Reduced Rings and Modules Arising from Morita Contexts

Qingbing Xu
Department of Basic courses, Chuzhou Polytechnic College
Chuzhou, Anhui, 239000, P.R. China

Yang Liu
School of Mathematics and Finance, Chuzhou University
Chuzhou Anhui, 239012, P.R. China

Mohammad Munir
Department of Mathematics, Government Postgraduate College No.1
Abbottabad, 22010, Pakistan

Kausar Nasreen
Department of Mathematics, Yildiz Technical University
Esenler, Istanbul, 34210, Turkey

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Abstract

In this paper, we study the reduced rings arising from Morita context
$\mathcal{M}(A, B) = (A, M, N, B, \varphi, \psi)$. Necessary and sufficient conditions are
investigated for the Morita ring $R$ to be reduced. In particular, the
reduced modules over Morita context rings are characterized.

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1 Introduction

Morita contexts appeared as a key ingredient in the work[9] of Morita that described equivalences between full categories of modules over rings with identities. One of the fundamental results in this direction says that the categories of left modules over the rings $R$ and $S$ are equivalent if and only if there exists a strict Morita context connecting $R$ and $S$. The concept of Morita context are extensively studied and often appear in a abundant of literature (see, e.g.,[10, 7, 5, 4]). In [2], Calci Mte Burak and Halicioglu Sait studied the primeness and semiprimeness of Morita context rings and modules. Necessary and sufficient conditions are investigated for an ideal of a Morita context to be a prime ideal and a semiprime ideal. This is the motivation of our work.

In this paper, the notation $\text{Nil}(R)$ and $\text{P}(R)$ stand for the set of all nilpotent element and prime radical. The following undefined symbols we refer to [14, 1, 11]. The latest related research can be found in [15, 13].

2 Prime and semiprime rings over Morita context

Some theorems in this section can be referred in [2]. These results will be proved in an alterative simple methods. We now list some conventions and preliminary results which will be useful in the sequel. Let $A$ and $B$ be associative rings with multiplicative identities, Morita context over $A$, $B$ is 6-tuple $\mathcal{M}(A, B) = (A, M, N, B, \varphi, \psi)$, where $BM_A$ and $AN_B$ are unital bimodules. $\varphi : N \otimes_B M \rightarrow A$ and $\psi : M \otimes_A N \rightarrow B$ are bimodule homomorphisms, satisfying the following associativity conditions:

$$m'\varphi(n \otimes m) = \psi(m' \otimes n)m$$

and

$$\varphi(n \otimes m)n' = n\psi(m \otimes n')$$

for all $m, m' \in M, n, n' \in N$, and $I = Im\varphi, J = Im\psi$ are two trace ideals of $A$ and $B$ respectively.

Associated to any Morita context $\mathcal{M}(A, B)$ is an associative ring, called the Morita ring of $\mathcal{M}(A, B)$, which incorporates all the information involved in the 6-tuple, defined to be the formal $2 \times 2$ matrix ring

$$R = \begin{pmatrix} A & N \\ M & B \end{pmatrix}.$$  

The formal matrix multiplication is given by

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} aa' + \varphi(n \otimes m') & an' + nb' \\ ma' + bm' & bb' + \psi(m \otimes n') \end{pmatrix}.$$
\( \mathcal{M}(A, B) \) is said to be a projective Morita context (injective Morita context) if the maps \( \varphi \) and \( \psi \) are epic (resp. monic). \( \mathcal{M}(A, B) \) is said to be semi-projective Morita context (semi-injective Morita context) if one of the maps \( \varphi \) or \( \psi \) is epic (resp. monic).

Note that \( \mathcal{M}(A, B) \) is an injective Morita context if \( N \otimes_B M \cong I = \text{Im}\varphi \) and \( M \otimes_A N \cong J = \text{Im}\psi \). \( \mathcal{M}(A, B) \) is a projective Morita context, we claim that \( A \) is Morita equivalent to \( B \). Common properties shared by Morita equivalence are termed as Morita invariants. For instance, being prime or semiprime are Morita invariant, while being reduced, commutative, domain, division rings or fields are not Morita invariant. Now, we use the Morita theory to simplify the following proof.

An ideal \( I \) in a ring \( R \) is called a prime ideal if \( I \neq R \), and for ideals \( A, B \subseteq R \), \( AB \subseteq I \) implies \( A \subseteq I \) or \( B \subseteq I \). Equivalently, if for \( a, b \in R \), \( aRb \subseteq I \) implies \( a \in I \) or \( b \in I \). The prime radical of a ring \( R \) is the intersection of all prime ideals of \( R \) is denoted by \( P(R) \). An ideal \( I \) in a ring \( R \) is said to be semiprime ideal if for any \( A \subseteq R \), \( A^2 \subseteq I \), implies \( A \subseteq I \). Equivalently, if for \( a \in R \), \( aRa \subseteq I \) implies \( a \in I \). It is clear that a prime ideal is always a semiprime ideal.

A ring \( R \) is called prime if 0 is a prime ideal, equivalently for any \( a, b \in R \), \( aRb = 0 \) implies \( a = 0 \) or \( b = 0 \). A ring \( R \) is said to be semiprime if it has no nonzero nilpotent ideal, equivalently for any \( a \in R \), \( aRa = 0 \) implies \( a = 0 \). Obviously, prime rings are semiprime, a ring \( R \) is semiprime if and only if \( P(R) = 0 \). A proper submodule \( N \) of a left \( R \)-module \( M \) is called prime if for any \( r \in R \) and \( m \in M \), \( rRm \subseteq N \) implies \( rM \subseteq N \) or \( m \in N \). It is easy to show that if \( N \) is a prime submodule of \( M \), then the annihilator of the module \( M/N \) is a two-sided prime ideal of \( R \). A module \( M \) is said to be prime if 0 is a prime submodule of \( M \), i.e., for any \( r \in R \) and \( m \in M \), \( rRm = 0 \) implies \( rM = 0 \) or \( m = 0 \).

**Lemma 2.1** Let \( \mathcal{M}(A, B) = (A, M, N, B, \varphi, \psi) \) be a Morita context, and its Morita ring is denoted by \( R = \begin{pmatrix} A & N \\ M & B \end{pmatrix} \).

1. If \( \psi \) is epic, then \( B \) and \( R \) are Morita equivalent.
2. If \( \varphi \) is epic, then \( A \) and \( R \) are Morita equivalent.

**Proof** We only prove (1). (2) can be proved similarly.

Let \( \mathcal{M}'(R, B) = (R, M', N', B, \varphi', \psi') \), its Morita ring is

\[
\mathcal{M}'(R, B) = \begin{pmatrix} R & N' \\ M' & B \end{pmatrix} = \begin{pmatrix} A & N & N' \\ M & B & B \\ M & B & B \end{pmatrix}.
\]

In which, \( M' = (M B), N' = \begin{pmatrix} N \\ B \end{pmatrix} \), \( \varphi' : N' \otimes_B M' \rightarrow R \) and \( \psi' : M' \otimes_R \)
$N' \rightarrow B$ are maps respectively. The maps are defined by

$$\varphi'(\begin{pmatrix} n \\ b \end{pmatrix} \otimes (m' b')) = \begin{pmatrix} \varphi(n \otimes m') \\ bm' \\ nb' \end{pmatrix}$$

and

$$\psi'(m b \otimes \begin{pmatrix} n' \\ b' \end{pmatrix}) = \psi(m \otimes n') + bb'.$$

It is easy to check the associativity conditions, hence $\mathcal{M}'(R, B)$ is a Morita context.

For an arbitrary $b \in B$, $\psi'(\begin{pmatrix} 0_M b \\ 0_N \end{pmatrix}) = b$ which shows that $\psi'$ is epic.

On the other hand, assume that the map $\psi$ is epic, thus an arbitrary element $b$ is of the form $b = \sum \psi(m_i \otimes n_i)$, then each element of $R$ can be expressed in the form

$$\begin{pmatrix} a \\ n \\ b \end{pmatrix} = \varphi'(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (a \ n)) + \varphi'(\begin{pmatrix} 0 \\ m \end{pmatrix} \otimes (1 \ n)) + \sum \varphi'(\begin{pmatrix} 0 \\ m_i \end{pmatrix} \otimes (0 \ n_i)).$$

This shows that the map $\varphi'$ is also epic. So $\mathcal{M}'(R, B)$ is a projective Morita context, $R$ and $B$ are Morita equivalent.

**Theorem 2.2** Let $R = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a Morita ring of $\mathcal{M}(A, B) = (A, M, N, B, \varphi, \psi)$, $\varphi$ or $\psi$ is epic, the following statements are equivalent.

1. $R$ is a prime ring.
2. $A$ and $B$ are prime rings and $M$, $N$ are prime modules.
3. $A$ or $B$ is a prime ring.

**Proof** (1)⇒(2) Let $a_1, a_2 \in A$, such that $a_1 A a_2 = 0$. Then we have

$$\begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} A & N \\ M & B \end{pmatrix} \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 A a_2 \\ 0 \\ 0 \end{pmatrix}.$$

As $R$ is a prime ring, so $a_1 = 0$ or $a_2 = 0$, hence $A$ is a prime ring. Similarly, we can prove $B$ is also a prime ring.

Now let $a \in A$ and $n \in N$, such that $a A n = 0$. Then

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} A & N \\ M & B \end{pmatrix} \begin{pmatrix} 0 \\ n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a A n \\ 0 \end{pmatrix}.$$

Thus $a = 0$ or $n = 0$. This implies that $a N = 0$ or $n = 0$. Therefore, $N$ is a prime left $A$–module.
let \( b \in B \) and \( n \in N \) with \( nBb = 0 \). Since
\[
\begin{pmatrix}
0 & n \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
A & N \\
M & B
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & b
\end{pmatrix} = \begin{pmatrix}
0 & nBb \\
0 & 0
\end{pmatrix}.
\]
By (1), \( b = 0 \) or \( n = 0 \). Thus \( Nb = 0 \) or \( n = 0 \). Hence \( N \) is a prime right \( B \)-module. Similarly, \( BMA \) is a prime module.

\( 2 \Rightarrow 3 \) is trivial.

\( 3 \Rightarrow 1 \) For the Morita context \( M(A, B) \), the \( \psi \) is epic. by Lemma 2.1, \( M(R, B) \) is a projective Morita context. Hence \( R \) and \( B \) are Morita equivalent.

Since \( B \) is prime, so \( R = \begin{pmatrix}
A \\
M \\
B
\end{pmatrix} \) is also a prime ring. Similarly, \( \varphi \) is epic, \( A \) is a prime ring, \( R \) is a prime ring too.

**Theorem 2.3** Let \( R = \begin{pmatrix}
A \\
M \\
B
\end{pmatrix} \) be a Morita ring of \( M(A, B) = (A, M, N, B, \varphi, \psi) \), \( \varphi \) or \( \psi \) is epic, the following statements are equivalent.

1. \( R \) is a semiprime ring.
2. \( A \) and \( B \) are semiprime rings.
3. \( A \) or \( B \) is a semiprime ring.

**Proof** Similar to Theorem 2.2.

**Theorem 2.4** Let \( M(A, B) = (A, M, N, B, \varphi, \psi) \) be a Morita context, \( M(I, J) = (I, M_1, N_1, J, \varphi_1, \psi_1) \) be an ideal of \( M(A, B) \), and the \( \varphi_1 \) or \( \psi_1 \) is epic. the following two statements are equivalent.

1. \( M(I, J) \) is a prime ideal.
2. \( I \) and \( J \) are prime ideals of \( A \) and \( B \), and \( N_1 = \{ n \in N : nM \subseteq I \} = \{ n \in N : Mn \subseteq J \} \), \( M_1 = \{ m \in M : Mm \subseteq I \} = \{ m \in M : mN \subseteq J \} \).

**Proof** \( 1 \Rightarrow 2 \) As \( M(I, J) \) is a prime ideal of \( M(A, B) \), let \( a_1, a_2 \in A \) with \( a_1Aa_2 \subseteq I \), we have
\[
\begin{pmatrix}
a_1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
A & N \\
M & B
\end{pmatrix}
\begin{pmatrix}
a_2 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
a_1Aa_2 & 0 \\
0 & 0
\end{pmatrix}.
\]
So
\[
\begin{pmatrix}
a_1Aa_2 & 0 \\
0 & 0
\end{pmatrix} \subseteq \begin{pmatrix}
I \\
M_1
\end{pmatrix}
\begin{pmatrix}
N_1 \\
J
\end{pmatrix}
\]
Since \( M(I, J) \) is a prime ideal,
\[
\begin{pmatrix}
a_1 & 0 \\
0 & 0
\end{pmatrix} \in \begin{pmatrix}
I \\
M_1
\end{pmatrix}
\begin{pmatrix}
N_1 \\
J
\end{pmatrix}
\text{ or } \begin{pmatrix}
a_2 & 0 \\
0 & 0
\end{pmatrix} \in \begin{pmatrix}
I \\
M_1
\end{pmatrix}
\begin{pmatrix}
N_1 \\
J
\end{pmatrix}.
\]
Then \( a_1 \in A \) or \( a_2 \in A \), hence \( I \) is a prime ideal of \( A \). The proof of the latter part is given in [2, Theorem 2.7].
(2)⇒(1) For the Morita context $\mathcal{M}(I, J)$, the $\psi_1$ is epic. By Lemma 2.1, $\mathcal{M}(R_1, J)$ is a projective Morita context. Hence $R_1$ and $J$ are Morita equivalent. Since $J$ is prime, so $R_1 = \begin{pmatrix} I & N_1 \\ M_1 & J \end{pmatrix}$ is also prime. Similarly, $\varphi_1$ is epic, $I$ is a prime ideal, $R_1$ is also prime.

**Theorem 2.5** Let $\mathcal{M}(A, B) = (A, M, N, B, \varphi, \psi)$ be a Morita context, $\mathcal{M}(I, J) = (I, M_1, N_1, J, \varphi_1, \psi_1)$ be an ideal of $\mathcal{M}(A, B)$, and the $\varphi_1$ or $\psi_1$ is epic. the following two statements are equivalent.

(1) $\mathcal{M}(I, J)$ is semiprime.

(2) $I$ and $J$ are semiprime ideals of $A$ and $B$, and $N_1 = \{n \in N : nM \subseteq I\} = \{n \in N : Mn \subseteq J\}$, $M_1 = \{m \in M : Mm \subseteq I\} = \{m \in M : mN \subseteq J\}$.

It well known that simple, semisimple, von Neumann regular, Noetherian, Artinian, self-injective, quasi-Frobenius, primitive, semiprimitive, (semi-)hereditary, nonsingular, coherent, semiprimary, perfect, semiperfect, semilocal are all Morita invariant properties. The above techniques can simplify the proof of the following problems.

**Theorem 2.6** Let $R = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a Morita ring of $\mathcal{M}(A, B) = (A, M, N, B, \varphi, \psi)$. Then

(1) $R$ is a left perfect ring if and only if $A$ and $B$ are left perfect.

(2) $R$ is a semiperfect ring if and only if $A$ and $B$ are semiperfect.

(3) $R$ is a semilocal ring if and only if $A$ and $B$ are semilocal.

(4) $R$ is a semiprimary ring if and only if $A$ and $B$ are semiprimary.

(5) $R$ is a regular ring if and only if $A$ and $B$ are regular.

**Proof** See [14].

### 3 Reduced rings and reduced module over Morita context

A ring $R$ is said to be reduced if $R$ has no nonzero nilpotent elements, or equivalently, if $a^k = 0$ for some $k \geq 1$ implies $a = 0$ in $R$. Any reduced ring is a semiprime ring. Every commutative semiprime ring is a reduced ring.

**Lemma 3.1** Let $R = \begin{pmatrix} A & N \\ M & B \end{pmatrix}$ be a ring of Morita context $\mathcal{M}(A, B)$ with zero trace ideals. Then,

(1) $\text{Nil}(R) = \begin{pmatrix} \text{Nil}(A) & N \\ M & \text{Nil}(B) \end{pmatrix}$.

(2) $R/\text{Nil}(R) \cong A/\text{Nil}(A) \times B/\text{Nil}(B)$. 


Proof (1) If \((a \ m \ b) \in \text{Nil}(R)\), then \((a \ m \ b)^k = 0\) for some \(k \geq 1\). Since 
\[
(a \ m \ b)^k = \begin{pmatrix} a & n \\ m & b \end{pmatrix}^k\
\]
for some \(n \in N, \ m \in M\). So \(a \in \text{Nil}(A), \ b \in \text{Nil}(B)\).

Conversely, if \(a^t = 0, \ b^t = 0\), for any \((a \ n \ m \ b) \in R\), we have
\[
(a \ n \ m \ b)^k = \begin{pmatrix} a^k & \sum_{j=1}^k a^{k-j}n^{j-1}m^j \\ b^k & \sum_{j=1}^k a^{k-j}m^{j-1}n^j \end{pmatrix}.
\]
When \(j \leq t\), then \(2t - j \geq t\), \(a^{2t-j} = 0\), \(b^{2t-j} = 0\), when \(j > t\), then \(j - 1 \geq t\), \(a^{j-1} = 0\), \(b^{j-1} = 0\), put \(k = 2t\), Then \((a \ n \ m \ b)^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). So \((a \ n \ m \ b) \in \text{Nil}(R)\).

(2) for any \((a \ n \ m \ b) \in R\), there exists an isomorphism,
\[
f: \begin{pmatrix} A \ N \\ M \ B \end{pmatrix} \rightarrow \begin{pmatrix} A/\text{Nil}(A) & 0 \\ 0 & B/\text{Nil}(B) \end{pmatrix}
\]
\[
(a \ n \ m \ b) \mapsto \begin{pmatrix} a + \text{Nil}(A) \\ b + \text{Nil}(B) \end{pmatrix}
\]
and \(\ker f = \text{Nil}(R)\).

Lemma 3.2 If \(R_1\) and \(R_2\) are reduced rings, then the direct product \(R_1 \times R_2\) is also reduced.

Proof Check directly.

Lemma 3.3 A ring \(R\) is reduced if and only if \(eRe\) is reduced for all idempotent \(e \in R\).

Proof See [4, Lemma 3.1].

Proposition 3.4 Let \(R\) be a ring of Morita context \(M(A, B)\), if \(R\) is reduced, then \(R\) has no divisors of zero.

Proof For any \((a \ n \ m \ b) \in R\), suppose \((a \ n \ m \ b)^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\), for some positive integer \(k\). Since \(R\) is reduced, then \((a \ n \ m \ b)^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). Replacing 
\[
(a \ n \ m \ b) \text{ by } (a \ m \ b) (a' \ n' \ m' \ b')
\]
in the relation \((a \ n \ m \ b)^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\), we have
\[
[(a \ n \ m \ b) (a' \ n' \ m' \ b')]^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
for some \( \begin{pmatrix} a & n \\ m & b \end{pmatrix}, \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} \in R \). Since \( R \) is reduced, then
\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix} \begin{pmatrix} a' & n' \\ m' & b' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
But \( \begin{pmatrix} a & n \\ m & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), therefore, \( R \) has no zero divisors.

**Proposition 3.5** Let \( R \) be a ring of Morita context \( \mathcal{M}(A,B) \), which is not necessarily with zero trace ideals. If \( R \) is reduced, then \( A \) and \( B \) are reduced.

**Proof** For any \( \begin{pmatrix} a & n \\ m & b \end{pmatrix} \in \text{Nil}(R) \), by Lemma 3.1,
\[
\begin{pmatrix} a & n \\ m & b \end{pmatrix}^k = \begin{pmatrix} a^k & * \\ * & b^k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]
As \( R \) is reduced, so \( a^k = b^k = a = b = 0 \). Hence \( a \in \text{Nil}(A) \), \( b \in \text{Nil}(B) \), \( A \) and \( B \) are reduced.

But the converse of the Proposition 3.5 is not true. In fact, although \( A \) and \( B \) are reduced, the ring \( R \) of Morita context \( \mathcal{M}(A,B) \) although with zero trace ideals need not to be reduced. By Lemma 3.1, \( R/\text{Nil}(R) \cong A/\text{Nil}(A) \times B/\text{Nil}(B) \). If \( A \) and \( B \) are reduced, then \( R/\text{Nil}(R) \cong A \times B \). So \( \text{Nil}(R) = \begin{pmatrix} 0 & N \\ M & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), thus, \( R \) is not a reduced ring.

A natural question arises under which conditions does the converse of proposition hold? We have the following theorem.

**Theorem 3.6** Let \( R \) be the ring of \( \mathcal{M}(A,B) \), \( R \) is a reduced ring if and only if \( A \) and \( B \) are reduced, and \( M = N = 0 \).

**Proof** By Proposition 3.5, for any \( m \in M \), \( n \in N \), since \( R \) is a reduced ring, we have
\[
\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \in \text{Nil}(R) \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \in \text{Nil}(R),
\]
so \( m = n = 0 \) and \( M = N = 0 \). Conversely, If \( M = N = 0 \), then \( \text{Nil}(R) \cong \text{Nil}(A) \times \text{Nil}(B) \). Since \( A \) and \( B \) are reduced, \( R \cong A \times B \) is also reduced.

**Proposition 3.7** If \( R \) is a reduced ring of Morita context \( \mathcal{M}(A,B) \), then \( R \) is a semiprime ring.
Proof It is trivial.

It is well known that Morita ring $R$ is a noncommutative ring. Therefore a semiprime ring is not necessarily reduced ring. Counter example see [11, Corollary 2.5].

Let $R$ be a ring, and let $s \in R$ be central. The 4-tuple becomes a ring with the following operations:

$$
\begin{pmatrix}
  a & n \\
  m & b
\end{pmatrix}
+ \begin{pmatrix}
  a' & n' \\
  m' & b'
\end{pmatrix} = \begin{pmatrix}
  a + a' & n + n' \\
  m + m' & b + b'
\end{pmatrix},
$$

$$
\begin{pmatrix}
  a & n \\
  m & b
\end{pmatrix}
\begin{pmatrix}
  a' & n' \\
  m' & b'
\end{pmatrix} = \begin{pmatrix}
  aa' + smn' & an' + nb' \\
  ma + bm' & smn' + bb'
\end{pmatrix}.
$$

This ring is denoted by $K_s(R)$. A Morita context ring $\begin{pmatrix}
  A & N \\
  M & B
\end{pmatrix}$ with $A = B = M = N = R$ is called a generalized matrix ring over $R$.

**Proposition 3.8** If a generalized matrix ring $K_s(R)$ is reduced and $s$ is a non zero divisor, then $R$ is reduced.

**Proof** For any $a, b, n, m \in R$ and $\begin{pmatrix}
  a & n \\
  m & b
\end{pmatrix} \in K_s(R)$ with $\begin{pmatrix}
  a & n \\
  m & b
\end{pmatrix}^2 = \begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}$. Since

$$
\begin{pmatrix}
  a & n \\
  m & b
\end{pmatrix}^2 = \begin{pmatrix}
  a^2 + smn & * \\
  * & b^2 + smn
\end{pmatrix} = \begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}
$$

and $K_s(R)$ is reduced, so $m = n = 0$, $a = b = 0$, thus $R$ is a reduced ring. But the converse statement is not true. For example, Let $R = Z$ and $s = 1$. $K_s(R)$ is the ring of $2 \times 2$ matrices over the ring $Z$. Then $R$ is reduced and $s$ is no zero divisor, but $K_s(R)$ is not reduced.

A left $R$–module $M$ is reduced[8], if whenever $a \in R$, $m \in M$, $a^2m = 0$ implies $aRm = 0$. Let $X$ be a left $A$–module and $Y$ a left $B$–module. We assume that there is an $R$–module homomorphism $f : N \otimes_B Y \to X$ and an $B$–module homomorphism $g : M \otimes_A X \to Y$ such that

$$
n(mx) = (nm)x, m(ny) = (mn)y, \quad m \in M, n \in N, x \in X, y \in Y.
$$

where $mx$ denotes $g(m \otimes x)$, and $ny$ denotes $f(n \otimes y)$. The group of column vectors $\begin{pmatrix}
  X \\
  Y
\end{pmatrix}$ is turned into a left $R$–module if we take the product of the matrix by a column as the module multiplication,

$$
\begin{pmatrix}
  a & n \\
  m & b
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix} = \begin{pmatrix}
  ax + ny \\
  mx + by
\end{pmatrix}$$
The homomorphisms $f$ and $g$ are called the homomorphisms of module multiplication.

**Theorem 3.9** A left $R$–module $\begin{pmatrix} X \\ Y \end{pmatrix}$ is reduced if and only if $f$ and $g$ are the zero homomorphisms and $A X$ and $B Y$ are reduced.

**Proof** Suppose $\begin{pmatrix} X \\ Y \end{pmatrix}$ be reduced, let $x \in X$, $a \in A$, $y \in Y$, $b \in B$, such that $a^2 x = 0$, $b^2 y = 0$, then

$$
\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}^2 \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Since $\begin{pmatrix} X \\ Y \end{pmatrix}$ is reduced, we have

$$
\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & N \\ M & B \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} a Ax \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$
$$
\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} A & N \\ M & B \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ bBy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

So $A X$ and $B Y$ are reduced.

Let $x \in X$, $n \in N$, $y \in Y$, $m \in M$, as $\begin{pmatrix} X \\ Y \end{pmatrix}$ is reduced, we have

$$
\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}^2 \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

So we have the following two equations

$$
\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & N \\ M & B \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} f(n \otimes x)M \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$
$$
\begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \begin{pmatrix} A & N \\ M & B \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ g(m \otimes y)N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Hence $f = 0$ and $g = 0$.

Conversely, let $\begin{pmatrix} x \\ y \end{pmatrix} \in \begin{pmatrix} X \\ Y \end{pmatrix}$, $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in R$ be such that

$$
\begin{pmatrix} a & n \\ m & b \end{pmatrix}^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a^2 x \\ b^2 y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$
Since $AX$ and $BY$ are reduced, so we have $aAx = 0$, $bBy = 0$. Thus, for any 
\[
\begin{pmatrix}
    a' & n' \\
    m' & b'
\end{pmatrix} \in R,
\]
\[
\begin{pmatrix}
    a & n \\
    m & b
\end{pmatrix}^2
\begin{pmatrix}
    a' & n' \\
    m' & b'
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix} = \begin{pmatrix}
    aa'x + ng(m \otimes x) + af(n' \otimes y) + f(n \otimes y)b' \\
    bb'y + a'g(m' \otimes x) + bf(m' \otimes x) + mf(n' \otimes y)
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix}.
\]

So, the module $\begin{pmatrix} X \\ Y \end{pmatrix}$ over $R$ is reduced.

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References


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