6k^2i^2 + 6k^2j^2 + 6k^2l^2 + 6k^2s^2 + 6l^2i^2 + 6l^2j^2 + 6l^2k^2 + 6l^2s^2 + 6s^2i^2 + 6s^2j^2 + 6s^2k^2 + 6s^2l^2 \text{ and integers} h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda \text{ and } (h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda) = 1 \text{ and } p \equiv 3(\text{mod} 16) \text{ in } E_{-2p} \text{ then, we have} 

\[ \text{rank}(E_{-2(100h^4+11i^4+j^4+k^4+l^4+s^4+t^4+u^4+ \ldots +2\eta^2\lambda^2+2i^2\delta^2+2i^2\lambda^2+2\delta^2\lambda^2)}(Q)) = 1. \]

(4). Denote prime \( p \) as \( p = 16h^4 + 3i^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^4 + i^4 + \lambda^4 + 8h^2j^2 + 8h^2k^2 + 8h^2l^2 + 8h^2s^2 + 8h^2t^2 + 8h^2u^2 + 8h^2v^2 + 8h^2w^2 + 8h^2\eta^2 + 8h^2\lambda^2 + 2i^2j^2 + 2i^2k^2 + 2i^2l^2 + 2i^2s^2 + 2i^2t^2 + 2i^2u^2 + 2i^2v^2 + 2i^2w^2 + 2i^2\eta^2 + 2i^2\lambda^2 + 2i^2\delta^2 + 2i^2\beta^2 + 2j^2k^2 + 2j^2l^2 + 2j^2s^2 + 2j^2t^2 + 2j^2u^2 + 2j^2v^2 + 2j^2w^2 + 2j^2\eta^2 + 2j^2\lambda^2 + 2j^2\delta^2 + 2j^2\beta^2 + 2k^2l^2 + 2k^2s^2 + 2k^2t^2 + 2k^2u^2 + 2k^2v^2 + 2k^2w^2 + 2k^2\eta^2 + 2k^2\lambda^2 + 2k^2\delta^2 + 2k^2\beta^2 + 2l^2s^2 + 2l^2t^2 + 2l^2u^2 + 2l^2v^2 + 2l^2w^2 + 2l^2\eta^2 + 2l^2\lambda^2 + 2l^2\delta^2 + 2l^2\beta^2 + 2s^2t^2 + 2s^2u^2 + 2s^2v^2 + 2s^2w^2 + 2s^2\eta^2 + 2s^2\lambda^2 + 2s^2\delta^2 + 2s^2\beta^2 + 2t^2s^2 + 2t^2u^2 + 2t^2v^2 + 2t^2w^2 + 2t^2\eta^2 + 2t^2\lambda^2 + 2t^2\delta^2 + 2t^2\beta^2 + 2u^2v^2 + 2u^2w^2 + 2u^2\eta^2 + 2u^2\lambda^2 + 2u^2\delta^2 + 2u^2\beta^2 + 2v^2w^2 + 2v^2\eta^2 + 2v^2\lambda^2 + 2v^2\delta^2 + 2v^2\beta^2 + 2w^2\eta^2 + 2w^2\lambda^2 + 2w^2\delta^2 + 2w^2\beta^2 + 2\eta^2\lambda^2 + 2\eta^2\delta^2 + 2\eta^2\beta^2 + 2\lambda^2\delta^2 + 2\lambda^2\beta^2 + 2\delta^2\beta^2) \]

\[ \text{for } \Gamma \text{ then, it is sufficient toenumerate the rank of } E_{-2p}. \text{ Frist of all, there exists 14 variables(whose degree is 4) in coefficient of } e^4. \text{ The square terms are from } j^4 \text{ to } \lambda^4(\text{containing } 16h^4). \text{ The remanent term which is not a square is } 3i^4. \text{ Next, we ought to observe the arithmetical value } -2M^4 + 3i^4e^4. \text{ And the relation } -2M^4 + 3i^4e^4 = i^4 \text{ should be valid. Henceforth, if two integers } M \text{ and } \epsilon \text{ are selected as } i \text{ and 1 then, we are confronted with } -2i^4 + 3i^4 \text{ and it is matched to our purpose. In the next step, it is necessary to consider other terms. The terms whose absolute value in coefficient is 8 are from } -8h^2i^2 \text{ to } 8h^2\lambda^2 \text{ and the number of these are 13. The symbol of } h^2i^2 \text{ is only negative and that of others are positive. Accordingly, there must exist one term whose coefficient is negative in variables from } h \text{ to } \lambda \text{ in integer } N. \text{ And we confront twelve negative terms from } -2i^2j^2 \text{ to } -2i^2\lambda^2 \text{ in the above. Remaining terms’ coefficients are all } 2 \text{ from} \]
Compositions of primes in elliptic curves $y^2 = x^3 - 2px$,

$2j^2k^2$ to $2\delta^2\lambda^2$). Now, we should determine the components of value $N$. From $16h^4$, take $4h^2$ and from the existence $-8h^2j^2$ choose the value $-\lambda^2$. Furthermore, select the squares $j^2$ and $k^2$ and $l^2$ and $s^2$ and $t^2$ and $u^2$ and $v^2$ and $w^2$ and $\eta^2$ and $\iota^2$ and $\delta^2$ and $\lambda^2$. To sum up, if the value $N$ is gotten as the form $4h^2 - i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + i^2 + \delta^2 + \lambda^2$ then, we attain our aim. Thus, the triple $(i, j, 4h^2 - i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + i^2 + \delta^2 + \lambda^2)$ is produced as the solution of equation 4). On this account, we are confronted with $\#\alpha(\Gamma) = 4$. Wherefore, we get that $r4.2$.

Hence, $\text{rank}(E_{-2(16h^4+3l^4+j^4+k^4+l^4+s^4+\cdots+2\eta^2\lambda^2+2\iota^2\delta^2+2\iota^2\lambda^2+2\delta^2\lambda^2)}(Q)) = 1$ is gotten.

(2) It is enough that we only examine the solvability of equation 4)$N^2 = -2M^4 + (6h^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^2 + \iota^2 + \delta^2 + \lambda^2 - 4h^2j^2 + 4h^2k^2 + 4h^2l^2 + 4h^2s^2 + 4h^2t^2 + 4h^2u^2 + 4h^2v^2 + 4\cdot h^2w^2 + 4h^2\eta^2 + 4h^2\delta^2 + 4h^2\lambda^2 - 2h^2i^2 - 2h^2j^2 - 2h^2k^2 - 2h^2l^2 - 2h^2s^2 - 2h^2u^2 - 2h^2v^2 - 2h^2w^2 - 2h^2\eta^2 - 2h^2\delta^2 - 2h^2\lambda^2 - 2j^2k^2 + 2j^2l^2 + 2j^2s^2 + 2j^2t^2 + 2j^2u^2 + 2j^2v^2 + 2j^2w^2 + 2j^2\eta^2 + 2j^2\delta^2 + 2j^2\lambda^2 + 2k^2\delta^2 + 2k^2\lambda^2 + 2\iota^2s^2 + 2\iota^2t^2 + 2\iota^2u^2 + 2\iota^2v^2 + 2\iota^2w^2 + 2\iota^2\eta^2 + 2\iota^2\delta^2 + 2\iota^2\lambda^2 + 2s^2\delta^2 + 2s^2\lambda^2 + 2t^2\delta^2 + 2t^2\lambda^2 + 2u^2v^2 + 2u^2\eta^2 + 2u^2\delta^2 + 2u^2\lambda^2 + 2v^2w^2 + 2v^2\eta^2 + 2v^2\delta^2 + 2v^2\lambda^2 + 2w^2\delta^2 + 2w^2\lambda^2 + 2\iota^2\delta^2 + 2\iota^2\lambda^2 + 2\delta^2 + 2\delta^2\lambda^2)e^4$ for $\Gamma$ due to [2]. There exist squares from $i^4$ to $\lambda^4$, hence if we can find square form in term for $h^4$ then, we can expect the advent of polynomial’s square in resultant. In numerical value $-2M^4 + 6h^4 e^4$ we choose $M$ and $e$ as $h$ and 1 then, we have $-2h^4 + 6h^4$. It is the thing we pursued. Moreover, there are 13 terms from $-4h^2i^2$ to $4h^2\lambda^2$. And there also exist 12 terms from $-2i^2j^2$ to $-2i^2s^2$. Until the term $2\delta^2\lambda^2$ the number of terms are decreased one by one. All these are matched to the previous 13 squares(in the last containing $4h^2$) if we guess the emergence of square in resultant(the components are variables $h, i, j, k, l, s, t, u, v, w, \eta, \iota, \delta, \lambda$). Thereby, from substituting the two values $h$ and 1 into $M$ and $e$ respectively we can attain the square. On this account, the value $N$ is gotten as $N = 2h^2 - i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + \iota^2 + \delta^2 + \lambda^2$. For that reason, the triple $(h, 1, 2h^2 - i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + \iota^2 + \delta^2 + \lambda^2)$ is educed as the solution of relating equation 4). So we take that $\#\alpha(\Gamma) = 4$. For this reason, there is induced the result $\text{rank}(E_{-2(6h^4+i^4+j^4+k^4+l^4+s^4+\cdots+2i^2\delta^2+2\iota^2\lambda^2+2\delta^2\lambda^2)}(Q)) = 1$ because of r4.2.

(3) We only investigate whether there is a solution in 4)$N^2 = -2M^4 + (100h^4 + 11i^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^2 + \iota^2 + \delta^2 + \lambda^2 + 60h^2i^2 - 20h^2j^2 + 20h^2k^2 + 20h^2l^2 + 20h^2s^2 + 20h^2t^2 + 20h^2u^2 + 20h^2v^2 + 20h^2w^2 + 20h^2\eta^2 + 20h^2\delta^2 + 20h^2\lambda^2 - 6i^2j^2 + 6i^2k^2 + 6i^2l^2 + 6i^2s^2 + 6i^2t^2 + 6i^2u^2 + 6i^2v^2 + 6i^2w^2 + 6i^2\eta^2 + 6i^2\delta^2 + 6i^2\lambda^2 - 2i^2j^2 + 2i^2k^2 + 2i^2l^2 + 2i^2s^2 + 2i^2t^2 + 2i^2u^2 + 2i^2v^2 + 2i^2w^2 + 2i^2\eta^2 + 2i^2\delta^2 + 2i^2\lambda^2 - 2j^2\delta^2 - 2j^2\lambda^2 - 2\iota^2\delta^2 - 2\iota^2\lambda^2 - 2\delta^2 - 2\delta^2\lambda^2)e^4$.


\( j^2i^2 - 2j^2 \delta^2 - 2j^2 \lambda^2 + 2k^2l^2 + 2k^2s^2 + 2k^2t^2 + 2k^2u^2 + 2k^2v^2 + 2k^2w^2 + 2k^2\eta^2 + 2k^2\epsilon^2 + 2k^2\delta^2 + 2k^2\lambda^2 + 2l^2s^2 + 2l^2t^2 + 2l^2u^2 + 2l^2v^2 + 2l^2w^2 + 2l^2\eta^2 + 2l^2\epsilon^2 + 2l^2\delta^2 + 2l^2\lambda^2 + 2s^2t^2 + 2s^2u^2 + 2s^2v^2 + 2s^2w^2 + 2s^2\eta^2 + 2s^2\epsilon^2 + 2s^2\delta^2 + 2s^2\lambda^2 + 2t^2u^2 + 2t^2v^2 + 2t^2w^2 + 2t^2\eta^2 + 2t^2\epsilon^2 + 2t^2\delta^2 + 2t^2\lambda^2 + 2u^2v^2 + 2u^2w^2 + 2u^2\eta^2 + 2u^2\epsilon^2 + 2u^2\delta^2 + 2u^2\lambda^2 + 2v^2w^2 + 2v^2\eta^2 + 2v^2\epsilon^2 + 2v^2\delta^2 + 2v^2\lambda^2 + 2w^2\eta^2 + 2w^2\epsilon^2 + 2w^2\delta^2 + 2w^2\lambda^2 + 2\eta^2\delta^2 + 2\eta^2\lambda^2 + 2\lambda^2 \delta^2 + 2\delta^2 \lambda^2 \) for \( \Gamma \) or not due to [2]. The term \( 11i^4 \) coefficient of \( e^4 \) is not a square. From arithmetical value \(-2M^4 + 11i^4 e^4 \) if two integers \( M \) and \( e \) are chosen as \( i \) and \( l \) then, we attain that \(-2i^4 + 11i^4 \) and it is a square \( 9i^4 \). If it is proper that we select these two values then, it needs to check other part. First of all, there are thirteen terms from \( 60h^2l^2 \) to \( 20h^2\lambda^2 \). If the terms \( 10h^2, 3i^2, -j^2, k^2, l^2, s^2, t^2, u^2, v^2, w^2, \eta^2, \epsilon^2, \delta^2, \lambda^2 \) are assigned as the component of value \( N \) then, above squares can be educed. Moreover, remaining terms from \(-6i^2j^2 \) to \( 2\delta^2 \lambda^2 \) also can be induced. Eventually, the solution of relating equation 4) is gotten as the triple \((l, 10h^2 + 3i^2 - j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + l^2 + \delta^2 + \lambda^2) \). Henceforth, we see that \#\( \alpha(\Gamma) \) = 4. Therefore, \( \text{rank}(E_{-2(100h^4 + 11i^4 + j^4 + k^4 + l^4 + s^4 + \ldots + 2i^2\lambda^2 + 2\delta^2 \lambda^2)}(Q)) = 1 \) is gotten from \( r4.2 \).

(4). Because we confront the result in [2], there is remained only the relating equation \( 4) N^2 = -2M^4 + (16h^4 + 3i^4 + j^4 + k^4 + l^4 + s^4 + t^4 + u^4 + v^4 + w^4 + \eta^4 + \epsilon^4 + \delta^4 + \lambda^4 + 8h^2i^2 + 8h^2j^2 + 8h^2k^2 + 8h^2l^2 + 8h^2s^2 + 8h^2t^2 + 8h^2u^2 + 8h^2v^2 + 8h^2w^2 + 8h^2\eta^2 + 8h^2\epsilon^2 + 8h^2\delta^2 + 8h^2\lambda^2 + 2i^2j^2 + 2i^2k^2 + 2i^2l^2 + 2i^2s^2 + 2i^2t^2 + 2i^2u^2 + 2i^2v^2 + 2i^2w^2 + 2i^2\eta^2 + 2i^2\epsilon^2 + 2i^2\delta^2 + 2i^2\lambda^2 + 2j^2k^2 + 2j^2l^2 + 2j^2s^2 + 2j^2t^2 + 2j^2u^2 + 2j^2v^2 + 2j^2w^2 + 2j^2\eta^2 + 2j^2\epsilon^2 + 2j^2\delta^2 + 2j^2\lambda^2 + 2k^2l^2 + 2k^2s^2 + 2k^2t^2 + 2k^2u^2 + 2k^2v^2 + 2k^2w^2 + 2k^2\eta^2 + 2k^2\epsilon^2 + 2k^2\delta^2 + 2k^2\lambda^2 + 2l^2s^2 + 2l^2t^2 + 2l^2u^2 + 2l^2v^2 + 2l^2w^2 + 2l^2\eta^2 + 2l^2\epsilon^2 + 2l^2\delta^2 + 2l^2\lambda^2 + 2s^2t^2 + 2s^2u^2 + 2s^2v^2 + 2s^2w^2 + 2s^2\eta^2 + 2s^2\epsilon^2 + 2s^2\delta^2 + 2s^2\lambda^2 + 2t^2u^2 + 2t^2v^2 + 2t^2w^2 + 2t^2\eta^2 + 2t^2\epsilon^2 + 2t^2\delta^2 + 2t^2\lambda^2 + 2u^2v^2 + 2u^2w^2 + 2u^2\eta^2 + 2u^2\epsilon^2 + 2u^2\delta^2 + 2u^2\lambda^2 + 2v^2w^2 + 2v^2\eta^2 + 2v^2\epsilon^2 + 2v^2\delta^2 + 2v^2\lambda^2 + 2w^2\eta^2 + 2w^2\epsilon^2 + 2w^2\delta^2 + 2w^2\lambda^2 + 2\eta^2\delta^2 + 2\eta^2\lambda^2 + 2\lambda^2 \delta^2 + 2\delta^2 \lambda^2 \) for \( \Gamma \) that must be treated the solvability. Only \( 3i^4 \) is not a square form between the terms(whose degree is 4). In numerical value \(-2M^4 + 3i^4 e^4 \), if we take \( M \) and \( e \) as \( i \) and \( l \) then, we gain \(-2i^4 + 3i^4 \) and this is a square. There exist 13 terms from \( 8h^2i^2 \) to \( 8h^2\lambda^2 \). If the terms \( 4h^2, i^2, j^2, k^2, l^2, s^2, t^2, u^2, v^2, w^2, \eta^2, \epsilon^2, \delta^2, \lambda^2 \) are supposed as the component of value \( N \) then, above squares can be derived. Furthermore, if we determine a bundle in each case from \( 2i^2j^2 \) to \( 2i^2\lambda^2 \), from \( 2j^2k^2 \) to \( 2j^2\lambda^2 \), from \( 2k^2l^2 \) to \( 2k^2\lambda^2 \), from \( 2l^2s^2 \) to \( 2l^2\lambda^2 \), from \( 2s^2t^2 \) to \( 2s^2\lambda^2 \), from \( 2t^2u^2 \) to \( 2t^2\lambda^2 \), from \( 2u^2v^2 \) to \( 2u^2\lambda^2 \), from \( 2v^2w^2 \) to \( 2v^2\lambda^2 \), from \( 2w^2\eta^2 \) to \( 2w^2\lambda^2 \), from \( 2\eta^2\delta^2 \) to \( 2\eta^2\lambda^2 \), from \( 2\delta^2 \lambda^2 \) and \( 2\delta^2 \lambda^2 \) then, the condition for being appeared the square form is completed. Resultantly, the integer \( N \) is induced as \( 4h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + i^2 + \delta^2 + \lambda^2 \). Henceforth, the solution of equation 4) is given as the triple \((i, 1, 4h^2 + i^2 + j^2 + k^2 + l^2 + s^2 + t^2 + u^2 + v^2 + w^2 + \eta^2 + i^2 + \delta^2 + \lambda^2) \). As a result, we take that \#\( \alpha(\Gamma) \) = 4.
Accordingly, it is gotten that $r4.2$ and it follows the consequence
\[
\text{rank}(E_{−2}(16h^4+3t^4+j^4+k^4+t^4+...+2t^2\delta^2+2t^2\lambda^2))(Q)) = 1.
\]

Above primes $p$ were composed of 14 variables and 105 terms\cdots\ (F). The components of $p$ were all terms for square. With maintaining the square (the degree of 4) there is change in symbols of terms for $(\ )^2$. We can also obtain primes $p$ whose numbers of variables and terms are more than above\cdots\ (G). For attaining rank 1 in $E_{−2p}$: $y^2 = x^3 − 2px$, $p$ should be $p \equiv 3, 11 (mod 16)$. In specially, in forms $p \equiv 3, 11 (mod 16)$ the primes $p$ of (F) are appeared often. It is also same in primes $p$ of (G). These are tendency of computation. It doesn’t mean (F), (G) are emerged only in forms $p \equiv 3, 11 (mod 16)$. There is remained the possibility in $p \equiv 5, 13 (mod 16)$. In these primes the forms of (F) and (G) can be induced but compared with $p \equiv 3, 11 (mod 16)$ it is more difficult. From above results, calculating rank of elliptic curve $E_{−2p}$: $y^2 = x^3 − 2px$ became the treatment related to complex polynomial.

**Remark 2.2.** In curves $E_{−p}$ and $E_{−4p}$ we also can search the systematized result of rank 1 where prime $p$ is composed of many variables.

**Remark 2.3.** In [4], Mecklenburg surveyed the rank of $E$: $y^2 = x^3 + c$. The homomorphisms $\alpha$ and $\alpha'$ were defined as $\alpha$: $E(Q) → Q' / Q(\sqrt{c})^2$ and $\alpha'$: $E'(Q) → Q' / Q(\sqrt{c})^2$ and $\alpha$, $\alpha' = \sqrt{c}$ and there were given that $|im \alpha| \cdot |im \alpha' | = 3^{r+1}$ if $c$ or $−27c$ is a perfect square and $|im \alpha| \cdot |im \alpha' | = 3^r$ if neither $c$ nor $−27c$ are perfect squares([4]). It is well-known that two homomorphisms $\alpha$ and $\bar{\alpha}$ and formula $2^r = \frac{\bar{\alpha}(\Gamma)\bar{\alpha}(\Gamma)}{\alpha}$ were applied to the form $y^2 = x^3 + ax^2 + bx$ ([5]). But in reality, these are used in calculating the rank of curve of the form $y^2 = x^3 + Ax$ (Above theorems are case of this curve.). In form of the curve $y^2 = x^3 + c$ there is left the possibility that the formulas $|im \alpha| \cdot |im \alpha'| = 3^{r+1}$ and $|im \alpha| \cdot |im \alpha'| = 3^r$ induce various results of generalized rank.

## 3 Examples

In section 3, we will submit examples of previous theorem. From [1], the primality can be examined.

Several examples from theorem 2.1(1) to (4) are derived as follows:
There deduced two examples in each case. Since the primes $p$ in theorem 2.1 are composed of various terms finding examples is difficult compared with the prime of the form $p = Hu^4 + Iu^2v^2 + Kv^4$.

References


Received: April 11, 2023; Published: May 22, 2023