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Laplacian Spectrum and Brouwer's Conjecture of a Graph

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Abstract

The aim of this paper is to focus on the GroneMerris Theorem and Brouwers Conjecture and give the results of some famous graphs. For example, a threshold graph, a star graph, $(K_4 - e) + K_2$. Moreover, we investigated some common properties between nonplanar graphs $K_{3,3}$, K_5 in this regard. We have relied on [1], [6] and [9].

Mathematics Subject Classification: 05C30, 05C50

Keywords: Graph, Laplacian matrix, Laplacian spectrum, Degree sequence, Conjugate of degree sequence

1 Introduction

Let G be a simple undirected graph with a vertex set $V(G)$ and $n(G)$ be the number of vertices in that set. The edge set is denoted by $E(G)$, and the number of edges in the graph G is denoted by $m(G)$. We define its Laplacian matrix as:

$$L(G) = D(G) - A(G).$$

Where $A(G)$ is the adjacency matrix of the graph G, whose (i, j)-entry is equal to 1 if v_i is adjacent to v_j, and equal to 0, otherwise. The matrix $D(G)$ is the diagonal matrix of the graph G namely:

$$d_i(G) = |\{j : v_i \textit{ is adjacent to } v_j\}|.$$

Recall that the **Handshaking Lemma** is $\sum_{i=1}^{n} d_i(G) = 2m(G)$. We denote the eigenvalues (Spectrum) of the Laplacian matrix $L(G)$ by:

$$0 = \mu_n \leq \mu_{n-1} \leq ..... \leq \mu_1.$$

These eigenvalues are real because the Laplacian matrix is symmetric and positive semi-definite. Let k be a natural number, and define $S_k(G) = \sum_{i=1}^{k} \mu_i(G)$ to be the sum of the k largest Laplacian eigenvalues of G (we sometimes write $\mu_i = \mu_i(G)$). In 1994, Grone and Merris presented a conjecture [5]. The conjecture asserts that the relationship between the conjugate degrees of the graph G that is $d_i^*(G) = |\{j : d_j \geq i\}|$ for $i \in \{1, 2, 3, ..., n\}$ and the sum of the k largest Laplacian eigenvalues (Spectrum) of the graph G. In 2011 Hua Bai gave a complete proof of this conjecture in his paper [3]. It is now called the Grone-Merris Theorem. Let us recall this important theorem.

Theorem 1.1 *For the graph G mentioned above, and for any natural number k such that $1 \leq k \leq n$, we have:*

$$S_k(G) \leq \sum_{i=1}^{k} d_i^*(G).$$

There are many interesting cases and examples of graphs related to this theorem. That we have studied and understood in my master's dissertation. In this paper we collect all the results about this conjecture. In Section 3 we investigated more things of this conjecture. It is clear that the Grone-Merris Conjecture (theorem) is related to the following conjecture which is due to Andries Brouwer [2].

Corollary 1.2 *For each natural number k and any graph G with $n(G)$ vertices and $m(G)$ edges.*

$$S_k(G) = \sum_{i=1}^{k} \mu_i \leq m(G) + \binom{k+1}{2},$$

where $\binom{k+1}{2}$ is the binomial coefficients, which is equal to $\frac{k(k+1)}{2}$. In this paper, we investigated Brouwer's Conjecture for a regular graph and tree. Moreover, we find some common properties between $K_{3,3}$ and K_5 in this regard.

In this paper we organized as follows. In Section 2, we study the basic concepts and main results of the graph theory. In Section 3, we have thoroughly investigated and given the main results of several classes of graphs in the Grone-Merris Conjecture (theorem) and Majorization Theory. We find some common properties between nonplanar graphs $K_{3,3}$ and K_5 in particular for this conjecture. Finally, in Section 4, we give the main results for the Brouwer's Conjecture of some special graphs.

2 Basic Concepts of Graph Theory

In this introductory part, we shall introduce some basic definitions, theories, notions, and examples in graph theory that are related to my work. We can see [7] and [10].

Definition 2.1 *A graph $G = (V, E)$ is a mathematical structure consisting of two finite sets $V(G)$ and $E(G)$. The elements of $V(G)$ are called vertices (or nodes, point). The elements of $E(G)$ are called edges. Each edge has a set of two vertices associated with it, which are called its endpoints.*

Figure 1: Example of graph.

Definition 2.2 *A* ***connected graph*** *is a graph that is in one piece, so that any two vertices are connected by a path.*

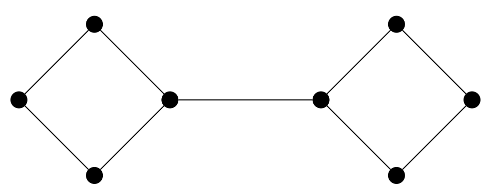


Figure 2: Example of a connected graph.

Definition 2.3 *A* ***disconnected graph*** *is a graph in which there does not exist any path between at least one pair of vertices.*

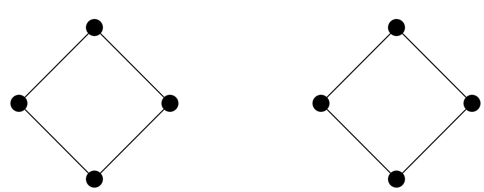


Figure 3: Example of a disconnected graph.

Definition 2.4 *A tree is an undirected graph in which any two vertices are connected by exactly one path or equivalent a connected acyclic undirected graph.*

Definition 2.5 *A star that is on n vertices is a tree consisting of one vertex that is adjacent to the remaining $n-1$ vertices.*

Definition 2.6 *The degree of a vertex in a graph G is the number of edges that are incident to the vertex and denoted by $deg(v) = d(v)$.*

Definition 2.7 ***A complete graph*** *is a simple graph such that every pair of vertices is joined by an edge and denoted by K_n. On otherwise the graph of order n and size $\binom{n}{2}$ edge is the complete graph.*

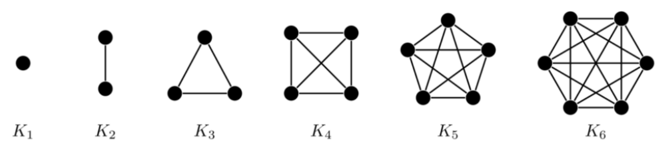


Figure 4: The first six complete graphs.

Definition 2.8 ***A complement graph*** *or* ***inverse*** *of a graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and anly if they are not adjacent in G.*

Definition 2.9 *The graph G is* ***a regular graph*** *if and only if every vertex in the graph has the same degree r.*

Definition 2.10 ***A bi-partite*** *is composed of two independent sets of vertices (bi-partition), with p and q vertices respectively and some edges joining pairs of vertices u and v such that u and v not same partition, denoted by $K_{p,q}$.*

Definition 2.11 ***A complete bi-partite graph*** *is simple bipartite graph such that every vertex in one of the bipartition subset is joined to every vertex in the other bipartiteon subset.*

Definition 2.12 *A* ***threshold graph*** *is a graph obtained from the graph K_0 by a sequence of operations of the form add an isolated vertex or take the complement.*

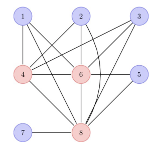


Figure 5: Example of threshold graph.

Definition 2.13 ***A planar graph*** *is a graph that can be drawn in the plane without any edges crossing.*

Theorem 2.14 *(Euler's Formula:) Let G be a connected planar simple graph with n vertices, m edges and r regions then $n + r = m + 2$.*

Lemma 2.15 *If $G = (V, E)$ is a connected planar graph and $|V| > 2$ then $|E| \leqslant |V| - 6$.*

Corollary 2.16 *The graphs K_5 and $K_{3,3}$ are nonplanar and they are the smallest nonplanar graphs.*

proof:

- K_5 is a nonplanar.
  We know that K_5 has 5 vertecies and 10 edges. Let us prove that by contradiction. Suppose that K_5 is a planar graph. From Lemma 2.15 we have $|V| = 5 > 2$ then $10 \leqslant 3(5) - 6$. this is a contradiction because $10 \nleqslant 9$. So K_5 is a nonplanar graph.

- $K_{3,3}$ is a nonplanar.
  The complete bipartite graph $K_{3,3}$ has 6 vertices, 9 edges and 7 regions. Let us prove that by contradiction. Suppose that $K_{3,3}$ is a planar graph, from Theorem 2.14 $n + r = m + 2$, this is a contradiction because $6 + 7 \neq 2 + 9$. So $K_{3,3}$ is a nonplanar graph. ■

Definition 2.17 *A graph $F = (V(F), E(F))$ is a **subgraph** of a graph $G = (V(G), E(G))$ if and only if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$.*

Definition 2.18 *A spanning subgraph is a subgraph obtained only by edge deletions. In other words, the vertex set of the subgraph is the entire vertex set of the original graph.*

Definition 2.19 *The graph union of two graphs $G = (V, E), G' = (V', E')$ is the graph $G \cup G'$ whose vertex-set and edge-set are disjoint unions respectively, of the vertex-set and edge-set of G and G'.*

Definition 2.20 *The graph join of two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ is obtained from the graph union by adding an edge between each vertex in G_1 and G_2.*

3 Grone-Merris Conjecture (Theorem) and Majorization Theory

In this section, we will discuss several classes of graphs in which equality holds for all k in the previous Conjecture (theorem) 1.1, including threshold graphs, trees, and complete graphs. Moreover, we find some common properties between $K_{3,3}$ and K_5 in particular for this conjecture. We can see [1], [4] [6], [8], [9] and [11].

Definition 3.1 *Let* $(a) = \{a_1, a_2, a_3, ..., a_r\}$ *and* $(b) = \{b_1, b_2, b_3, ..., b_s\}$ *be finite nonincreasing sequences of real numbers, where* (a) *and* (b) *contain respectively* r, s *real numbers, then we say that* (a) ***majorizes*** (b) *if:*

$$a_1 \geq b_1,$$

$$a_1 + a_2 \geq b_1 + b_2,$$

$$a_1 + a_2 + a_3 \geq b_1 + b_2 + b_3,$$

and so on. Then at the end $\sum_{i=1}^{r} a_i = \sum_{i=1}^{s} b_i$. *This relation is a partial order relation, then we denote it by* $(a) \succeq (b)$.

Definition 3.2 *If* $d(G) = (d_1, d_2, ..., d_k)$ *is a degree sequence of a graph* G*, then we say that* $d^*(G) = (d_1^*, d_2^*, ..., d_k^*)$ *is the conjugate of the degree sequence* $d(G)$ *where* $d_i^*(G) = |\{j : d_j \geq i\}|$ *for* $i \in \{1, 2, 3, ..., n\}$.

We shall use Definition 3.2 frequently.

3.1 Threshold Graphs (maximal graph)

Threshold graphs are graphs that can be constructed recursively by adding isolated vertices then taking the graph complement. In the Grone-Meriss Conjecture the equality holds if and only if the graph is a threshold graph.
The following theorems are from [9]. It make insight into the relation between the Laplacian eigenvalues (spectrum) and the conjugate of degree sequence in the threshold graph.

Theorem 3.3 *If* G *is a threshold graph on* n *vertices with Laplacian matrix* $L(G)$*, then all eigenvalues are integers.*

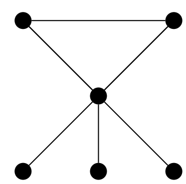
Theorem 3.4 *If* G *is a threshold graph (maximal graph) on* n *vertices with Laplacian matrix* $L(G)$*, and* $d(G) = (d_1, d_2, ..., d_n)$ *is the degree sequence of* G*, and* $\mu_1 \leq \mu_2 \leq ..... \leq \mu_n$ *are the eigenvalues of the Laplacian matrix* $L(G)$*, then* $\mu_1 = 0$ *and* $\mu_{n-i+1} = d_i^*(G)$ *for* $1 \leq i \leq n - 1$.

As a special case of the above theorem, we have the following corollary:

Corollary 3.5 *If G is a threshold graph (maximal graph) on n vertices with degree sequence $d(G) = (d_1, d_2, ..., d_n)$, and $L(G)$ is the Laplacian matrix of the graph G with eigenvalues (spectrum) $\mu_1 \leq \mu_2 \leq ..... \leq \mu_n$. Then $\mu_1 = 0$ and $\mu_i = d^*_{n-i+1}(G)$ for $2 \leq i \leq n$.*

We shall discuss the following examples regarding the above theorem:

Example 3.6 *Consider the threshold graph G with $n = 6$ vertices:*



Recall that the degree sequence is $d(G) = (5, 2, 2, 1, 1, 1)$. We observe that $d^*(G) = (6, 3, 1, 1, 1, 0)$. Then we want to compute the eigenvalues of the Laplacian matrix $L(G)$. First, the Laplacian matrix is:

$$\underset{D(G)}{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}} - \underset{A(G)}{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}} = \underset{L(G)}{\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}}$$

The characteristic polynomial of $L(G)$ is $det(L(G) - \mu I)) = 0$

$$\Rightarrow \det\left(\begin{bmatrix} 2-\mu & -1 & -1 & 0 & 0 & 0 \\ -1 & 2-\mu & -1 & 0 & 0 & 0 \\ -1 & -1 & 5-\mu & -1 & -1 & -1 \\ 0 & 0 & -1 & 1-\mu & 0 & 0 \\ 0 & 0 & -1 & 0 & 1-\mu & 0 \\ 0 & 0 & -1 & 0 & 0 & 1-\mu \end{bmatrix}\right) = 0$$

then $\mu^6 - 12\mu^5 + 48\mu^4 - 82\mu^3 + 63\mu^2 - 18\mu.$
$= \mu(\mu^5 - 12\mu 4 + 48\mu^3 - 82\mu^2 + 63\mu - 18).$
$= \mu(\mu - 1)(\mu^4 - 11\mu^3 + 37\mu^2 - 45\mu + 18).$
$= \mu(\mu - 1)^2(\mu^3 - 10\mu^2 + 27\mu - 18).$
$= \mu(\mu - 1)^3(\mu 2 - 9\mu + 18).$
$= \mu(\mu - 1)^3(\mu - 3)(\mu - 6) = 0.$
The spectrum of the Laplacian matrix is $\mu(G) = (6, 3, 1, 1, 1, 0)$. We obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	6	6
2	9	9
3	10	10
4	11	11
5	12	12
6	12	12

We observe that the Grone Merris Conjecture (theorem) holds. Moreover, it seems clear that the equality is verified from the table. This indicates that the graph in this example is a maximal graph according to the theorem.

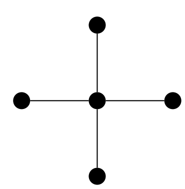
3.2 Trees

In graph theory, a star is a special type of tree where a star is a graph with n vertices consisting of one vertex that is adjacent to the remaining $n-1$ vertices. In this part, we will study the equality of the Grone-Merris Conjecture.

Theorem 3.7 *Let T be a tree then the equality holds in the Grone-Merris Theorem for all k if and only if T is a star.*

proof: Let T be a tree with n vertices the Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} > 0$, and the degree sequence $d_1 \geq d_2 \geq ..... \geq d_n$. We have $d_1^*(G) = n$ as $d_i(G) \geq 1$ for all i. If equality occurs in the Grone-Merris Theorem for all i, then we must have $\mu_1 = d_1^* = n$. This implies that $\mu_1 = n$. So that T is join of two graphs (n is an eigenvalues of G if and only if G is join of two graphs).
Let $T = G_1 + G_2$, $|V(G_i)| \geq 1$ for $i = 1, 2$. As T is a tree, so it contains no cycle. We have $G_1 = \overline{K_1}$ and $G_2 = \overline{K_{n-1}}$, Which gives $T = \overline{K_1} + \overline{K_{n-1}} = K_{1,n-1}$, which clearly is a star. ■

Example 3.8



Let $G = S_5$ be the star with $n = 5$ vertices. The Laplacian eigenvalues of S_5 are $\{5, 1, 1, 1, 0\}$. The degree sequence $d(G)$ is $\{1, 1, 1, 1, 4\}$, it is clear that the conjugate degree sequence $d^*(G)$ of this graph is $\{5, 1, 1, 1, 0\}$. So the equality holds for all $k = 1, ..., 5$ in the Grone Merris Conjecture (theorem).

3.3 Complete Graphs

The equality occurs in the Grone-Merris Conjecture (theorem) for the complete graph. This class is significant among several classes of graphs, every pair of vertices in complete graphs is joined by an edge.

Theorem 3.9 *For complete graphs, the equality holds in the Grone-Merris Conjecture for all natural numbers k.*

proof: If G be a complete graph K_n on n vertices. The Laplacian matrix which has spectrum $(0^1, n^{n-1}) = \{0, n, n, ...\}$, and conjugate degree sequence $d^*(K_n) = \{n, n, ..., 0\}$.
If $k = 1$ then the eigenvalues of $L(G)$ to be $\mu_1 = 0$ and $d_1^*(G) = 0$ then the $\sum_{i=1}^{1} \mu_{1-1+1} \leq \sum_{i=1}^{1} d_1^*(G)$ are holds and the equality holds in $k = 1$
If $k = 2$ then the eigenvalues of $L(G)$ is $\mu(G) = (0, 2)$. We see that the degree sequence $d(G) = (1, 1)$, and $d^*(G) = (2, 0)$ we obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	0	0
2	2	2

Observe that $\mu(G) \leq d^*(G)$ the Grone-Merris Conjecture holds. Moreover, the equality holds for all k.
If $k = 3$ complete graph with three vertices. Observe that its degree sequence $d(G) = (2, 2, 2)$, and the conjecate of degree sequence $d^*(G) = (3, 3, 0)$. Computing the eigenvalues of $L(G)$ is $\mu(G) = (3, 3, 0)$. We obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	3	3
2	6	6
3	6	6

If $k = 4$ complete graph with four vertices. Observe that its degree sequence $d(G) = (3, 3, 3, 3)$, we see that $d^*(G) = (4, 4, 4, 0)$. We know that the Laplacian spectrum of complete graph is $0^1, n^{n-1}$ then we have $\mu(G) = (4, 4, 4, 0)$. We obtain the following table:

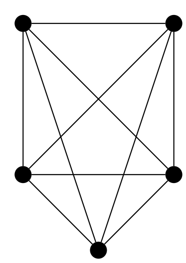
k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	4	4
2	8	8
3	12	12
4	12	12

Thus not only does the Grone-Merris Conjecture hold, but we have equality in $\sum_{i=1}^{k} \mu_{n-i+1} \leq \sum_{i=1}^{k} d_i^*(G)$ for all k. So if $k = n$ complete graph with n vertices then we know that the degree sequence $d(G) = (n-1)^n$ and $d^*(G) = \mu(G) = (0^1, n^{n-1})$. We obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	μ_{n-i+1}	d_1^*
2	μ_{n-2+1}	d_2^*
.	.	.
.	.	.
.	.	.
n	μ_{n-i+1}	d_n^*

It is clear that the Laplacian spectrum and the conjugate degree sequence are the same. Thus, in the Grone-Merris Conjecture, equality holds for all natural numbers k. ■

Example 3.10



Consider the above graph. We observe that the graph $G = K_5$ is the complete graph with five vertices. Take note that its degree sequence is $d(G) = (4,4,4,4,4)$, and the conjugate of degree sequence is $d^*(G) = (5,5,5,5,0)$. We know that the Laplacian spectrum of a complete graph are $(0^1, n^{n-1})$, then we have $\mu(G) = (5,5,5,5,0)$. We obtain the following table:

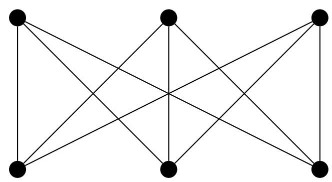
k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	5	5
2	10	10
3	15	15
4	20	20
5	20	20

Thus, not only does the Grone-Merris Conjecture (theorem) hold, but we have equality in $\sum_{i=1}^{k} \mu_{n-i+1} \leq \sum_{i=1}^{k} d_i^*$ for all $k = 1, ..., 5$.

proposition 3.11 *The two graphs K_5 and $K_{3,3}$ have the following common properties:*

- *Both are regular graphs.*
- *Both are nonplannar. We can see proof of Corollary 2.16.*
- *Both achieve the Grone-Merris Conjecture (theorem), but the equality occurs in K_5 without $K_{3,3}$. We can see that in Example 3.10, and we can see that for $K_{3,3}$ in the following example.*
- *Both achieve the Brouwer Conjecture, but there is no equality. We can see that in Section 4, Subsection 4.3.*

Example 3.12



Consider the complete bipartite graph $G = K_{3,3}$ with $n = 6$ vertices the degree sequence is $d(G) = (3,3,3,3,3,3)$, and the conjugate of the degree sequence is $d^*(G) = (6,6,6,0,0,0)$. The Laplacian spectrum of the complete bipartite graph can be obtained as $0^1, m^{n-1}, n^{m-1}, (m+n)^1$, then we have $\mu(G) = (6,3,3,3,3,0)$. We obtain the following table:

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	6	6
2	9	12
3	12	18
4	15	15
5	18	18
6	18	18

We obeserve that the Grone Merris Conjecture holds for all $k = 1, ..., 6$. Moreover, the equality holds when $k = 1, 5, 6$.

3.4 More Graph We Study

Let G be $(K_4 - e) + K_2$ then the equality holds in the Grone-Merris Conjecture. Proof: Let $G_1 = K_4 - e$ on $n_1 = 4$ vertices, and $G_2 = K_2$ on $n_2 = 2$ vertices. The Laplacian eginvalues of G_1 are $\{0, \lambda_i\}$ for all $(i = 1, ..., n_1 - 1)$, and the Laplacian eginvalues of G_2 are $\{0, \mu_i\}$ for all $(i = 1, ..., n_2 - 1)$. If $G_1 + G_2$ the Laplacian eginvalues are $(n_1 + n_2, n_2 + \lambda_i, n_1 + \mu_j, 0)$, where $1 \le i \le n_1 = 4$ and $1 \le j \le n_2 = 2$. Thus the Laplacian spectrum of $K_4 - e$ is $\{4, 4, 2, 0\}$ and the Laplacian spectrum of K_2 is $\{2, 0\}$, so the Laplacian spectrum of $G = G_1 + G_2 = (K_4 - e) + K_2$ is $\{6, 6, 6, 6, 4, 0\}$. It is clear that the degree sequence of the graph G is $\{5, 5, 5, 5, 4, 4\}$, thus the conjugate degree sequence is $\{6, 6, 6, 6, 4, 0\}$. So the equality holds for all k in the Grone Merris Conjecture. ■

4 The Brouwer Conjecture for Special Graphs

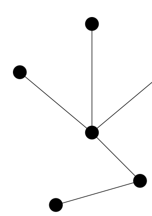
In this section, we provide further evidence for Brouwer's Conjecture by remarking that it holds for several classes of graphs. The following subsection asserts that to prove this conjecture for Tree, Regular graph and Complete graph K_5 and Complete bipartite graph $K_{3,3}$. We can see [1], [2], [4], [8] and [9].

4.1 Trees

The following theorem from [9] gives an affirmative answer to the Brouwer Conjecture for trees with n vertices and $n-1$ edges. Trees are undirected graphs in which any two vertices are connected by exactly one path.

Theorem 4.1 *If T be a tree with n vertices, then the sum of $S_k(T)$ is bounded by $S_k(T) \leq m(T) + 2k - 1$ for $1 \leq k \leq n$.*

Example 4.2 *Consider the following tree below:*



We observe that the adjacency matrix of this graph is: $A(G) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

The degree matrix is: $D(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

The Laplacian matrix is $D(G) - A(G) = L(G)$*:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

We see that the characteristic polynomial of the Laplacian matrix is det $(L(G)-\mu I) = 0$*:*

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\mu & 0 & 0 & -1 & 0 & 0 \\ 0 & 1-\mu & 0 & -1 & 0 & 0 \\ 0 & 0 & 1-\mu & -1 & 0 & 0 \\ -1 & -1 & -1 & 4-\mu & -1 & 0 \\ 0 & 0 & 0 & -1 & 2-\mu & -1 \\ 0 & 0 & 0 & 0 & -1 & 1-\mu \end{bmatrix}\right) = 0.$$

$= \mu^6 - 10\mu^5 + 33\mu^4 - 46\mu^3 + 28\mu^2 - 6\mu.$
$= \mu.(\mu^5 - 10\mu^4 + 33\mu^3 - 46\mu^2 + 28\mu - 6).$
$= \mu.(\mu-1)^2(\mu^4 - 9\mu^3 + 24\mu^2 - 22\mu + 6).$
$= \mu.(\mu-1)^2(\mu^3 - 8\mu^2 + 16\mu - 6).$
$= \mu.(\mu-1)^2.(\mu - 0.486).(\mu - 2.428).(\mu - 5.086) = 0.$
The Laplacian spectrum is $\{0, 1, 1, 0.486, 2.428, 5.086\}$*, and* $m(T) = n - 1 = 6 - 1 = 5$*. We obtain the following table:*

k	$S_k(G) = \sum_{i=1}^{k} \mu_i$	$m(T) + 2k - 1$
1	0	6
2	1	8
3	2	10
4	2.486	12
5	4.854	14
6	9.94	16

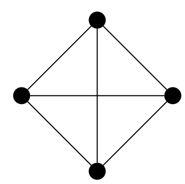
We obeserve that $S_k(G) = \sum_{i=1}^{k} \mu_i \leq m(T) + 2k - 1$*. It holds for all* $k = 1, ..., 6$*.*

4.2 Regular Graph

We now establish the Brouwer Conjecture of regular graphs. Before we proceed, we will introduce the concept for the regular graph, whose vertices all have an equal degree. In the next theorem, we will prove that Brouwer Conjugcture holds for regular graphs, which is proved by [8]

Theorem 4.3 *The Brouwer Conjecture holds for regular graphs.*

Example 4.4 *Consider the following 3-regular graph G below. We will prove that the Brouwer Conjecture holds without equality.*



The adjacency matrix of this graph is: $A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$

The degree matrix is: $D(G) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$

The Laplacian matrix is $D(G) - A(G) = L(G)$*:*

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Recall that the characteristic polynomial of the Laplacian matrix is $det(L(G) - \mu I) = 0$*:*

$$\det\left(\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}\right) = \det\left(\begin{bmatrix} 3-\mu & -1 & -1 & -1 \\ -1 & 3-\mu & -1 & -1 \\ -1 & -1 & 3-\mu & -1 \\ -1 & -1 & -1 & 3-\mu \end{bmatrix}\right) = 0$$

$= \mu^4 - 12\mu^3 + 48\mu^2 - 64\mu = \mu.(\mu^3 - 12\mu^2 + 48\mu - 64).$
$= \mu.(\mu - 4)^3 = 0.$ *Then the Laplacian spectrum is:*$\{0, 4, 4, 4\}$. *We obtain the following table:*

k	$S_k(G) = \sum_{i=1}^{k} \mu_i$	$m(G) + \binom{k+1}{2}$.
1	0	7
2	4	9
3	8	12
4	12	16

We obeserve that $S_k(G) = \sum_{i=1}^{k} \mu_i \leq m(G) + \binom{k+1}{2}$. *It holds for all* $k = 1, ..., 4$. *It is clear there is no equality.*

4.3 Brouwer Conjecture for Nonplanar Graphs K_5 and $K_{3,3}$

In this section, we shall study the last common properties between complete graph K_5 and complete bipartite graph $K_{3,3}$. Now first we study the complete graph $K5$ via the next example.

Example 4.5 *In Example 3.10, recall that the degree sequnce of complete graph* K_5 *is* $d(G) = (4, 4, 4, 4, 4)$, *and the conjugate of degree sequence is* $d^*(G) = (5, 5, 5, 5, 0)$. *We know that the Laplacian spectrum of* K_5 *is* $\mu(G) = (5, 5, 5, 5, 0)$. *Observe that there are* 10 *edges in* K_5, *then we obtain the following table to compute the Brouwer Conjecture:*

k	$S_k(G) = \sum_{i=1}^{k} \mu_i$	$m(G) + \binom{k+1}{2}$.
1	5	11
2	10	13
3	15	16
4	20	20
5	20	25

We obeserve that the Brouwer Conjecture holds for all $k = 1, ..., 5$. *Morover, the equalty holds when* $k = 4$.

Example 4.6 *Revisiting the graph in Example 3.12, we show that the degree sequence is* $d(G) = (3, 3, 3, 3, 3, 3)$, *and the conjugate of this degree sequence is* $d^*(G) = (6, 6, 6, 0, 0, 0)$. *The Laplacian spectrum of this graph is* $\mu(G) = (6, 3, 3, 3, 3, 0)$. *We obtain the following table:*

k	$\sum_{i=1}^{k} \mu_{n-i+1}$	$\sum_{i=1}^{k} d_i^*$
1	6	10
2	9	12
3	12	15
4	15	19
5	18	24
6	18	30

We observe that the Brouwer Conjecture holds for all $k = 1, ..., 6$. Moreover, there is no equality in this graph.

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