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Star Modules and Equivalences

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Abstract

We study characterizations of s-Σ-quasi-projective modules and properties of special star modules (tilting modules and Σ-quasi-projective self-generators) and characterize Σ-self-static star modules in terms of equivalences.

Mathematics Subject Classification: 16D90

Keywords: star module; Σ-self-static; s-Σ-quasi-projective modules

1 Introduction

The notion of classical star modules (i.e., $*$-modules) arises from the paper [9] of Menini and Orsatti, where the following result was proved: let R and S be rings, let $\mathcal{C} \subseteq R$-Mod closed under direct sums and factors and let $\mathcal{D} \subseteq S$-Mod closed under submodules and containing S, then any equivalence between $\mathcal{C}$ and $\mathcal{D}$ is represented by an R-module P with $S = \text{End}_R P$ via a pair of functors $H_P = \text{Hom}_R(P, -)$ and $T_P = P_S \otimes -$. Such modules are then called $*$-modules. Moreover, in this case $\mathcal{C} = \text{Gen}(P)$ and $\mathcal{D} = \text{Cogen}(P^*)$ (see the following definitions). Classical star modules give generalizations of both Fuller's quasi-progenerators [7] and classical tilting modules [3]. In fact, quasi-progenerators (i.e. finitely generated quasi-projective self-generators) are just classical star modules which generate all of its submodules [2] while classical tilting modules are just classical star modules which generate all the injectives [3].

In [14, 16], infinitely generated star modules (i.e., 1-star modules in [14] and self-tilting modules in [16] respectively) was studied. Many homological characterizations of classical star modules were extended to the general settings, except the characterizations using equivalences.

It is easy to see that a star module P which is not selfsmall can not represent an equivalence between $\mathrm{Gen}(P)$ and $\mathrm{Cogen}(P^*)$. However, there still exists an intimate relation between equivalences and star modules which are Σ-self-static (it's a question whether all star modules are Σ-self-static). Namely, let $\mathcal{C} \subseteq R$-Mod closed under direct sums and factors and let $\mathcal{D} \subseteq S$-Mod closed under kernels and images of homomorphisms and containing S. Then there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$ if and only if there exists a Σ-self-static star module ${}_RP$ such that $\mathcal{C} = \mathrm{Gen}(P)$ and $\mathcal{D} = \mathrm{Copres}(P^*)$ (Theorem 3.7). Σ-self-static tilting modules are also characterized in term of equivalences, as well as Σ-quasi-projective self-generators (Theorem 3.7).

We also give characterizations of tilting modules over commutative domain (Proposition 2.9), s-Σ-quasi-projective modules (Proposition 2.4) and Σ-quasi-projective self-generators (Proposition 2.11). It was also shown that every infinitely generated tilting module is faithfully balanced (Lemma 2.10).

Throughout this paper, R will be an associated ring with nonzero identity and all modules will be unitary and mean left modules without explicit mentions. Denote by R-Mod the class of left R-modules. If $f : X \to Y$ and $g : Y \to Z$ are homomorphisms, we denote by fg the composition of f and g.

From now on, we fix $P \in R$-Mod with $S = \mathrm{End}_R P$ and let $P^* = \mathrm{Hom}_R(P, Q)$, where Q is an injective cogenerator in R-Mod. Note that P is also a right S-module and that $P^* \in S$-Mod.

We denote that $P^{\perp_1} := \{M \in R\text{-Mod} \mid \mathrm{Ext}^1_R(P, M) = 0\}$. Also we denote by $\mathrm{Add}P$ ($\mathrm{add}P$) the class of direct summands of (finite) direct sums of copies of P.

$M \in R$-Mod is presented (respectively generated) by P if there exists an exact sequence $P_2 \to P_1 \to M \to 0$ (respectively $P_1 \to M \to 0$) with $P_i \in \mathrm{Add}P$. Denote by $\mathrm{Pres}(P)$ (respectively $\mathrm{Gen}(P)$) the class of modules presented (respectively generated) by P. Dually, assuming that $K \in S$-Mod, we denote by $\mathrm{Adp}K$ the class of direct summands of direct products of copies of K. $N \in S$-Mod is copresented (respectively cogenerated) by K if there exists an exact sequence $0 \to N \to K_1 \to K_2$ (respectively $0 \to N \to K_1$) with $K_i \in \mathrm{Adp}K$. Denote by $\mathrm{Copres}(K)$ (respectively $\mathrm{Cogen}(K)$) the class of modules copresented (respectively cogenerated) by K.

P is Σ-quasi-projective (respectively w-Σ-quasi-projective [2], s-Σ-quasi-projective [11]) if the functor $\mathrm{Hom}_R(P, -)$ preserves the exactness of any exact sequence $P_1 \to M \to 0$ (respectively $0 \to M_1 \to P_1 \to M \to 0$ with $M_1 \in \mathrm{Gen}(P)$, $P_2 \to P_1 \to M \to 0$) with $P_i \in \mathrm{Add}P$. Clearly, P is Σ-quasi-projective implies that P is s-Σ-quasi-projective, and the later implies that P is w-Σ-quasi-projective.

Denote that $H_P = \mathrm{Hom}_R(P, -)$, $T_P = P \otimes_S -$. Then (T_P, H_P) is a pair of adjoint functors and there are the following canonical homomorphisms for any $M \in R$-Mod and any $N \in S$-Mod:

$$\rho_M : T_P H_P M \to M \text{ by } p \otimes f \to f(p) ;$$
$$\sigma_N : N \to H_P T_P N \text{ by } n \to [p \to p \otimes n] .$$

It is well known (see for instance [2]) that for any $N \in S$-Mod σ_N is injective if and only if $N \in \mathrm{Cogen}(P^*)$ and that for any $M \in R$-Mod ρ_M is surjective if and only if $M \in \mathrm{Gen}(P)$. Note that $H_P M \in \mathrm{Copres}(P^*)$ for any $M \in R$-Mod and that $T_P N \in \mathrm{Pres}(P)$ for any $N \in S$-Mod (see for instance [2]).

Following [15], we say that P is Σ-self-static provided that $\rho_{P^{(X)}}$ is an isomorphism for any set X. Recall that P is selfsmall if, for any set X, the canonical homomorphism $\mathrm{Hom}_R(P, P^{(X)}) \simeq \mathrm{Hom}_R(P, P)^{(X)}$ is an isomorphism. Clearly selfsmall modules are Σ-self-static. The converse fails in general (Example 3.9).

We denote by $\mathrm{Subgen}(P)$ the class of submodules of modules in $\mathrm{Gen}(P)$.

2 Star modules

We begin with the following useful lemma which is also known as the generalized Schanuel's Lemma.

Lemma 2.1 *Let* $0 \to K_1 \to P_1 \to M \to 0$ *and* $0 \to K_2 \to P_2 \to M \to 0$ *be exact with* $P_1, P_2 \in \mathrm{Ad}P$. *If the induced sequences* $0 \to H_P K_1 \to H_P P_1 \to H_P M \to 0$ *and* $0 \to H_P K_2 \to H_P P_2 \to H_P M \to 0$ *are also exact, then* $K_1 \oplus P_2 \simeq K_2 \oplus P_1$.

Proof. By diagram chasing, one can prove that $K_1 \oplus P_2 \simeq T \simeq K_2 \oplus P_1$ with T a fibered product of $P_1 \to M$ and $P_2 \to M$.

Lemma 2.2 *The following are equivalent for an R-module P:*

(1) P is w-Σ-quasi-projective .

(2) For any exact sequence $0 \to K \to L \to N \to 0$ *with* $K \in \mathrm{Gen}(P)$ *and* $L \in \mathrm{Pres}(P)$, *the induced sequence* $0 \to H_P K \to H_P L \to H_P N \to 0$ *is also exact.*

(3) For any exact sequence $0 \to K \to P' \to N \to 0$ *with* $P' \in \mathrm{Add}P$ *and* $N \in \mathrm{Pres}(P)$, $K \in \mathrm{Gen}(P)$ *if and only if the induced sequence* $0 \to H_P K \to H_P P' \to H_P N \to 0$ *is exact.*

(4) For any exact sequence $0 \to K \to L \to N \to 0$ *with* $L, N \in \mathrm{Pres}(P)$, $K \in \mathrm{Gen}(P)$ *if and only if the induced sequence* $0 \to H_P K \to H_P L \to H_P N \to 0$ *is exact.*

Proof. (1) $\Leftrightarrow$ (2). By [16, 3.2(1)].

(4) $\Rightarrow$ (3) $\Rightarrow$ (1). Obviously.

(1) $\Rightarrow$ (3). Clearly we need only to show that if the induced sequence $0 \to H_P K \to H_P P' \to H_P N \to 0$ is exact then $K \in \mathrm{Gen}(P)$.

Since $N \in \mathrm{Pres}(P)$, we have an exact sequence $0 \to N' \to P_N \to N \to 0$ with $P_N \in \mathrm{Add}P$ and $N' \in \mathrm{Gen}(P)$. By assumptions, P is w-Σ-quasi-projective, so the induced sequence $0 \to H_P N' \to H_P P_N \to H_P N \to 0$ is also exact. It follows that $N' \oplus P' \simeq K \oplus P_N$ by Lemma 2.1. Hence $K \in \mathrm{Gen}(P)$.

(3) $\Rightarrow$ (4). Similarly we need to show that if the induced sequence $0 \to H_P K \to H_P L \to H_P N \to 0$ is exact then $K \in \mathrm{Gen}(P)$.

Since $L \in \mathrm{Pres}(P)$, we have an exact sequence $0 \to L' \to P_L \to L \to 0$ with $P_L \in \mathrm{Add}P$ and $L' \in \mathrm{Gen}(P)$. By assumptions the induced sequence $0 \to H_P L' \to H_P P_L \to H_P L \to 0$ is also exact. Note that we can construct the following exact commutative diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L' & = & L' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M & \to & P_L & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & K & \to & L & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By applying the functor H_P to the diagram, we observe that the induced sequence $0 \to H_P M \to H_P P_L \to H_P N \to 0$ is exact by arguments above. It follows that $M \in \mathrm{Gen}(P)$ by assumptions. Hence $K \in \mathrm{Gen}(P)$, too.

Corollary 2.3 *Assume the R-module P is w-Σ-quasi-projective. Then the functor* H_P *preserves short exact sequences in* $\mathrm{Pres}(P)$.

We now give some characterizations of s-Σ-quasi-projective modules.

Proposition 2.4 *The following are equivalent for an R-module P:*

(1) P is s-Σ-quasi-projective.

(2) For any exact sequence $P' \to L \to N \to 0$ *with* $P' \in \mathrm{Add}P$ *and* $L \in \mathrm{Pres}(P)$, *the induced sequence* $H_P P' \to H_P L \to H_P N \to 0$ *is also exact.*

(3) For any exact sequence $K \to L \to N \to 0$ *with* $K, L \in \mathrm{Pres}(P)$, *the induced sequence* $H_P K \to H_P L \to H_P N \to 0$ *is also exact.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Obviously.

(1) $\Rightarrow$ (2). Let $X = \mathrm{Ker}(L \to N)$ and $Y = \mathrm{Ker}(P' \to L)$. Then we have two exact sequence $0 \to Y \to P' \to^{\pi_X} X \to 0$ and $0 \to X \to L \to N \to 0$ with $X \in \mathrm{Gen}(P)$. Since $L \in \mathrm{Pres}(P)$, we have an exact sequence $0 \to L' \to P_L \to L \to 0$ with $P_L \in \mathrm{Add}P$ and $L' \in \mathrm{Gen}(P)$. By assumptions and Lemma 2.2, we obtain the induced sequences $0 \to H_PL' \to H_PP_L \to H_PL \to 0$ and $0 \to H_PX \to H_PL \to H_PN \to 0$ are exact. We now show that the induced sequence $0 \to H_PY \to H_PP' \to^{H_P\pi_X} H_PX \to 0$ is also exact and then the conclusion follows.

As in the proof of Lemma 2.2, we have the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L' & = & L' & & & & \\
 & & \downarrow i & & \downarrow & & & & \\
0 & \to & M & \to & P_L & \to & N & \to & 0 \\
 & & \downarrow \pi & & \downarrow & & \| & & \\
0 & \to & X & \to & L & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By applying the functor H_P to the diagram, we observe that the induced sequences $0 \to H_PM \to H_PP_L \to H_PN \to 0$ and $0 \to H_PL' \to H_PM \to H_PX \to 0$ are exact by above arguments and the Snake Lemma. Since $L' \in \mathrm{Gen}(P)$, we also have an exact sequence $0 \to L'' \to P_{L'} \to^{\pi_{L'}} L' \to 0$. Then we can construct the following diagram, where $\theta_M : P' \to M$ is such that $\theta_M\pi = \pi_X$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L'' & \to & M' & \to & Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_{L'} & \xrightarrow{(1,0)} & P_{L'} \oplus P' & \xrightarrow{\binom{0}{1}} & P' & \to & 0 \\
 & & \pi_{L'}\downarrow & & \binom{\pi_{L'}i}{\theta_M}\downarrow & & \downarrow \pi_X & & \\
0 & \to & L' & \xrightarrow{i} & M & \xrightarrow{\pi} & X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It is easy to check that the above diagram is exact commutative. Note that the middle column is exact after applying the functor H_P, since $M \to P_L$ is monomorphism and P is s-Σ-quasi-projective. Hence, combining arguments above, we obtain that the induced sequence $0 \to H_P Y \to H_P P' \to H_P X \to 0$ is also exact by applying the functor H_P to the diagram.

(2) $\Rightarrow$ (3). Let $X = \mathrm{Ker}(L \to N)$ and $Y = \mathrm{Ker}(K \to L)$. Then we have two exact sequence $0 \to Y \to K \to X \to 0$ and $0 \to X \to L \to N \to 0$ with $X \in \mathrm{Gen}(P)$. Since $K \in \mathrm{Pres}(P)$, we have an exact sequence $0 \to K' \to P_K \to K \to 0$ with $P_K \in \mathrm{Add}P$ and $K' \in \mathrm{Gen}(P)$. By assumptions, we easily obtain that the induced sequence $0 \to H_P K' \to H_P P_K \to H_P K \to 0$ is exact. Also we have the following exact commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K' & = & K' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M & \to & P_K & \to & X & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & Y & \to & K & \to & X & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By assumptions we have that the induced sequence $0 \to H_P M \to H_P P_K \to H_P X \to 0$ is exact since $X \hookrightarrow L$. By applying the functor H_P to the diagram we obtain that the induced sequence $0 \to H_P Y \to H_P K \to H_P X \to 0$ is exact. It follows that the induced sequence $H_P K \to H_P L \to H_P N \to 0$ is exact.

The following characterizations of star modules in [14] is often used.

Proposition 2.5 *The following are equivalent for an R-module P.*

(1) P is a star module, i.e., $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ *and P is w-Σ-quasi-projective.*

(2) For any $K \leq P^{(X)}$ with X a set, the canonical homomorphism $\mathrm{Ext}^1_R(P, K) \to \mathrm{Ext}^1_R(P, P^{(X)})$ *is injective if and only if* $K \in \mathrm{Gen}(P)$.

(3) For any $K \leq L$ with $L \in \mathrm{Gen}(P)$*, the canonical homomorphism* $\mathrm{Ext}^1_R(P, K)$
$\to \mathrm{Ext}^1_R(P, L)$ *is injective if and only if* $K \in \mathrm{Gen}(P)$.

Lemma 2.6 *Let P be an R-module. Consider the following statements.*

(1) $\mathrm{Gen}(P) \subseteq P^{\perp_1}$.

(2) P is w-Σ-quasi-projective and $\mathrm{Pres}(P)$ *is closed under extensions.*

(3) $\mathrm{Pres}(P) \subseteq P^{\perp_1}$.

Then, for any P, (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). For any exact sequence $0 \to K \to P_N \to N \to 0$ with $P_N \in \mathrm{Add}P$ and $K \in \mathrm{Gen}(P)$, we have that the induced sequence $0 \to H_PK \to H_PP_N \to H_PN \to 0$ is exact since $\mathrm{Ext}^1_R(P, K) = 0$. Hence P is w-Σ-quasi-projective.

To see that $\mathrm{Pres}(P)$ is closed under extensions, we first show that $\mathrm{Gen}(P)$ is closed under extensions. Consider now any exact sequence $0 \to K \to L \to N \to 0$ with $K, N \in \mathrm{Gen}(P)$. Let then $0 \to N' \to P_N \to N \to 0$ for some $P_N \in \mathrm{Add}P$, we obtain the following pullback diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N' & = & N' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & X & \to & P_N & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & L & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By assumptions we easily have that the middle row splits. It follows that $X \simeq P_N \oplus K \in \mathrm{Gen}(P)$. Therefore $L \in \mathrm{Gen}(P)$ too.

Now consider any exact sequence $0 \to U \to V \to W \to 0$ with $U, W \in \mathrm{Pres}(P)$. Then we have two exact sequences $0 \to U' \to P_U \to U \to 0$ and $0 \to W' \to P_W \to W \to 0$ with $P_U, P_W \in \mathrm{Add}P$ and $U', W' \in \mathrm{Gen}(P)$. As in the proof of Lemma 2.4, we can construct the following exact commutative diagram since $U \in P^{\perp_1}$ by assumptions.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U' & \to & V' & \to & W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_U & \to & P_U \oplus P_W & \to & P_W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U & \to & V & \to & W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Gen}(P)$ is closed under extensions, we have that $V' \in \mathrm{Gen}(P)$. Hence $V \in \mathrm{Pres}(P)$ and $\mathrm{Pres}(P)$ is closed under extensions.

(2) $\Rightarrow$ (3). Take any $K \in \mathrm{Pres}(P)$ and any extension $0 \to K \to L \to P \to 0$ of P by K. Then $L \in \mathrm{Pres}(P)$. Since P is w-Σ-quasi-projective, the induced sequence $0 \to H_PK \to H_PL \to H_PP \to 0$ is exact by Corollary 2.3. Hence

the extension splits and $K \in P^{\perp_1}$.

Following [13], we call an R-module P quasi-tilting if P is a star module with $\mathrm{Gen}(P) \subseteq P^{\perp_1}$. Lemma 2.6 suggests the following characterizations of quasi-tilting modules which also extends [4, Proposition 2.1].

Corollary 2.7 *Let P be an R-module. The following are equivalent.*

(1) P is quasi-tilting.

(2) $\mathrm{Gen}(P) = \mathrm{Pres}(P) \subseteq P^{\perp_1}$.

(3) P is a star module and $\mathrm{Gen}(P)$ *is closed under extensions.*

(4) $\mathrm{Subgen}P \cap P^{\perp_1} = \mathrm{Gen}(P)$.

Proof. By Lemma 2.6, we need only to prove that (1) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (4). Clearly $\mathrm{Gen}(P) \subseteq \mathrm{Subgen}P \cap P^{\perp_1}$ by definitions. On the other hand, for any $M \in \mathrm{Subgen}P \cap P^{\perp_1}$, we have an exact sequence $0 \to M \to M' \to M'' \to 0$ with $M' \in \mathrm{Gen}(P)$ and that the functor H_P preserves the exactness. Since P is a star module, we immediately obtain that $M \in \mathrm{Gen}(P)$ by applying Proposition 2.5. Therefore, we also have that $\mathrm{Subgen}P \cap P^{\perp_1} \subseteq \mathrm{Gen}(P)$.

(4) $\Rightarrow$ (1). It is easy to see that, for any $K \leq L$ with $L \in \mathrm{Gen}(P)$, we obtain that the canonical homomorphism $\mathrm{Ext}^1_R(P, K) \to \mathrm{Ext}^1_R(P, L)$ is injective if and only if $K \in P^{\perp_1}$ by assumptions. Since $K \in \mathrm{Subgen}P$ obviously, we have that $K \in P^{\perp_1}$ if and only if $K \in \mathrm{Subgen}P \cap P^{\perp_1} = \mathrm{Gen}(P)$. Now the conclusion follows from Proposition 2.5 and the definition of quasi-tilting modules.

Recall that an R-module P is (classical) tilting provided that its (finitely generated) projective dimension ≤ 1, $\mathrm{Ext}^1_R(P, P^{(X)}) = 0$ for any set X and there is an exact sequence $0 \to R \to P_1 \to P_2 \to 0$ with $P_1, P_2 \in \mathrm{Add}P$ ($\mathrm{add}P$) [6]. By results in [6,14], we have the following characterizations of tilting modules.

Proposition 2.8 *The following are equivalent for an R-module P.*

(1) P is tilting.

(2) P is a star module and $E(R) \in$ $\mathrm{Gen}(P)$*.*

(3) $\mathrm{Gen}(P) = P^{\perp_1}$.

By [5], all classical tilting modules over a commutative ring are progenerators. This fails in the general case, that is, infinitely generated tilting modules over a commutative ring need not be projective generators, see for instance [16]. However, another result ([5, Proposition 2.1]) about tilting modules over commutative domains holds in the general case, as we will show in the following.

Proposition 2.9 *Let R be a commutative domain and P be an R-module. Then P is tilting if and only if P is a faithful star module.*

Proof. The necessary part follows from Proposition 2.8. The sufficient part may be proved by the way similar to the proof of [5, Proposition 2.1]. For reader's convenience we present here the skeleton.

By Proposition 2.8, it is enough to prove that $Q = Q(R) \in \text{Gen}(P)$. Let $S = \text{End}_R P$. Note that the natural inclusion $R \hookrightarrow S$ (since P is faithful) gives the inclusion $Q = R \otimes_R Q \hookrightarrow S \otimes_R Q$ as Q is flat as an R-module, so $S \otimes_R Q \neq 0$. Then $0 \neq T_P(S \otimes_R Q) = P \otimes_R Q \in \text{Gen}(P)$. Since there is an epimorphism of Q-vector-spaces: $P \otimes_R Q \to Q$, we finally obtain that $Q \in \text{Gen}(P)$. Now the conclusion follows from Proposition 2.8.

It is well known that classical tilting modules are faithfully balanced. The following lemma shows that infinitely generated tilting modules are also faithfully balanced.

Lemma 2.10 *Assume that P is a tilting R-module. Then P is faithfully balanced.*

Proof. By [6, Proposition 2.17], there is a cardinal X such that the R-module $P^{(X)}$ is faithfully balanced. Hence we have that $R \simeq \text{BiEnd}(P^{(X)}) \simeq \text{BiEnd(P)}$ canonically (see [1, 14.2]), i.e., P is faithfully balanced.

The following result characterizes Σ-quasi-projective self-generators.

Proposition 2.11 *The following are equivalent for an R-module P.*

(1) P is a Σ-quasi-projective (Σ-)self-generator.

(2) P is a Σ-quasi-projective star module.

(3) P is a star module and a self-generator.

(4) P is a star module and a Σ-self-generator.

(5) P is a star module and $\text{Gen}(P) = \text{Subgen}(P)$.

(6) P is a s-Σ-quasi-projective star module.

Proof. (1) $\Rightarrow$ (5). $\text{Gen}(P) = \text{Subgen}(P)$ follows from [7, Lemma 2.2]. It is easy to check that $\text{Gen}(P) = \text{Pres}(P)$. Since P is clearly w-Σ-quasi-projective, P is then a star module.

(5) $\Rightarrow$ (4) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). Consider any exact sequence $0 \to M' \to P \to M \to 0$. Since P is a self-generator, we obtain that $M' \in \text{Gen}(P)$. Hence the induced sequence $0 \to H_P M' \to H_P P \to H_P M \to 0$ is exact since P is w-Σ-quasi-projective by assumptions. It follows that P is quasi-projective. Then P is a self-generator is equivalent to say that P is a Σ-self-generator by [17].

(4) $\Rightarrow$ (2). By a way similar as in the proof of (3) $\Rightarrow$ (4), we easily obtain that P is Σ-quasi-projective.

(2) $\Rightarrow$ (6). Obviously.

(6) $\Rightarrow$ (1). Consider any exact sequence $0 \to M' \to P_M \to M \to 0$ with $P_M \in \mathrm{Add}P$. Since P is a star module, we have $M \in \mathrm{Pres}(P)$. Now by applying Proposition 2.4 to the exact sequence $P_M \to M \to 0 \to 0$, we obtain that the induced sequence $H_P P_M \to H_P M \to 0$ is exact since P is s-Σ-quasi-projective. It follows that the induced sequence $0 \to H_P M' \to H_P P_M \to H_P M \to 0$ is exact. Hence P is Σ-quasi-projective. Moreover, $M' \in \mathrm{Gen}(P)$ by Proposition 2.5. Hence P is also a Σ-self-generator.

3 Equivalences

Let R, S be rings. In this section we fix the following notions.

(1) $P^* := H_P Q$, where Q is an injective cogenerator in R-Mod.

(2) $\mathcal{C} \subseteq R$-Mod and $\mathcal{C}$ is closed under direct sums and cokernels of homomorphisms (in $\mathcal{C}$).

(3) $\mathcal{D} \subseteq S$-Mod and $\mathcal{D}$ is closed under kernels and images of homomorphisms (in $\mathcal{D}$) and contains S.

Lemma 3.1 *Let P be Σ-self-static and w-Σ-quasi-projective. Then the pair of functor (H_P, T_P) defines an equivalence between* $\mathrm{Pres}(P)$ *and* $\mathrm{Copres}(P^*)$.

Proof. For any $M \in \mathrm{Pres}(P)$, there is an exact sequence $0 \to K \to P_M \to M \to 0$ with $K \in \mathrm{Gen}(P)$. Since P is w-Σ-quasi-projective, the functor H_P preserves its exactness. Hence ρ_M is an isomorphism by [15, 3.7], since P is also Σ-self-static. This shows that $\mathrm{Pres}(P)$ coincides with the class of modules M such that ρ_M is an isomorphism. It follows that $H_P : \mathrm{Pres}(P) \rightleftharpoons \mathrm{Copres}(P^*) : T_P$ is an equivalence from [12] or [15, 4.3].

Corollary 3.2 *Let P be a Σ-self-static star module. Then the functor T_P preserves short exact sequences in* $\mathrm{Copres}(P^*)$.

Proof. Since $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ for a star module P, the conclusion then follows from Lemma 3.1 and [5, Proposition 1.1].

We give now some properties of two categories involved in the equivalences in Lemma 3.1.

Lemma 3.3 *Let P be Σ-self-static and w-Σ-quasi-projective. Then*

(1) $\mathrm{Pres}(P)$ *is closed under cokernels of homomorphisms.*

(2) $\mathrm{Copres}(P^*)$ *is closed under kernels and images of homomorphisms.*

Proof. (1) By [16, 3.2(2)]. Note that the conclusion also holds for any w-Σ-quasi-projective modules P.

(2) Consider any homomorphism $f : K_1 \to K_2$ in $\mathrm{Copres}(P^*)$. Let $X = \mathrm{Ker} f$ and $Y = \mathrm{Im} f$ and consider the exact sequence $0 \to X \to K_1 \to Y \to 0$. By applying the functor T_P we obtain an induced exact sequence $0 \to M \to T_P K_1 \to T_P Y \to 0$ for some $M \in \mathrm{Gen}(P)$. By Lemma 2.2, the induced sequence $0 \to H_P M \to H_P T_P K_1 \to H_P T_P Y \to 0$ is exact. Hence we have the following exact commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & X & \to & K_1 & \to & Y & \to & 0 \\ & & \downarrow h & & \downarrow \sigma_{K_1} & & \downarrow \sigma_Y & & \\ 0 & \to & H_P M & \to & H_P T_P K_1 & \to & H_P T_P Y & \to & 0 \end{array}$$

Since σ_{K_1} is an isomorphism and σ_Y is a monomorphism, we have that $X \simeq H_P M$ and that σ_Y is an isomorphism. Hence, $X, Y \in \mathrm{Copres}(K)$.

Proposition 3.4 *Assume that there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$ defined by a pair of functors (H, T). Then there exists a Σ-self-static w-Σ-quasi-projective R-module P such that $\mathcal{C} = \mathrm{Pres}(P)$, $\mathcal{D} = \mathrm{Copres}(P^*), H \simeq H_P$, $T \simeq T_P$ and $S \simeq \mathrm{End}_R P$ naturally.*

Proof. Let ${}_R P = T(S)$. Then a standard proof (e.g.[9]) shows that $H \simeq H_P$, $T \simeq T_P$ and $S \simeq \mathrm{End}_R P$ canonically.

Since $\mathcal{C}$ is closed under direct sums, we have that $P^{(X)} \simeq T_P H_P P^{(X)}$ for any set X, i.e., P is Σ-self-static. Moreover, since $\mathcal{C}$ is closed under cokernels of homomorphisms, we then have that $\mathrm{Pres}(P) \subseteq \mathcal{C}$. However, $\mathcal{C} = T_P H_P \mathcal{C} \subseteq \mathrm{Pres}(P)$ from the categories equivalence. Hence $\mathcal{C} = \mathrm{Pres}(P)$. The remained part is to show that P is w-Σ-quasi-projective.

Consider now any exact sequence $0 \to M \to P_N \to^f N \to 0$ with $P_N \in \mathrm{Add} P$ and $M \in \mathrm{Gen}(P)$. By applying the functor H_P, we obtain an induced sequence $0 \to H_P M \to H_P P_N \to L \to 0$, where $L = \mathrm{Im} H_P(f)$. Note that $H_P P_N, H_P N \in \mathcal{D}$ by the equivalence, so we have that $L \in \mathcal{D}$ since $\mathcal{D}$ is closed under images of homomorphisms. Now by applying the functor T_P, we obtain the following exact commutative diagram.

$$\begin{array}{ccccccccc} & & T_P H_P M & \to & T_P H_P P_N & \to & T_P L & \to & 0 \\ & & \downarrow \rho_M & & \downarrow \rho_{P_N} & & \downarrow h & & \\ 0 & \to & M & \to & P_N & \to & N & \to & 0 \end{array}$$

Since ρ_M is an epimorphism and ρ_{P_N} is an isomorphism, we have that $h = T_P(i)\rho_N$ is an isomorphism, where i denotes the induced canonical homomorphism $L \to H_P N$. Since $N \in \mathrm{Pres}(P)$ clearly, we have that ρ_N is an isomorphism by the equivalence. It follows that $T_P(i)$, and hence, $H_P T_P(i)$ is an isomorphism. Note that $L, H_P N \in \mathcal{D}$, so we obtain that $i = H_P T_P(i)$ by the equivalence. This shows that i is an isomorphism. Hence the induce sequence $0 \to H_P M \to H_P P_N \to H_P N \to 0$ is exact, and consequently, P is w-Σ-quasi-projective.

Lemma 3.5 *Assume that P is Σ-self-static s-Σ-quasi-projective. Then* $\mathrm{Copres}(P^*)$ *is closed under kernels, cokernels and images of homomorphisms. In particular,* $\mathrm{Copres}(P^*)$ *contains all finitely presented S-modules.*

Proof. It's sufficient to show that $\mathrm{Copres}(P^*)$ is closed under cokernels of homomorphisms. Consider any exact sequence $K_1 \to^f K_2 \to N \to 0$ with $K_1, K_2 \in \mathrm{Copres}(P^*)$ and $N = \mathrm{Coker} f$. By applying the functor T_P, we obtain an induced exact sequence $T_P K_1 \to^{T_P(f)} T_P K_2 \to T_P N \to 0$. Since P is s-Σ-quasi-projective, we have that the induced sequence $H_P T_P K_1 \to^{H_P T_P(f)} H_P T_P K_2 \to H_P T_P N \to 0$ is exact by Proposition 2.4. Hence we obtain that $N = \mathrm{Coker} f \simeq \mathrm{Coker} H_P T_P(f) = H_P T_P N \in \mathrm{Copres}(P^*)$ by the equivalence as in Lemma 3.1.

Proposition 3.6 *Assume moreover $\mathcal{D}$ is closed under cokernels of homomorphisms. Let the pair of functors (H, T) define an equivalence between $\mathcal{C}$ and $\mathcal{D}$. Then there exists a Σ-self-static s-Σ-quasi-projective R-module P such that $\mathcal{C} = \mathrm{Pres}(P)$, $\mathcal{D} = \mathrm{Copres}(P^*), H \simeq H_P$, $T \simeq T_P$ and $S \simeq \mathrm{End}_R P$ naturally.*

Proof. Similarly as in Proposition 3.4, we need to show that P is s-Σ-quasi-projective. Consider any exact sequence $P_2 \to P_1 \to N \to 0$ with $P_1, P_2 \in \mathrm{Add} P$. Then we have an induced exact sequence $H_P P_2 \to H_P P_1 \to L \to 0$ for some L. By applying the functor T_P we obtain an exact sequence $T_P H_P P_2 \to T_P H_P P_1 \to T_P L \to 0$. Note that $L \in \mathcal{D} = \mathrm{Copres}(P^*)$ since $\mathcal{D}$ is closed under cokernels of homomorphisms, so we obtain that the induced canonical homomorphism $i : L \to H_P N$ is an isomorphism by a way similar as in Proposition 3.4. It follows that the induced sequence $H_P P_2 \to H_P P_1 \to H_P N \to 0$ is exact and that P is s-Σ-quasi-projective.

Theorem 3.7 *Assume moreover $\mathcal{C}$ is closed under factors.*

*(1) There is an equivalence between $\mathcal{C}$ and $\mathcal{D}$ if and only if there exists a Σ-self-static star module $P \in R$-*Mod *such that $\mathcal{C} =$* $\mathrm{Gen}(P)$ *and $\mathcal{D} =$* $\mathrm{Copres}(P^*)$.

(2) Assume moreover $\mathcal{C}$ contains all injective R-modules. Then there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$ if and only if there exists a Σ-self-static tilting R-module P such that $\mathcal{C} = \mathrm{Gen}(P)$ and $\mathcal{D} = \mathrm{Copres}(P^*)$.

*(3) Assume moreover $\mathcal{D}$ is closed under cokernels of homomorphisms. Then there is an equivalence between $\mathcal{C}$ and $\mathcal{D}$ if and only if there exists a Σ-quasi-projective self-generator $P \in R$-*Mod *such that $\mathcal{C} =$* $\mathrm{Gen}(P)$ *and $\mathcal{D} =$* $\mathrm{Copres}(P^*)$.

Proof. (1). The necessity follows from Proposition 3.4 and the fact $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ which is easily obtained from our assumptions. The sufficient part follows from the definition of star modules and Lemma 3.1.

(2). By (1) and Proposition 2.8.

(3). By [15], every Σ-self-generator is Σ-self-static. Hence the conclusion follows from (1), Proposition 2.11, Lemma 3.5, and Proposition 3.6.

Corollary 3.8 *Let P be an R-module with $S = \mathrm{End}_R P$. Then P is Σ-quasi-projective self-generator if and only if P is a Σ-self-static star module and P_S is flat.*

Proof. The necessary part follows from Proposition 2.11 and [7]. We need to show the sufficient part.

By Proposition 2.11, it is enough to show that P is a self-generator. Now take any $M \leq P$ and consider the exact sequence $0 \to M \to P \to^f N \to 0$. Then we obtain an induced sequence $0 \to H_P M \to H_P P \to^{H_P(f)} H_P N$. Since P_S is flat, we have that the induced sequence $0 \to T_P H_P M \to T_P H_P P \to^{T_P H_P(f)} T_P H_P N$ is exact. Note that $P, N \in \mathrm{Gen}(P)$ and there is an equivalence between $\mathrm{Gen}(P)$ and $\mathrm{Copres}(P^*)$ by Theorem 3.7, so we have that $M = \mathrm{Ker} f \simeq \mathrm{Ker} T_P H_P(f) = T_P H_P M$ canonically. It follows that $M \in \mathrm{Gen}(P)$, i.e. P is a self-generator.

We end this note with some examples.

Example 3.9 *(1) P is Σ-self-static provided that P is projective or P is a Σ-self-generator.*

(2) Let p be a prime and $P = \boldsymbol{Z}(p^\infty)$ be the prüfer p-group. Then P as a $\boldsymbol{Z}$-module is a Σ-self-static star module, moreover, it is quasi-tilting (but not tilting).

(3) $Q \oplus Q/Z$ is tilting in $\boldsymbol{Z}$-Mod.

(4) Consider $P = \boldsymbol{Z}(p^\infty)$ as a canonical $\boldsymbol{J}_p$-module, where $\boldsymbol{J}_p$ is the ring of p-adic integers. Then P is Σ-self-static tilting.

Proof. (1) follows from [15, 3.4] and (3) follows from [16].

(2). Note that in this case Wisbauer [16] had shown that P is a star module and that Orsatti [10] had shown that P is Σ-self-static. Moreover, $\mathrm{Gen}(P) \subseteq \mathrm{Ker}\mathrm{Ext}^1_{\mathbf{Z}}(P, -)$ by [8], hence P is quasi-tilting.

(4). In fact, by [5] (see also [9]) we have that P is Σ-self-static and that $\mathrm{Gen}(P) = \mathrm{Pres}(P)$. Since $\mathrm{Gen}(P) = \mathcal{I}$ by [8], where $\mathcal{I}$ denote the class of all injective $\mathbf{J}_p$-modules, we obtain that $\mathcal{I} = \mathrm{Gen}(P) = \mathrm{Pres}(P) \subseteq \mathrm{Ext}^1_{\mathbf{J}_p}(P, -)$. Hence, by Proposition 2.8, P is tilting.

References

[1] F.D. Anderson and K.R. Fuller, *Rings and Categories of Modules*, 1st ed., Springer-Verlag, New York, 1974.

[2] R. Colpi, Some remarks on equivalences between categories of modules, *Comm. in Algebra*, **18** (1990), 1935-1951. https://doi.org/10.1080/00927879008824002

[3] R. Colpi, Tilting modules and $*$-moduels, *Comm. Algebra*, **21** (1993), no. 4, 1095-1102.

[4] R. Colpi, G. D'Este, and A. Tonolo, Quasi-tilting modules and counter equivalences, *J. Algebra*, **191** (1997), no. 2, 461–494. https://doi.org/10.1006/jabr.1997.6873

[5] R. Colpi and C. Menini, On the structure of $*$-modules, *J. Algebra*, **158** (1993), no. 2, 400-419. https://doi.org/10.1006/jabr.1993.1138

[6] R. Colpi and J. Trlifaj, Tilting modules and tilting torsion theories, *J. Algebra*, **178** (1995), no. 2, 614-634. https://doi.org/10.1006/jabr.1995.1368

[7] K.R. Fuller, Density and equivalence, *J. Algebra*, **29** (1974), 528-550. https://doi.org/10.1016/0021-8693(74)90088-x

[8] E. Gregorio and A. Tonolo, Weakly tilting bimodules, *Forum Math.*, **13** (2001), 589-614. https://doi.org/10.1515/form.2001.024

[9] C. Menini, A. Orsatti, Representable equivalences between categories of modules and applications, *Rend. Sem. Mat. Univ. Padova*, **82** (1989), 203-231.

[10] A. Orsatti, Equivalenze rappresentabili tra categorie di moduli, *Rend. Sem. Mat. Fiz.*, **60** (1990), 243-260. https://doi.org/10.1007/bf02925089

[11] M. Sato, Fuller's Theorem on equivalences, *J. Algebra*, **52** (1978), 174-184. https://doi.org/10.1016/0021-8693(78)90275-2

[12] M. Sato, On equivalences between module categories, *J. Algebra*, **59** (1979), no. 2, 412-420. https://doi.org/10.1016/0021-8693(79)90137-6

[13] J. Trlifaj, Dimension estimates for representable equivalences of module categories, *J. Algebra*, **193** (1997), 660-676. https://doi.org/10.1006/jabr.1996.7005

[14] J. Wei, n-star modules and n-tilting modules, *J. Algebra*, **283** (2005), 711–722. https://doi.org/10.1016/j.jalgebra.2004.09.032

[15] R. Wisbauer, *Static Modules and Equivalences*, Interactions between ring theory and representations of algebras (Murcia), Lecture Notes in Pure and Appl. Math., Vol. 210, Dekker, New York, 2000, 423–449.

[16] R. Wisbauer, Tilting in module categories, *Abelian Groups, Module Theory, and Topology*, D. Dikranjan, L. Salce (ed.), Marcel Dekker LNPAM 201,421-444 (1998).

[17] B. Zimmerman-Huisgen, Endomorphism rings of self-generators, *Pacific J. Math.*, **61** (1978), 587-602. https://doi.org/10.2140/pjm.1975.61.587

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