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On Representation Theory of Lie Triple Systems

Asmaa M. Alshahrani

Umm Al-Qura University
College of Applied Sciences
Department of Mathematics
Makkah, Saudi Arabia

Abstract

The present work concerns certain aspects of Lie algebras, Lie triple systems and Jordan triple systems. We summarize the latest results on the universal enveloping algebras of Lie algebras and Lie triple systems. In particular, we study a Casimir element as the element in the center of the universal enveloping algebras. Using this element, we characterize the semi-simple Lie triple systems among the quadratic Lie triple systems. We define Jordan's triple systems relationship with algebras and Lie algebras. Finally, we prove some theorems, examples and facts on all of the above.

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1 Introduction

Associative algebra, Jordan algebra and Lie algebra are three types of algebras. Associative algebra $\mathcal{A}$ has the product $a \cdot b$ such that

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \ \forall\ a,\ b,\ c \in \mathcal{A}.$$

Jordan algebra is $\mathcal{A}$ with the Jordan product $\{a,\ b\} = ab + ba$. The vector space $\mathcal{A}$ over a field $\mathbb{F}$ with the commutator $[a,\ b] = a \boxdot b = ab - ba$ with certain conditions is a Lie algebra. We can say that for any associative algebra of characteristic which is not 2, there is a Jordan algebra if and only if it is commutative. Also, there is a Lie algebra for any associative algebra, which is the commutator as the Lie bracket. There is an associative algebra called universal enveloping algebra which preserves the representation theory for any Lie algebra. On one hand, a subspace $\mathcal{M}$ of an associative algebra $\mathcal{A}$, which is closed under $\{\{a,\ b\},\ c\}$ is called Jordan triple systems and a subspace $\mathcal{M}$ of an associative algebra $\mathcal{A}$, closed under ternary operation $[[a,\ b],\ c]$ is called a Lie triple system. The main motivation of this work is to continue the use of both Lie algebra and Jordan algebra to expand upon these sciences. The starting point is the following three questions:

1. Do all properties of Lie algebras apply to Lie triple systems?
2. Can we give a new characterization of a semi-simple and perfect Lie triple system?
3. Is there a relationship between Jordan triple systems, Lie triple systems and Lie algebras?

In fact, as an algebraic system. The Lie triple system is a natural generalization of Lie algebra. Lie triple systems have arisen initially in Cartan's study of Riemannian geometry, where real simple Lie algebras classifications have been employed to classify an important category of Riemannian manifolds, namely the symmetric spaces. On the other hand, Lister had developed notions of radical, semi-simplicity and solvability as defined for Lie triple systems, including proofs of the existence of a semi-simple subsystem complementary to the radical and of the decomposition of a semi-simple system into the direct sum of simple ideals, in his article: a structure theory of Lie triple systems. In 2002 Zhang and Zhao, they study the invariant symmetric bilinear forms on Lie triple systems. In 2010 Ni, Junna and Jianhua wrote some notes on quadratic Lie triple systems. In 2018 Ahmed Alghamdi and Amir Baklouti studied the representation of semi-simple Jordan and Lie triple systems. In this article, all vector spaces have finite dimensions over the fields $\mathbb{C}$ or $\mathbb{R}$, which are both of course of characteristic zero. The article is divided into seven sections. Starting from section two contains the basic concepts on which the work depends. Section four is devoted to the Lie triple system. Section six contains Jordan triple systems. The main results in this work are the following:
In Section 3, we give several examples of the Casimir element in Lie algebra and solve them using Schur's lemma. In Section 5, we define a universal enveloping algebra of the Lie triple system $\mathcal{L}$ and from it, we define the center of the universal enveloping algebra of the Lie triple system $\mathcal{L}$. The Casimir operator or Casimir element or Casimir invariant is an element in the center of the universal enveloping algebra of the Lie triple system. We define the

Casimir element of the Lie triple system. Some theorems and observations will be proven and mentioned to help understand this concept. We generalization of Schur's lemma in the setting of the Lie triple system. We define the perfect Lie triple system and proved some theorems and facts related to it. Here, we characterize the semi-simple Lie triple systems among the quadratic Lie triple systems using the Casimir element. In Section 7, we define the relationship between algebra, Lie triple system and Jordan triple system. We prove that the special linear Lie algebra of trace equal to zero (traceless) is the Jordan triple system.

2 Lie Algebras

In this section we recall some definitions, notions and facts which can be found in [3], [4] and [5]. The definition of Lie algebra is given below.

Definition 2.1. *A Lie algebra is a vector space L over a field $\mathbb{F}$, with a bilinear map (multiplication) $[*,\ *] : L \times L \to L$ satisfying the following two axioms:*

(L1) For all $a \in L$, we have $[a,\ a] = 0_L$ (alternatively).

(L2) For all $a,\ b,\ c \in L$, we have: $[a,\ [b,\ c]] + [b,\ [c,\ a]] + [c,\ [a,\ b]] = 0_L$ (Jacobi identity).

Definition 2.2. *Let $(L,\ \boxdot)$ be a Lie algebra of finite dimension over a field $\mathbb{F}$. Let $x \in L$ be a fixed element. Define a map $ad_x : L \to L$ as follows:*

$$ad_x(y) = x \boxdot y = [x,\ y].$$

For all $y \in L$. Simply the effect of the map ad_x is to multiply each element y of L on the left by the fixed element x of L. Then ad_x is a linear map from L to L. That is

$$ad_x \in \mathrm{End}_{\mathbb{F}}(L) = \{f : L \to L : f \text{ is an endomorphism } \}.$$

That we have a map which send each x in L to the linear map,

$$ad_x \in \mathrm{End}_{\mathbb{F}}(L) = \{f : L \to L;\ f \ \text{ is an endomorphism } \}.$$

Let us write the new map as

$$Ad : L \to \ End_{\mathbb{F}}(L) = \{f : L \to L :\ f \text{ is an endomorphism } \}.$$

Defined by $Ad(x) = ad_x \in End_{\mathbb{F}}(L)$.

Definition 2.3. *Let L be a finite-dimensional Lie algebra with basis $\{y_1,\ y_2, \cdots,\ y_n\}$ and the scalars C_{ij}^k given by:*

$$[y_i,\ y_j] = \sum_k C_{ij}^k y_k,\ \textit{for } 1 \leqslant i,\ j \leqslant n,\ C_{ij}^k \in \mathbb{F}.$$

Then the universal enveloping algebra is the associative algebra, generated by $y_1,\ y_2,\ \cdots,\ y_n$ subject to the relations

$$[y_i,\ y_j] = y_i y_j - y_j y_i = \sum_k^n C_{ij}^k y_k\ ,\ \textit{for } 1 \leqslant i,\ j \leqslant n,\ C_{ij}^k \in \mathbb{F}.$$

Note: the coefficients C_{ij}^k are called structure constants.

Definition 2.4. *Let L be a Lie algebra over a field $\mathbb{F}$. The center of $U(L)$ is*

$$Z(U(L)) = \{a \in U(L) : [a,\ b] = 0,\ \forall\ b \in U(L)\}\,.$$

Definition 2.5. *Let L be a finite-dimensional Lie algebra over a field $\mathbb{F}$. If there exists a symmetric bilinear form $B : L \times L \longrightarrow \mathbb{F}$ such that:*

$$B(a,\ b) = Tr\ (ad\ a \circ ad\ b),\ \forall\ a,\ b \in L.$$

Then we call B the Killing form of L.

Remark 2.1. *If the Killing form satisfies: $B\Big([a,\ b],\ c\Big) = B\Big(a,\ [b,\ c]\Big),\ \forall\ a,\ b,\ c \in L$, then we call it invariance (associative).*

Lemma 2.1. 1. *The Killing form B is called non-degenerate if and only if the $n\times n$ matrix whose $i,\ j$ entry is $B(x_i,\ x_j)$ has a non-zero determinant.*

2. *The Killing form B is called degenerate if and only if the $n \times n$ matrix whose $i,\ j$ entry is $B(x_i,\ x_j)$ has a zero determinant.*

Proof. The proof can be found in [4]. □

Example 2.1. *We compute the Killing form of $\mathfrak{sl}(2,\ \mathbb{C})$. We know that the basis of $\mathfrak{sl}(2,\ \mathbb{C})$ is*

$$\left\{ e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_{1(-1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

First, we find $ad\ e_{12},\ ad\ e_{21}$, and $ad\ e_{1(-1)}$.

$$\begin{aligned} [e_{12},\ e_{12}] &= e_{12}e_{12} - e_{12}e_{12} = 0e_{12} + 0e_{1(-1)} + 0e_{21}. \\ [e_{12},\ e_{1(-1)}] &= e_{12}e_{1(-1)} - e_{1(-1)}e_{12} = -2e_{12} + 0e_{1(-1)} + 0e_{21}. \\ [e_{12},\ e_{21}] &= e_{12}e_{21} - e_{21}e_{12} = 0e_{12} + e_{1(-1)} + 0e_{21}. \end{aligned}$$

Therefore,

$$\begin{aligned} ad\ e_{12}(e_{12}) &= 0e_{12} + 0e_{1(-1)} + 0e_{21}. \\ ad\ e_{12}(e_{1(-1)}) &= -2e_{12} + 0e_{1(-1)} + 0e_{21}. \\ ad\ e_{12}(e_{21}) &= 0e_{12} + e_{1(-1)} + 0e_{21}. \\ &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} [e_{21},\ e_{12}] &= e_{21}e_{12} - e_{12}e_{21} = 0e_{12} - e_{1(-1)} + 0e_{21}. \\ [e_{21},\ e_{1(-1)}] &= e_{21}e_{1(-1)} - e_{1(-1)}e_{21} = 0e_{12} + 0e_{1(-1)} + 2e_{21}. \\ [e_{21},\ e_{21}] &= e_{21}e_{21} - e_{21}e_{21} = 0e_{12} + 0e_{1(-1)} + 0e_{21}. \end{aligned}$$

Consequently,

$$\begin{aligned} ad\ e_{21}(e_{12}) &= 0e_{12} - e_{1(-1)} + 0e_{21}. \\ ad\ e_{21}(e_{1(-1)}) &= 0e_{12} + 0e_{1(-1)} + 2e_{21}. \\ ad\ e_{21}(e_{21}) &= 0e_{12} + e_{1(-1)} + 0e_{21}. \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} [e_{1(-1)},\ e_{12}] &= e_{1(-1)}e_{12} - e_{12}e_{1(-1)} = 2e_{12} + 0e_{1(-1)} + 0e_{21}. \\ [e_{1(-1)},\ e_{1(-1)}] &= e_{1(-1)}e_{1(-1)} - e_{1(-1)}e_{1(-1)} = 0e_{12} + 0e_{1(-1)} + 0e_{21}. \\ [e_{1(-1)},\ e_{21}] &= e_{1(-1)}e_{21} - e_{21}e_{1(-1)} = 0e_{12} + 0e_{1(-1)} - 2e_{21}. \end{aligned}$$

So,

$$\begin{aligned} ad\ e_{1(-1)}(e_{12}) &= 2e_{12} + 0e_{1(-1)} + 0e_{21}. \\ ad\ e_{1(-1)}(e_{1(-1)}) &= 0e_{12} + 0e_{1(-1)} + 0e_{21}. \\ ad\ e_{1(-1)}(e_{21}) &= 0e_{12} + 0e_{1(-1)} - 2e_{21}. \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

Now, we find the Killing form:

$$B = \begin{array}{cc} \begin{array}{ccc} \downarrow e_{12} & \downarrow e_{1(-1)} & \downarrow e_{21} \end{array} & \\ \begin{pmatrix} Tr\ (ad\ e_{12} \circ ad\ e_{12}) & Tr\ (ad\ e_{12} \circ ad\ e_{1(-1)}) & Tr\ (ad\ e_{12} \circ ad\ e_{21}) \\ Tr\ (ad\ e_{1(-1)} \circ ad\ e_{12}) & Tr\ (ad\ e_{1(-1)} \circ ad\ e_{1(-1)}) & Tr\ (ad\ e_{1(-1)} \circ ad\ e_{21}) \\ Tr\ (ad\ e_{21} \circ ad\ e_{12}) & Tr\ (ad\ e_{21} \circ ad\ e_{1(-1)}) & Tr\ (ad\ e_{21} \circ ad\ e_{21}) \end{pmatrix} & \begin{array}{l} \leftarrow e_{12} \\ \leftarrow e_{1(-1)} \\ \leftarrow e_{21} \end{array} \end{array}$$

$$= \begin{pmatrix} Tr\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 0 \end{pmatrix} \\ Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

We want to find the determinant of matrix B to determine if it is degenerate or non-degenerate.

$$|B| = \begin{vmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{vmatrix} = 0\begin{vmatrix} 8 & 0 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 4 & 0 \end{vmatrix} + 4\begin{vmatrix} 0 & 8 \\ 4 & 0 \end{vmatrix} = 4(-32) = -128 \neq 0.$$

Hence, B is non-degenerate.

We review the definition of representation of Lie algebra L over a field $\mathbb{F}$. Let V be a vector space. The linear representation of this Lie algebra is

$$\rho : L \to End_{\mathbb{F}}(V) \text{ or } \rho : L \to \mathfrak{gl}_{\mathbb{F}}(V),$$

such that $\rho([x,\ y]) = [\rho(x),\ \rho(y)] = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x),\ \forall\ x,\ y \in L$.

This part will go over the fundamental knowledge that we will use to find the Casimir operator of Lie algebra. The Casimir operator or Casimir element or Casimir invariant is an element in the center of the universal enveloping algebra of a Lie algebra.

Definition 2.6. *Let L be a finite-dimensional semi-simple Lie algebra over a field $\mathbb{F}$. Let $U(L)$ be the universal enveloping algebra of L and $B : L \times L \to \mathbb{F}$ be the Killing form on L, such that B is non-degenerate. Let $\{x_i : i = 1, \cdots,\ dim_{\mathbb{F}}(L)\}$ be a basis of L and $\{y_i : i = 1, \cdots,\ dim_{\mathbb{F}}(L)\}$ be the dual basis of L, defined by*

$$B(x_i,\ y_j) = \delta_{ij} = \begin{cases} 1 & if\ \ i = j \\ 0 & if\ \ i \neq j. \end{cases}$$

The element $C := \displaystyle\sum_{i=1}^{dim_{\mathbb{F}}(L)} x_i y_i$ of $U(L)$ is called the Casimir element of the Lie algebra L.

Definition 2.7. *Let $\rho : L \to End_{\mathbb{F}}(V)$ be a faithful representation of L on a vector space V. The Casimir invariant of ρ is defined by the map $C : V \to V$ such that*

$$\rho(C) = \sum_{i=1}^{dim_{\mathbb{F}}(L)} \rho(x_i) \circ \rho(y_i).$$

Theorem 2.1. *Let L be a Lie algebra over the field $\mathbb{C}$. Let C be the Casimir operator $C : V \to V$ of the representation of L. Then the commutator*

$$[C,\ \rho(a)] = 0_{U(L)},\ \forall\ a \in L.$$

That is the element C commutes with every endomorphism in $Im(L)$.

Proof. Note that the bracket is that on $End_{\mathbb{C}}(V)$. Then $[C,\ \rho(a)] = \left[\sum_{i,\ j=1}^{\dim_{\mathbb{C}}(L)} \rho(x_i) \circ \rho(y_j),\ \rho(a) \right]$. Hence,

$$\begin{aligned} C\rho(a) - \rho(a)C &= \sum_{i=1}^{\dim_{\mathbb{C}}(L)} \rho(x_i) \circ \rho(y_j) \circ \rho(a) - \rho(a) \circ \rho(x_i) \circ \rho(y_j) \\ &= \sum_{i=1}^{\dim_{\mathbb{C}}(L)} \rho(x_i) \circ [\rho(y_j), \rho(a)] - [\rho(a), \rho(x_i)] \circ \rho(y_j) \\ &= \sum_{i=1}^{\dim_{\mathbb{C}}(L)} \rho(x_i) \circ \rho([y_j, a]) - \rho([a, x_i]) \circ \rho(y_j) \end{aligned}$$

If $[y_j,\ a] = \sum_{i=1}^{\dim_{\mathbb{C}}(L)} n_{ij} y_i$ and $[a,\ x_i] = \sum_{j=1}^{\dim_{\mathbb{C}}(L)} m_{ij} x_j$, since the Killing form B is invariant. Then

$$n_{ij} = B([y_j,\ a],\ x_i) = (y_j,\ [a,\ x_i]) = m_{ij}.$$

Hence,

$$\begin{aligned} C\rho(a) - \rho(a)C &= \sum_{i,\ j=1}^{\dim_{\mathbb{C}}(L)} n_{ij} \rho(x_i) \circ \rho(y_i) - m_{ij} \rho(x_j) \circ \rho(y_j) \\ &= \sum_{i,\ j=1}^{\dim_{\mathbb{C}}(L)} n_{ij} \rho(x_i) \circ \rho(y_j) - \sum_{i,\ j=1}^{\dim_{\mathbb{C}}(L)} m_{ij} \rho(x_i) \circ \rho(y_j) \\ &= 0. \end{aligned}$$

□

The following result Lemma is a generalization of Schur's in the Lie algebra setting.

Lemma 2.2. *Let L be a Lie algebra over the field $\mathbb{C}$. If the representation ρ with module V of L is irreducible, then any operator M which commutes with every endomorphism, can be written as follows:*

$$M = c_\rho .\ id_V,$$

for some non-zero constant $c_\rho \in \mathbb{C}$.

We apply the Lemma 2.2 in the examples. The following proposition gives the constant c_ρ the development of Schur's Lemma.

Proposition 2.2. *The Casimir operator of the representation $\rho : L \to End_{\mathbb{F}}(V)$ of a Lie algebra over a field $\mathbb{F}$ is $C_\rho = c_\rho .\ id_V$, where $c_\rho = \dfrac{dim_{\mathbb{F}}(L)}{dim_{\mathbb{F}}(V)}$.*

Proof. We have: $Tr(C) = Tr(c_\rho .\ id_V) = c_\rho . Tr(id_V) = c_\rho .\ dim_{\mathbb{F}}(V)$. (1)
Also,

$$\begin{aligned} Tr(C) &= Tr\Big(\sum_{i=1}^{dim_{\mathbb{F}}(L)} \rho(x_i) \circ \rho(y_i) \Big) \\ &= \sum_{i=1}^{dim_{\mathbb{F}}(L)} B(x_i,\ y_i) \\ &= \sum_{i=1}^{dim_{\mathbb{F}}(L)} \delta_{ii} = dim_{\mathbb{F}}(L). \end{aligned} \tag{2}$$

Hence, by (1) and (2), $dim_{\mathbb{F}}(L) = c_\rho .\ dim_{\mathbb{F}}(V) \Rightarrow c_\rho = \dfrac{dim_{\mathbb{F}}(L)}{dim_{\mathbb{F}}(V)}$. □

3 Main Results of This Section

In this part, we give several examples of the Casimir operator of Lie algebra.

Example 3.1. *The Casimir operator of the special unitary group $SU(2)$. We can find the Casimir operator by Schur's Lemma as follows:*

$$\rho_{spin} : SU(2) \to End_{\mathbb{R}}(\mathbb{C}^2) \cong Mat(2,\ \mathbb{C})$$
$$\rho_{spin}(\sigma_x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \rho_{spin}(\sigma_y) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \rho_{spin}(\sigma_z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These three matrices are a basis of the associated $\mathfrak{su}(2)$ *over* $\mathbb{R}$. *Now, we study the Killing form:*

$$\begin{aligned}
B_{\rho_{spin}}(\sigma_x,\ \sigma_x) &= Tr\big(\rho_{spin}(\sigma_x)\circ\ \rho_{spin}(\sigma_x)\big) = Tr\big(\rho_{spin}(\sigma_x)^2\big)\\
&= Tr(\sigma_x)^2 = Tr\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^2 = Tr\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = Tr(id_{\mathbb{C}^2}) = 4\\
B_{\rho_{spin}}(\sigma_y,\ \sigma_y) &= Tr\big(\rho_{spin}(\sigma_y)\circ\ \rho_{spin}(\sigma_y)\big) = Tr\big(\rho_{spin}(\sigma_y)^2\big)\\
&= Tr(\sigma_y)^2 = Tr\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}^2 = Tr\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = Tr(id_{\mathbb{C}^2}) = 4\\
B_{\rho_{spin}}(\sigma_z,\ \sigma_z) &= Tr\big(\rho_{spin}(\sigma_z)\circ\ \rho_{spin}(\sigma_z)\big) = Tr\big(\rho_{spin}(\sigma_z)^2\big)\\
&= Tr(\sigma_z)^2 = Tr\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 = Tr\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = Tr(id_{\mathbb{C}^2}) = 4.
\end{aligned}$$

So, $B_{\rho_{spin}} = \begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}$. *We use that because*

$$B_{\rho_{spin}}(\sigma_x,\ \sigma_y) = Tr\big(\rho_{spin}(\sigma_x)\circ\ \rho_{spin}(\sigma_y)\big) = 0 \quad (i \neq j).$$

The rest of the computation is in the same way.
The dual basis is $y_i = \dfrac{1}{4}\sigma_i$. *We find the Casimir operator of* $\mathfrak{su}(2)$ *as follows:*

$$\begin{aligned}
C_{\rho_{spin}} &= \sum_{i=1}^{3}\rho_{spin}(\sigma_i)\circ\rho_{spin}(y_i) = \sum_{i=1}^{3}\rho_{spin}(\sigma_i)\circ\rho_{spin}(\frac{1}{4}\sigma_i)\\
&= \frac{1}{4}\sum_{i=1}^{3}\rho_{spin}(\sigma_i)^2 = \frac{1}{4}\Big(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^2 + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}^2 + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2\Big)\\
&= \frac{1}{4}\Big(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big) = \frac{3}{4}(id_{\mathbb{C}^2}).
\end{aligned}$$

Therefore, $C_{\rho_{spin}} = c_{\rho_{spin}}.\ id_{\mathbb{C}^2},\ c_{\rho_{spin}} = \dfrac{dim_{\mathbb{F}}(L)}{dim_{\mathbb{F}}(V)} = \dfrac{dim\ \mathfrak{su}(2,\ \mathbb{C})}{dim_{\mathbb{R}}\mathbb{C}^2} = \dfrac{3}{4}$.

Example 3.2. *The Casimir operator of the special unitary group $SU(3)$. We have:*

$\rho_{spin} : SU(3) \to End_{\mathbb{C}}(\mathbb{C}^3)$

$$\rho_{spin}(\lambda_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho_{spin}(\lambda_2) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho_{spin}(\lambda_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rho_{spin}(\lambda_4) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \rho_{spin}(\lambda_5) = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \rho_{spin}(\lambda_6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rho_{spin}(\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \rho_{spin}(\lambda_8) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We study the Killing form:

$$\begin{aligned}
B_{\rho_{spin}}(\lambda_1,\ \lambda_1) &= Tr\big(\rho_{spin}(\lambda_1) \circ\ \rho_{spin}(\lambda_1)\big) = Tr\big(\rho_{spin}(\lambda_1)^2\big) = Tr(\lambda_1)^2 \\
&= Tr \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \\
B_{\rho_{spin}}(\lambda_2,\ \lambda_2) &= Tr\big(\rho_{spin}(\lambda_2) \circ\ \rho_{spin}(\lambda_2)\big) = Tr\big(\rho_{spin}(\lambda_2)^2\big) = Tr(\lambda_2)^2 \\
&= Tr \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \\
B_{\rho_{spin}}(\lambda_3,\ \lambda_3) &= Tr\big(\rho_{spin}(\lambda_3) \circ\ \rho_{spin}(\lambda_3)\big) = Tr\big(\rho_{spin}(\lambda_3)^2\big) = Tr(\lambda_3)^2 \\
&= Tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2 \\
B_{\rho_{spin}}(\lambda_4,\ \lambda_4) &= Tr\big(\rho_{spin}(\lambda_4) \circ\ \rho_{spin}(\lambda_4)\big) = Tr\big(\rho_{spin}(\lambda_4)^2\big) = Tr(\lambda_4)^2 \\
&= Tr \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 \\
B_{\rho_{spin}}(\lambda_5,\ \lambda_5) &= Tr\big(\rho_{spin}(\lambda_5) \circ\ \rho_{spin}(\lambda_5)\big) = Tr\big(\rho_{spin}(\lambda_5)^2\big) = Tr(\lambda_5)^2 \\
&= Tr \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 \\
B_{\rho_{spin}}(\lambda_6,\ \lambda_6) &= Tr\big(\rho_{spin}(\lambda_6) \circ\ \rho_{spin}(\lambda_6)\big) = Tr\big(\rho_{spin}(\lambda_6)^2\big) = Tr(\lambda_6)^2 \\
&= Tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 \\
B_{\rho_{spin}}(\lambda_7,\ \lambda_7) &= Tr\big(\rho_{spin}(\lambda_7) \circ\ \rho_{spin}(\lambda_7)\big) = Tr\big(\rho_{vec}(\lambda_7)^2\big) = Tr(\lambda_7)^2 \\
&= Tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}^2 = Tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2
\end{aligned}$$

$$\begin{aligned}
B_{\rho_{spin}}(\lambda_8,\ \lambda_8) &= Tr\big(\rho_{spin}(\lambda_8)\circ\ \rho_{spin}(\lambda_8)\big) = Tr\big(\rho_{spin}(\lambda_8)^2\big) = Tr(\lambda_8)^2 \\
&= (\frac{1}{\sqrt{3}})^2 Tr\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}^2 = (\frac{1}{3})Tr\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{pmatrix} = 2.
\end{aligned}$$

So, $B_{\rho_{spin}} = \begin{pmatrix}2&0&0&0&0&0&0&0\\0&2&0&0&0&0&0&0\\0&0&2&0&0&0&0&0\\0&0&0&2&0&0&0&0\\0&0&0&0&2&0&0&0\\0&0&0&0&0&2&0&0\\0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&2\end{pmatrix}$. *We use that because*

$$B_{\rho_{spin}}(\lambda_1,\ \lambda_2) = Tr\big(\rho_{spin}(\lambda_1)\circ\ \rho_{spin}(\lambda_2)\big) = 0 \quad (i \neq j).$$

The rest is the same way.
The dual basis is $y_i = \frac{1}{2}\lambda_i$. *Now, we find the Casimir operator of* $\mathfrak{su}(3)$ *as follows:*

$$\begin{aligned}
C_{\rho_{spin}} &= \sum_{i=1}^{8}\rho_{spin}(\lambda_i)\circ\rho_{spin}(y_i) \\
&= \sum_{i=1}^{8}\rho_{spin}(\lambda_i)\circ\rho_{spin}(\frac{1}{2}\lambda_i) \\
&= \frac{1}{2}\sum_{i=1}^{8}\rho_{spin}(\lambda_i)^2 = \frac{1}{2}\Big(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2+\lambda_6^2+\lambda_7^2+\lambda_8^2\Big) \\
&= \frac{1}{2}\begin{pmatrix}\frac{16}{3} & 0 & 0\\ 0 & \frac{16}{3} & 0\\ 0 & 0 & \frac{16}{3}\end{pmatrix} = \frac{8}{3}I_3.
\end{aligned}$$

Hence, $C_{\rho_{spin}} = c_{\rho_{spin}}.\ id_{\mathbb{C}^3},\ c_{\rho_{spin}} = \dfrac{dim_{\mathbb{F}}(L)}{dim_{\mathbb{F}}(V)} = \dfrac{dim\ \mathfrak{su}(3,\ \mathbb{R})}{dim_{\mathbb{C}}\mathbb{C}^3} = \dfrac{8}{3}$.

4 Lie Triple Systems

This section is devoted to the study of perfect Lie triple systems and the Casimir operator of Lie triple systems, in which we prove some associated facts. We refer to the following [1], [2] and [6].

Definition 4.1. *A vector space $\mathcal{L}$ over a field $\mathbb{F}$, with trilinear mapping:*

$$[*,\ *,\ *] : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

is a Lie triple system if all three of the following criteria are met:

$(Lt1)$*For all $a,\ b \in \mathcal{L},$ we have $[a,\ a,\ b] = 0_{\mathcal{L}}$, (alternatively).*
$(Lt2)$*For all $a,\ b,\ c \in \mathcal{L},$ we have $[a,\ b,\ c] + [b,\ c,\ a] + [c,\ a,\ b] = 0_{\mathcal{L}}$, (generalized Jacobi identity).*
$(Lt3)$*For all $u,\ v,\ a,\ b,\ c \in \mathcal{L}$, we have:*

$$[u,\ v,\ [a,\ b,\ c]] = [[u,\ v,\ a],\ b,\ c] + [a,\ [u,\ v,\ b],\ c] + [a,\ b,\ [u,\ v,\ c]],$$

(principal identity).

Lemma 4.1. *Let $\mathcal{L}$ be a Lie algebra over a field $\mathbb{F}$, with product $(a,\ b) \longmapsto [a,\ b]$ then $\mathcal{L}$ accompanied by $[a,\ b,\ c] \longmapsto [[a,\ b],\ c]$ is a Lie triple system over a field $\mathbb{F}$.*

Definition 4.2. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. The subspace $\mathfrak{h} \subseteq \mathcal{L}$ as well as $[\mathfrak{h},\ \mathcal{L},\ \mathcal{L}] \subseteq \mathfrak{h}$ is called an ideal of $\mathcal{L}$.*

Definition 4.3. *A simple Lie triple system $\mathcal{L}$ over a field $\mathbb{F}$ is a non-abelian Lie triple system, which has two and only two ideals, namely $\{0_{\mathcal{L}}\}$ and $\mathcal{L}$.*

Definition 4.4. *(These definitions are equivalent). If $\mathcal{L}$ is a finite-dimensional of a Lie triple system over a field $\mathbb{F}$ then:*

1. *$\mathcal{L}$ is called semi-simple if $\mathcal{L}$ is a direct sum of simple ideals.*
2. *$\mathcal{L}$ is called semi-simple if the Killing form is non-degenerate on $\mathcal{L}$ (see Definition 4.11).*
3. *$\mathcal{L}$ is called semi-simple if it has no non-zero solvable ideal (see Definition 4.5).*
4. *$\mathcal{L}$ is semi-simple if $\mathfrak{R}(\mathcal{L}) = 0$, (the maximal solvable ideal is called radical).*

For our purposes, we state the following definitions:

Definition 4.5. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$:*

1. *We define the ascending series $\mathcal{L}^{(n)}$ by $\mathcal{L}^{(0)} = \mathcal{L}$ and $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)},\ \mathcal{L}^{(n)},\ \mathcal{L}^{(n)}],\ \forall\ n \in \mathbb{N}$.*

$$\mathcal{L} = \mathcal{L}^{(0)} \supset \mathcal{L}^{(1)} \supset \mathcal{L}^{(2)} \supset \cdots$$

2. *Let B be an ideal in $\mathcal{L}$, we called B is solvable if there exist $n \in \mathbb{N}$ such that $B^{(n)} = 0_{\mathcal{L}}$.*

Quadratic Lie triple systems, on the other hand, are a generalization of quadratic Lie algebras. The definition of a quadratic Lie triple system is given below.

Definition 4.6. *A Lie triple system $\mathcal{L}$ over a field $\mathbb{F}$ is said to be quadratic, if $(\mathcal{L},\ [*,\ *,\ *])$ has a non-degenerate symmetric bilinear form B which satisfies:*

$$B([x,\ y,\ z],\ u) = B(z,\ [y,\ x,\ u]),\ \forall\ x,\ y,\ z,\ u \in \mathcal{L}.$$

In this case, B is said to be an invariant scalar product.

Definition 4.7. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. For $a,\ b \in \mathcal{L}$, we specify the left and right multiplications of $\mathcal{L}$:*

1. *The left multiplication of $\mathcal{L}$ is defined by $L(a,\ b)x = [a,\ b,\ x]$, $\forall\ x \in \mathcal{L}$.*

2. *The right multiplication of $\mathcal{L}$ is defined by $R(a,\ b)x = [x,\ a,\ b]$, $\forall\ x \in \mathcal{L}$.*

Definition 4.8. 1. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. Let $B : \mathcal{L} \times \mathcal{L} \to \mathbb{F}$ be a bi-linear form, an endomorphism f of $\mathcal{L}$ is called B-symmetric if*

$$B\big(f(a),\ b\big) = B\big(a,\ f(b)\big),\ \forall\ a,\ b \in \mathcal{L}.$$

2. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. Let $B : \mathcal{L} \times \mathcal{L} \to \mathbb{F}$ be a bi-linear form, an endomorphism f is of $\mathcal{L}$ called B-skew symmetric if*

$$B\big(f(a),\ b\big) = -B\big(a,\ f(b)\big),\ \forall\ a,\ b \in \mathcal{L}.$$

Definition 4.9. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. We consider the bilinear form*

$$B : \mathcal{L} \times \mathcal{L} \to \mathbb{F}$$

is said to be left invariant if $B(L(u,\ v)w,\ x) = B(x,\ L(v,\ u)w)$, $\forall\ x,\ u,\ v,\ w \in \mathcal{L}$. Where

$$L(u,\ v) : w \to [u,\ v,\ w]\ \in \mathcal{L}$$
$$L(v,\ u) : w \to [v,\ u,\ w]\ \in \mathcal{L}$$

Definition 4.10. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. We have taken into account the bilinear form*

$$B : \mathcal{L} \times \mathcal{L} \to \mathbb{F}$$

is said to be right invariant if $B(R(u,\ v)w,\ x) = B(x,\ R(v,\ u)w)$, $\forall\ x,\ u,\ v,\ w \in \mathcal{L}$. Where

$$R(u,\ v) : w \to [w,\ u,\ v]\ \in \mathcal{L}$$
$$R(v,\ u) : w \to [w,\ v,\ u]\ \in \mathcal{L}$$

Remark 4.1. *If B is both left and right invariant, then we say that B is an invariant bilinear form.*

Definition 4.11. *Let $\mathcal{L}$ be a Lie triple system over a field $\mathbb{F}$. Let $B : \mathcal{L}\times\mathcal{L} \to \mathbb{F}$ be a bilinear form on $\mathcal{L}$. We define the Killing form of the Lie triple system by:*

$$B(a,\ b) = \frac{1}{2} Tr\big(R(a,\ b) + R(b,\ a)\big),\ \forall\ a,\ b \in \mathcal{L}.$$

Note that: $B(a,\ b) = B(b,\ a)$ which means that B is symmetric.

5 Main Results of This Section

In the following, we present the results of the Lie triple system.

Definition 5.1. *Let $\mathcal{L}$ be a Lie triple system over the field $\mathbb{C}$. We can define a universal enveloping algebra of the Lie triple system, denoted by $U(\mathcal{L})$. Let $\mathcal{L}$ be finite-dimensional with basis $\{X_1, \cdots,\ X_n\}$ and the scalars a_{ij}^k (structure constants) given by*

$$[X_i,\ X_j,\ X_k] = \sum_r a_{ij}^k X_r,\ \textit{for}\ 1 \leqslant i,\ j,\ k \leqslant n,\ a_{ij}^k \in \mathbb{C}.$$

Then $U(\mathcal{L})$ can be defined as the unital associative algebra, on generators $X_1, \cdots,\ X_n$ subject to the relations

$$X_i X_j X_k - X_k X_j X_i = \sum_{r=1}^{n} a_{ij}^k X_r \quad ,\ \textit{for}\ 1 \leqslant i,\ j,\ k \leqslant n,\ a_{ij}^k \in \mathbb{C}.$$

Definition 5.2. *Let $\mathcal{L}$ be a Lie triple system over the field $\mathbb{C}$. The center of $U(\mathcal{L})$ is*

$$Z(U(\mathcal{L})) = \{a \in U(\mathcal{L}) : [a,\ b,\ c] = 0,\ \forall\ b,\ c \in U(\mathcal{L})\}.$$

The Casimir operator or Casimir element or Casimir invariant is an element in the center of the universal enveloping algebra of a Lie triple system. In the following, we define the Casimir element of the Lie triple systems.

Definition 5.3. *An endomorphism which commutes with all elements of a Lie triple system $\mathcal{L}$ over the field $\mathbb{C}$ is called Casimir operator if there exists $C \in U(\mathcal{L})$ such that*

$$[C,\ x,\ y] = 0,\ \forall\ x,\ y \in U(\mathcal{L}).$$

The endomorphism is called of order s if it is built from products of s elements;

$$C_s = \sum_{\beta_1,\ \beta_2, \cdots,\ \beta_s} f_{\beta_1,\ \beta_2, \cdots,\ \beta_s} X_1^{\beta_1} \cdots X_n^{\beta_s}.$$

Where $\{X_i : i \in \{1, \cdots, n\}\}$ is a basis of $\mathcal{L}$.
(Equivalently, let $\mathcal{L}$ be a finite-dimensional semi-simple Lie triple system over the field $\mathbb{C}$. Let $U(\mathcal{L})$ be the universal enveloping algebra of Lie triple system and $B : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ the Killing form on $\mathcal{L}$, such that B is non-degenerate. Let $\{x_i : i = 1, \cdots, dim_{\mathbb{C}}(\mathcal{L})\}$ be a basis of $\mathcal{L}$, $\{y_i : i = 1, \cdots, dim_{\mathbb{C}}(\mathcal{L})\}$ the dual basis of $\mathcal{L}$ and $\{z_i : i = 1, \cdots, dim_{\mathbb{C}}(\mathcal{L})\}$ the double dual basis of $\mathcal{L}$ defined by

$$B(x_i,\ y_j,\ z_k) = \frac{1}{2}\delta_{ij}\delta_{jk} = \frac{1}{2}\begin{cases} 1 & if \quad i = j = k \\ 0 & if \quad i \neq j \neq k. \end{cases}$$

The element of $U(\mathcal{L})$

$$C := \sum_{i=1}^{dim_{\mathbb{C}}(\mathcal{L})} x_i y_i z_i,\ \forall\ x_i \in \mathcal{L},\ y_i \in \mathcal{L}^*,\ z_i \in \mathcal{L}^{**} = \mathcal{L},$$

is called the Casimir operator of the Lie triple system $\mathcal{L}$).

Theorem 5.1. *Let C be the Casimir element of the Lie triple system $\mathcal{L}$. Then $C \in Z(U(\mathcal{L}))$.*

Lemma 5.1. *The Casimir element C satisfies $Cg = gC$, for all g in the Lie triple system $\mathcal{L}$.*

Proof. The proof is clear by Definition 5.3. □

Proposition 5.2. *For $g \in \mathcal{L}$ and $i \in \{1, \cdots, n\}$, consider the decompositions:*

$$[x_i,\ y_i,\ z_i] = \sum_{j=1}^{n} a_{ij} x_j y_j z_j,\ \ [y_i,\ z_i,\ g] = \sum_{j=1}^{n} b_{ij} y_j z_j$$

$$[z_i,\ g,\ x_i] = \sum_{j=1}^{n} c_{ij} z_j x_j, \quad [g,\ x_i,\ y_i] = \sum_{j=1}^{n} d_{ij} x_j y_j.$$

Then $a_{ij} = b_{ji},\ b_{ij} = c_{ji},\ c_{ij} = d_{ji}$.

Proof. Using the associativity of Killing form B, we have:

$$\begin{aligned} a_{ij} &= B\left([x_i,\ y_i,\ z_i],\ g\right) = B\left(x_i,\ [y_j,\ z_j,\ g]\right) = b_{ji} \\ b_{ij} &= B\left([y_i,\ z_i,\ g],\ x_j\right) = B\left(y_i, [z_j,\ g,\ x_j]\right) = c_{ji} \\ c_{ij} &= B\left([z_i,\ g,\ x_i],\ y_j\right) = B\left(z_i,\ [g,\ x_j,\ y_j]\right) = d_{ji}. \end{aligned}$$

□

Definition 5.4. *Let ρ be a faithful representation $\rho : \mathcal{L} \to End_{\mathbb{C}}(V^*)$ of $\mathcal{L}$ on a vector space V. The Casimir operator of ρ defined by: $C : V \to V$ such that*

$$\rho(C) = \sum_{i=1}^{\dim_{\mathbb{C}}(\mathcal{L})} \rho(x_i) \circ \rho(y_i) \circ \rho(z_i).$$

Lemma 5.2. *The Casimir invariant C does not depend on the choice of the basis $\{x_i\}$ (meaning that it is independent of the basis).*

Proof. We write basis $\{a_i\}$ as the column vector a. Let A be the transformation matrix from x to some other basis x'. Let D be the transformation matrix from y to the dual basis y' of x'. Let P be the transformation matrix from z to the double dual basis z' of x'. Then y' is dual to x' means $D^t = A^{-1}$ and z' is the double dual basis to x' means $P^{t^t} = A$. Hence, $C = \mathrm{Tr}(x(y)^t)$, because $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$. Consequently,

$$C' = \mathrm{Tr}(x'(y')^t) = \mathrm{Tr}(Ax(Dy)^t) = \mathrm{Tr}(Ax(y^t)D^t) = \mathrm{Tr}(x(y^t)D^tA) = \mathrm{Tr}(x(y^t)) = C.$$

$\square$

Theorem 5.3. *Let $\mathcal{L}$ be a Lie triple system over the field $\mathbb{C}$. Let C be the Casimir operator of a representation ρ of $\mathcal{L}$. Then*

$$[C,\ \rho(a),\ \rho(b)] = 0_{U(\mathcal{L})},\ \forall\ a,\ b \in \mathcal{L}.$$

This means that the element C commutes with every endomorphism of $Im(\mathcal{L})$.

Proof. If the Lie triple system $\mathcal{L}$ becomes of the Lie algebra L (i.e. $[A,\ B,\ C] = [[A,\ B],\ C]$), then by Theorem 2.1 $[C,\ \rho(a),\ \rho(b)] = [[C,\ \rho(a)],\ \rho(b)] = [0,\ \rho(b)] = 0$. $\square$

In the following, we start with a generalization of Schur's Lemma in the setting of Lie triple system and then show some of its details.

Lemma 5.3. *Let $\mathcal{L}$ be a Lie triple system over the field $\mathbb{C}$. If the representation ρ with module V of $\mathcal{L}$ is irreducible, then any operator M which commutes with every endomorphism can be written as:*

$$M = c_\rho \cdot id_V,$$

(for some non-zero constant $c_\rho \in \mathbb{C}$).

Proposition 5.4. *The Casimir operator of a representation ρ of a Lie triple system $\mathcal{L}$ over the field $\mathbb{C}$ is $C_\rho = c_\rho \cdot id_V$, where $c_\rho = \dfrac{1}{2}\dfrac{\dim_{\mathbb{C}}(\mathcal{L})}{\dim_{\mathbb{C}}(V)}$ and c_ρ comes from Schur's Lemma.*

Proof. We have $Tr(C) = Tr(c_\rho.\, id_V) = c_\rho.Tr(id_V) = c_\rho.\, dim_{\mathbb{C}}(V)$. (1)
Also, we can say that

$$\begin{aligned} Tr(C) &= Tr\Big(\sum_{i=1}^{dim_{\mathbb{C}}(\mathcal{L})} \rho(x_i)\circ\rho(y_i)\circ\rho(z_i)\Big) \\ &= \sum_{i=1}^{dim_{\mathbb{C}}(\mathcal{L})} B(x_i,\ y_i,\ z_i) \\ &= \frac{1}{2}\sum_{i=1}^{dim_{\mathbb{C}}(\mathcal{L})} \delta_{ii} = \frac{1}{2}dim_{\mathbb{C}}(\mathcal{L}). \end{aligned} \tag{2}$$

Hence, by (1) and (2), $dim_{\mathbb{C}}(\mathcal{L}) = \frac{1}{2}c_\rho.\, dim_{\mathbb{C}}(V) \Rightarrow c_\rho = \frac{1}{2}\frac{dim_{\mathbb{C}}(\mathcal{L})}{dim_{\mathbb{C}}(V)}$. □

In this part, we study the perfect Lie triple systems. We mention its definition and then we follow it with some facts and theorems. We circulated the scientific paper in the reference [2].

Definition 5.5. *The Lie triple system $\mathcal{L}$ over a field $\mathbb{F}$ is called perfect if*

$$[\mathcal{L},\ \mathcal{L},\ \mathcal{L}] = \mathcal{L}.$$

The following lemma is the motivation of the above definition.

Lemma 5.4. *The Lie triple system $\mathcal{L}$ is perfect if $[\mathcal{L},\ \mathcal{L},\ \mathcal{L}] = \mathcal{L}$.*

Proof. We can be sure assuming that $\mathcal{L}$ be a semi-simple Lie triple system. Then $\mathcal{L}$ can be written as a direct sum of simple ideals. We can write $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \cdots$, where $\mathcal{L}_1,\ \mathcal{L}_2,\ \mathcal{L}_3$ are simple ideals.

$$\begin{aligned} [\mathcal{L},\ \mathcal{L},\ \mathcal{L}] &= [\mathcal{L}_1\oplus\mathcal{L}_2\oplus\ \mathcal{L}_3,\ \mathcal{L}_1\oplus\mathcal{L}_2\oplus\ \mathcal{L}_3,\ \mathcal{L}_1\oplus\mathcal{L}_2\oplus\ \mathcal{L}_3] \\ &= [\mathcal{L}_1,\ \mathcal{L}_1,\ \mathcal{L}_1]\oplus[\mathcal{L}_2,\ \mathcal{L}_2,\ \mathcal{L}_2]\oplus[\mathcal{L}_3,\ \mathcal{L}_3,\ \mathcal{L}_3] \\ &= \mathcal{L}_1\oplus\mathcal{L}_2\oplus\mathcal{L}_3 = \mathcal{L}. \end{aligned}$$

Then $\mathcal{L}_i = [\mathcal{L}_i,\ \mathcal{L}_i,\ \mathcal{L}_i]$, $\forall\, i = 1,\ 2,\ 3$ and $\mathcal{L}_i$ are simple ideals. So, $[\mathcal{L},\ \mathcal{L},\ \mathcal{L}] = \mathcal{L}$. □

Lemma 5.5. *Let $\mathcal{L}$ be a semi-simple Lie triple system over a field $\mathbb{F}$. Let B be an invariant scalar product on $\mathcal{L}$. Then C is invertible, where C is the Casimir element of the Lie triple system.*

In the following theorem, we will use the concept of Levi subalgebra of $\mathcal{L}$ (notation S) which is the semidirect product of the solvable Lie triple system and the semisimple Lie triple system.

Theorem 5.5. *Let $\mathcal{L}$ be a perfect Lie triple system over a field $\mathbb{F}$, without any non-zero semi-simple ideal. Let $\mathfrak{R}$ be its radical. If $\mathcal{L}$ is quadratic, then $\mathfrak{R}^{\perp} = Z(\mathfrak{R})$.*

Proof. Let B be a non-degenerate symmetric bi-linear form (B invariant scalar product of $\mathcal{L}$). Suppose that u in $\mathfrak{R}$ and $v,\ w \in Z(\mathfrak{R})$. Then $u = \sum_{i=1}^{n} [x_i,\ y_i,\ r_i]$ where $r_i \in \mathfrak{R}$ and $x_i,\ y_i \in \mathcal{L}$, because $[\mathcal{L},\ \mathcal{L},\ \mathfrak{R}] = \mathfrak{R}$, then

$$B(u,\ v,\ w) = \sum_{i=1}^{n} B\left(x_i,\ y_i,\ [r_i,\ v,\ w]\right) = 0.$$

Hence, $Z(\mathfrak{R}) \subset \mathfrak{R}^{\perp}$. Let S be a Levi subalgebra of $\mathcal{L}$. Because $\left[\mathfrak{R}^{\perp},\ \mathcal{L},\ \mathcal{L}\right] = \mathfrak{R}^{\perp}$, then
$\mathfrak{R}^{\perp} = (S \cap \mathfrak{R}^{\perp}) \oplus (\mathfrak{R} \cap \mathfrak{R}^{\perp})$. So, $S \cap \mathfrak{R}^{\perp}$ is a Levi subalgebra of $\mathfrak{R}^{\perp}$ and $S \cap \mathfrak{R}^{\perp}$ is an ideal of S. Assume that $u,\ w \in \mathcal{L},\ x \in \mathfrak{R}$ and $v \in \mathfrak{R}^{\perp}$. Then $[u,\ w,\ x] \in \mathfrak{R}$ and $B([v,\ x,\ w],\ u) = B(v,\ [x,\ w,\ u]) = 0$ because B is non-degenerate. Then $[v,\ x,\ w] = 0$. So, $\left[\mathfrak{R}^{\perp},\ \mathfrak{R},\ \mathcal{L}\right] = \{0\} \Rightarrow \left[S \cap \mathfrak{R}^{\perp},\ \mathfrak{R},\ \mathcal{L}\right] = \{0\}$. Hence, $S \cap \mathfrak{R}^{\perp}$ is a semi-simple ideal of $\mathcal{L}$. Since $\mathcal{L}$ is a Lie triple system without any non-zero ideal, so $S \cap \mathfrak{R}^{\perp} = \{0\}$. Therefore, $\left[\mathfrak{R}^{\perp},\ \mathfrak{R},\ \mathcal{L}\right] = \{0\}$ and $\mathfrak{R}^{\perp} \subset \mathfrak{R}$ then $\mathfrak{R}^{\perp} \subset Z(\mathfrak{R})$. Hence, $\mathfrak{R}^{\perp} = Z(\mathfrak{R})$. $\square$

Theorem 5.6. *Let $\mathcal{L}$ be a perfect Lie triple system without any non-zero semi-simple ideal. Suppose that $\mathcal{L}$ is a quadratic and B is an invariant scalar product on $\mathcal{L}$. Let $\mathfrak{R}$ be its radical and S be the Levi subalgebra of $\mathcal{L}$. Then there exists a vector subspace V of $\mathfrak{R}$ which satisfies:*

1. $[S,\ V,\ -] \subset V$.
2. $\mathfrak{R} = V \oplus Z(\mathfrak{R})$.
3. $B_{|V \times V}$ *is non-degenerate.*
4. $B(S,\ V,\ -) = \{0\}$.

Proof. We proof that $B_{|A \times A}$ is non-degenerate, where $A = Z(\mathfrak{R}) \oplus S$. Let $u = v + w$, where $u \in A$, $v \in S$ and $w \in Z(\mathfrak{R})$ such that $B(u,\ A,\ -) = \{0\}$. So $B\left(v,\ w',\ -\right) = 0,\ \forall\ w' \in Z(\mathfrak{R})$. By Lemma 5.5, $v \in S$, so $v = 0$. Then $u = w$ and $B(w,\ S,\ -) = \{0\}$, which implies that $u = 0 + 0 = 0$. So, $B_{|A \times A}$ is non-degenerate. Therefore, $\mathcal{L} = A \oplus A^{\perp}$. Now, we proof that $\left[S,\ A^{\perp},\ -\right] \subset A^{\perp}$ and $A^{\perp} \subset \mathfrak{R}$. We know that $Z(\mathfrak{R}) \subset A$ and $A^{\perp} \subset \mathfrak{R}$. Let $v \in S,\ c,\ w \in A^{\perp}$ and $u \in A$, then

$$\begin{aligned} &B([v,\ c,\ w],\ u) + B(c,\ [v,\ w,\ u]) = 0 \\ &B([v,\ c,\ w],\ u) = -B(c,\ [v,\ w,\ u]), \end{aligned}$$

implies that $[v,\ c,\ w] \in A^{\perp}$. Thus $[S,\ A^{\perp},\ -] \subset A^{\perp}$. Now we show that the second part. Because $S \subset A$ we have $B(S,\ A^{\perp}, -) = \{0\}$. We know that if B is non-degenerate then $\mathcal{L} = A \oplus A^{\perp}$. Hence, $\mathcal{L} = S \oplus \left(Z(\mathfrak{R}) \oplus A^{\perp}\right)$. So, $\mathfrak{R} = Z(\mathfrak{R}) \oplus A^{\perp}$, $\mathfrak{R} \subset Z(\mathfrak{R}) \oplus A^{\perp}$ and $\mathcal{L} = S \oplus \mathfrak{R}$. Therefore the vector subspace we are looking for is $V = A^{\perp}$. □

Theorem 5.7. *Let $\mathcal{L}$ be a perfect Lie triple system over a field $\mathbb{F}$ without any non-zero semi-simple ideal. Let B be an invariant scalar product on $\mathcal{L}$. Then the Casimir operator C is nilpotent.*

A special case of the theorem is the following corollary:

Corollary 5.8. *Let $\mathcal{L}$ be a quadratic Lie triple system, perfect, which is not semi-simple and B be an invariant scalar product of $\mathcal{L}$. Then C is not invertible, where C is the Casimir element of the Lie triple system.*

Proof. There exists two ideals A and D of $\mathcal{L}$ such that $\mathcal{L} = M \oplus D \oplus A$, where M is a maximal semi-simple ideal of $\mathcal{L}$. We know that B is an invariant scalar product on $\mathcal{L}$, then $H = B_{|M\times M}$ is an invariant Scalar product on M, $N = B_{|D\times D}$ is an invariant scalar product on D and $P = B_{|A\times A}$ is an invariant scalar product on A. Then $B(M,\ D,\ A) = \{0\}$. So C is nilpotent and invertible. Let ρ be the representation of $U(\mathcal{L})$. Then:

1. $\rho\left(C_H\right)|_M$ is invertible, $\rho(C_H)(D) = \{0\}$ and $\rho\left(C_H\right)(A) = \{0\}$.
2. $\rho\left(C_N\right)_M = \{0\}$ and $\rho\left(C_N\right)|_D$ is nilpotent.
3. $\rho\left(C_H\right) \cdot \rho\left(C_N\right) \cdot \rho\left(C_P\right) = \rho\left(C_P\right) \cdot \rho\left(C_N\right) \rho(C_H) = 0$.
4. $\rho(C) = \rho\left(C_H\right) + \rho\left(C_N\right) + \rho\left(C_P\right)$.

Hence, $\rho\left(C_N\right)$ and $\rho\left(C_P\right)$ is nilpotent, then

$$\left(\rho\left(C_N\right)\right)^n = 0,\ \left(\rho\left(C_P\right)\right)^n = 0,\ \forall\ n \geqslant 1,\ n \in \mathbb{N}$$

and $(\rho(C))^n = \left(\rho\left(C_H\right)\right)^n$. So, D and A in $\ker(\rho(C))^n$. Because $\mathcal{L}$ is not semi-simple and D, A, non-zero, hence $(\rho(C))^n$ is not invertible $\Rightarrow \rho(C)$ is not invertible.

□

The following theorem is a characterization of the quadratic Lie triple system in terms of Casimir operators.

Theorem 5.9. *Let $\mathcal{L}$ be quadratic Lie triple system over a field $\mathbb{F}$. Let B be an invariant scalar product on $\mathcal{L}$. Then $\mathcal{L}$ is semi-simple if and only if the Casimir element corresponding to the form B is invertible.*

Proof. If $\mathcal{L}$ is semi-simple, then by Lemma 5.5, C is invertible. Conversely, suppose that C is invertible, then $Z(\mathcal{L}) = \{0\}$ and $[\mathcal{L},\ \mathcal{L},\ \mathcal{L}]^{\perp} = Z(\mathcal{L})$. So, $[\mathcal{L},\ \mathcal{L},\ \mathcal{L}] = \mathcal{L}$. (we know that if $\mathcal{L}$ is a perfect, quadratic Lie triple system which is not semi-simple and B be an invariant scalar product of $\mathcal{L}$, then C is not invertible). Hence, $\mathcal{L}$ is semi-simple and we have done. □

6 Jordan Triple Systems

In this section, we give an example and prove that the special linear Lie algebra of trace equal to zero (traceless) is the Jordan triple system. The reader can see [1].

Definition 6.1. *A vector space A over a field $\mathbb{F}$ which endowed with a trilinear mapping $(*,\ *,\ *)$: $A \times A \times A \longrightarrow A$ is a Jordan triple system if the following two conditions are satisfied:*
(Jt1) For all $u,\ v,\ w \in A$, we have $\{u,\ v,\ w\} = \{w,\ v,\ u\}$ (commutativity).
(Jt2) For all $u,\ v,\ w \in A$, we have $\{x,\ y,\ \{u,\ v,\ w\}\} - \{u,\ v,\ \{x,\ y,\ w\}\} = \{\{x,\ y,\ u\},\ v,\ w\} - \{u,\ \{y,\ x,\ v\},\ w\}$ (principal identity).

Lemma 6.1. *Let D be an associative algebra over a field $\mathbb{F}$ and set $A = M_{p\times q}(D),\ \ \forall\ p,\ q \in \mathbb{N}$, the $(p \times q)$ matrices over D. The set A is a vector space over $\mathbb{F}$ which is a Jordan triple system with respect to the product $\{a_1,\ a_2,\ a_3\} = a_1 a_2^t a_3 + a_3 a_2^t a_1$. Where a_2^t denotes the transpose matrix of a_2.*

7 Main Results of This Section

In this part, we present results for Jordan's triple system.

Example 7.1. *Let $D = \mathbb{C}G$, where G is the group of two elements C_2. Then D is an associative algebra (group algebra). Let $A = M_{2\times 3}(\mathbb{C}C_2)$ then we have to find a basis of A for matrix of type 2×3 where the number of basis is 6.*

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$a_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad a_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since every elements is linearly independent and can be written as the linear combination. First, we study Jordan's triple system conditions using the *relationship, we mentioned in the Lemma 6.1, first condition*

$$
\begin{aligned}
\{a_1, a_2, a_3\} &= \{a_3, a_2, a_1\} \\
\{a_1, a_2, a_3\} &= a_1 a_2^t a_3 + a_3 a_2^t a_1 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\{a_3, a_2, a_1\} &= a_3 a_2^t a_1 + a_1 a_2^t a_3 \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

The second condition

$$
\begin{aligned}
\{a_1, a_2, \{a_3, a_4, a_5\}\} &- \{a_3, a_4, \{a_1, a_2, a_5\}\} = \{\{a_1, a_2, a_3\}, a_4, a_5\} - \{a_3, \{a_2, a_1, a_4\}, a_5\} \\
L.\ H.\ S \Rightarrow &= \{a_1, a_2, a_3 a_4^t a_5 + a_5 a_4^t a_3\} - \{a_3, a_4, a_1 a_2^t a_5 + a_5 a_2^t a_1\} \\
&= \left\{a_1, a_2, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} - \left\{a_3, a_4, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right\} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \\
R.\ H.\ S \Rightarrow &= \{a_1 a_2^t a_3 + a_3 a_2^t a_1, a_4, a_5\} - \{a_3, a_2 a_1^t a_4 + a_4 a_1^t a_2, a_5\} \\
&= \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a_4, a_5\right\} - \left\{a_3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, a_5\right\} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

So, L. H. S=R. H. S. Therefore, it is Jordan triple system. But non-square matrices are not algebra. So, D is a group algebra and $M_{2\times 3}(\mathbb{C}C_2)$ is a Jordan triple system which is not an algebra over $\mathbb{C}$.

Theorem 7.1. *The special linear Lie algebra of trace zero (traceless) is a Jordan triple system.*

Proof. If $A = \mathfrak{sl}(2,\ \mathbb{C})$, then the basis of $\mathfrak{sl}(2,\ \mathbb{C})$ is

$$
\begin{aligned}
&e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_{1(-1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&e_{22} = e_{21} + e_{1(-1)} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \\
&e_{33} = e_{12} + e_{21} + e_{1(-1)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\end{aligned}
$$

The first condition:

$$\begin{aligned}
&\left\{e_{12}, e_{21}, e_{1(-1)}\right\} = \left\{e_{1(-1)}, e_{21}, e_{12}\right\}\\
&\left\{e_{12}, e_{21}, e_{1(-1)}\right\} = e_{12}e_{21}^{t}e_{1(-1)} + e_{1(-1)}e_{21}^{t}e_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\\
&\left\{e_{1(-1)}, e_{21}, e_{12}\right\} = e_{1(-1)}e_{21}^{t}e_{12} + e_{12}e_{21}^{t}e_{1(-1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

The second condition:

$$L.\ H.\ S \Rightarrow \left\{e_{12}, e_{21}, \left\{e_{1(-1)}, e_{22}, e_{33}\right\}\right\} - \left\{e_{1(-1)}, e_{22}, \left\{e_{12}, e_{21}, e_{33}\right\}\right\}$$

$$= \left\{\left\{e_{12}, e_{21}, e_{1(-1)}\right\}, e_{22}, e_{33}\right\} - \left\{e_{1(-1)}, \left\{e_{21}, e_{12}, e_{22}\right\}, e_{33}\right\} \Leftarrow R.\ H.\ S$$

$$\begin{aligned}
L.\ H.\ S &= \left\{e_{12}, e_{21}, e_{1(-1)}e_{22}^{t}e_{33} + e_{33}e_{22}^{t}e_{1(-1)}\right\} - \left\{e_{1(-1)}, e_{22}, e_{12}e_{21}^{t}e_{33} + e_{33}e_{21}^{t}e_{12}\right\}\\
&= \left\{e_{12}, e_{21}, \begin{pmatrix} 3 & 0 \\ 2 & -3 \end{pmatrix}\right\} - \left\{e_{1(-1)}, e_{22}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}\\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.\\
R.\ H.\ S &= \left\{e_{12}e_{21}^{t}e_{1(-1)} + e_{1(-1)}e_{21}^{t}e_{12}, e_{22}, e_{33}\right\} - \left\{e_{1(-1)}, e_{21}e_{12}^{t}e_{22} + e_{22}e_{12}^{t}e_{21}, e_{33}\right\}\\
&= \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, e_{22}, e_{33}\right\} - \left\{e_{1(-1)}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, e_{33}\right\} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

So, R. H. S=L. H. S. Hence, $\mathfrak{sl}(2,\ \mathbb{C})$ is a Jordan triple system. □

Proposition 7.2. *The relationship between Lie triple system $\mathcal{L}$ and Jordan triple system A is given by the following:*

$$\begin{aligned}
[a,\ b,\ c] &= \{a, b, c\} - \{b, a, c\}\\
&= ab^{t}c + cb^{t}a - ba^{t}c + ca^{t}b,\ \forall\ a,\ b,\ c \in \mathcal{L}.
\end{aligned}$$

Proof. The proof is clear by Definitions 6.1 and 4.1. □

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