Primitive Simple Permutation Group with $A_5$ as a Faithful Subconstituent

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Abstract

We determine all primitive simple permutation groups with suborbit length 5 which have a faithful subconstituent of $A_5$.

Mathematics Subject Classification: 05C25, 20B25

Keywords: primitive group, subconstituent, almost simple group

1 Introduction

Let $G$ be a primitive permutation group acting on the finite set $\Omega$. Denote $G_\alpha$ to be its vertex-stabilizer for $\alpha \in \Omega$ and $\Delta(\alpha)$ to be an orbit of $G_\alpha$ on $\Omega$, which is usually called a suborbit of $G$. Consider the action induced by $G_\alpha$ on $\Delta(\alpha)$, then its transitive constituent $G_{\Delta(\alpha)}^\alpha$ is a homomorphic image of $G_\alpha$, which is also called a subconstituent of $G$. The subconstituent of $G$ is called faithful if the kernel of $G_\alpha$ acting on $\Delta(\alpha)$ is trivial. Much excellent work has dealt with

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Zhengfei Wu was supported by the Basic Ability Improvement Project for Young and Middle Aged Teachers in Guangxi (2017KY1302) and Research Funds of Guangxi University Xingjian College of Science and Liberal Arts (Y2018ZKT01).

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those primitive permutation groups whose subconstituents are particular. For example, the primitive group with the suborbit of length 2 is a dihedral group of order $2p$ ($p$ is an odd prime number, see [1, Theorem 18.7]). The structure of $G$ would be more complicated when $G_\alpha$ has an orbit of length greater than 2. The classifications of primitive groups with the suborbit of length 3 were studied by Sims [2] and Wong [3]. Continuing with their work, the literatures [4, 5, 6] determined all primitive groups which have suborbits of length 4. So it is natural to consider the problem of the classification of all primitive groups which have suborbits of length 5. Suppose that $G$ has a suborbit $\Delta(\alpha)$ of length 5, Quirin [5] and Wang [8, 9, 10] determined all possible $G$ whose subconstituent is solvable or unfaithful. In this paper, the main goal is to determine the case when $G$ is a simple group and the subconstituent $G_\Delta(\alpha)$ is isomorphic to the alternating group $A_5$ and faithful.

2 Preliminaries

Lemma 2.1 [11] Let $p$ be an odd prime and $G \cong \text{PSL}(2, p^f)$. Then $G$ has exactly $|G|/12$ subgroups $K$ with the type $(2, 2)$ and either these subgroups are conjugate to each other if $p^{2f} \not\equiv 1 \pmod{16}$ or there are two conjugate classes with the same length if $p^{2f} \equiv 1 \pmod{16}$. Furthermore, $C_G(K) = K$, $|N_G(K)| = 12$ if $p^{2f} \not\equiv 1 \pmod{16}$ and $|N_G(K)| = 24$ if $p^{2f} \equiv 1 \pmod{16}$.

The following two lemmas determine all maximal subgroups of the projective special linear group $\text{PSL}(2, p^f)$ depending on the prime $p$ to be 2 or not.

Lemma 2.2 [12, 13] Let $q = 2^f \geq 4$. Then the maximal subgroup of $\text{PSL}(2, q)$ is one of the following cases:

(1) $\mathbb{Z}_2^f \rtimes \mathbb{Z}_{q-1}$;
(2) $D_{2(q-1)}$;
(3) $D_{2(q+1)}$;
(4) $\text{PGL}(2, q_0)$, where $q = q_0^r$ for some prime $r$ and $q_0 \neq 2$.

Lemma 2.3 [12, 13] Let $q = p^f \geq 5$, where $p$ is an odd prime. Then the maximal subgroup of $\text{PSL}(2, q)$ is one of the following cases:

(1) $\mathbb{Z}_p^f \rtimes \mathbb{Z}_{(q-1)/2}$;
(2) $D_{q-1}$, for $q \geq 13$;
(3) $D_{q+1}$, for $q \neq 7, 9$;
(4) $\text{PGL}(2, q_0)$, where $q = q_0^r$ (2 conjugate classes);
(5) $\text{PSL}(2, q_0)$, for $q = q_0^r$ where $r$ is an odd prime;
(6) $A_5$, for $q \equiv \pm 1 \pmod{10}$, where either $q = p$ or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$ (2 conjugate classes).
(7) $A_4$, for $q = p \equiv \pm 3 \pmod{8}$ and $q \not\equiv \pm 1 \pmod{10}$;

(8) $S_4$, for $q = p \equiv \pm 1 \pmod{8}$ (2 conjugate classes).

**Lemma 2.4** (Aschbacher’s Theorem [14]) Let $M$ be a subgroup of $GL(n, F_q)$ ($q = p^f$) with the center $Z$ and $V = V(n, F_q)$ the $n$-dimensional vector space over the field $F_q$. Then $M$ belongs to at least one of the following classes from $C_1$ to $C_9$:

- $C_1$: $M$ acts reducibly on $V$ and is a stabilizer of proper non-trivial $F_q$-subspaces.
- $C_2$: $M$ is the stabilizer of direct sum decompositions $V = W_1 \oplus W_2 \oplus \cdots \oplus W_t$, where $t \geq 2$ and $\dim W_i = d/t$ for each $i$.
- $C_3$: $M$ is the stabilizer of an $(n/t)$-dimensional vector space over the extension field $F_{q^t}$ ($t$ is a prime factor of $n$).
- $C_4$: $M$ is the stabilizer of tensor decompositions $V = V_1 \otimes V_2$ such that $\dim V_1 > \dim V_2 \geq 2$.
- $C_5$: $M$ is the stabilizer of an $n$-dimensional $F_{q^r}$-subspace of $V$ ($r$ is a prime).
- $C_6$: $M$ is the normalizer of extraspecial $r$-group ($r$ is a prime and $r \not= p$).
- $C_7$: $M$ is a stabilizer of the symmetric tensor decomposition $V = V_1 \otimes \cdots \otimes V_t$ with $t \geq 2$ and each $\dim V_i = m$, where $n = m^t$.
- $C_8$: $M$ normalizes a classical group in its natural representation.
- $C_9$: $M$ is absolutely irreducible and $M/(M \cap Z)$ is almost simple.

**3 Main result**

According to the theorem of classification of the finite simple groups, the non-abelian simple group $G$ might be an alternating group, a sporadic simple group, an exceptional group of Lie type or a classical simple group.

**Theorem 3.1** Let $G$ be an alternating group acting on $\Omega = \{1, 2, \cdots, n\}$ with a suborbit $\Delta(\alpha)$ of length 5 and have a faithful subconstituent $G_\alpha^{\Delta(\alpha)} \cong A_5$. Then $G \cong A_6$.

**Proof.** Let $H = \langle (1, 2) (3, 4), (1, 2, 3), (3, 4, 5) \rangle$ and $g = (1, 2) (5, 6)$. It is easy to check that $H = A_5$, $\langle H, g \rangle = A_6$ and $H \cap H^g = \langle (1, 2) (3, 4), (1, 2, 3) \rangle \cong A_4$. Denote $\Omega_1$ to be the set of right cosets of $H$ in $A_6$ and consider $A_6$ to act on $\Omega_1$ by right-multiplication. Then $Hg \cdot H$ is an orbit of $H$ with length 5, that is to say, $A_6$ has a suborbit of length 5 on $\Omega_1$. On the other hand, we know that except for $A_6$, the alternating group has no maximal subgroup which is isomorphic to $A_5$ according to the literature [15]. Thus $G \cong A_6$. □
**Theorem 3.2** Let $G$ be a sporadic simple group or an exceptional simple group of Lie type and have a maximal subgroup which is isomorphic to $A_5$, then $G$ is one of the following groups: $J_2$, $E_6(q)$, $E_7(q)$, $E_8(q)$ and $^2E_6(q)$.

**Proof.** By [16, 17], $J_2$ is the unique sporadic simple group that contains a maximal subgroup which is isomorphic to $A_5$. By [17], except for $E_6(q)$, $E_7(q)$, $E_8(q)$, $^2E_6(q)$, the exceptional simple group of Lie type has no such maximal subgroup. □

In the following section, suppose that $G$ is a classical simple group. We first consider the case when $G = \text{PSL}(2, p^f)$ ($p$ is a prime) in which all maximal subgroups have been determined by Lemma 2.2 and Lemma 2.3.

**Theorem 3.3** Suppose that $G = \text{PSL}(2, p^f)$ and $G$ has a maximal subgroup $H$ which is isomorphic to $A_5$. Denote $\Omega$ to be the set of right cosets of $H$ in $G$, and consider the action of $G$ on $\Omega$ by right-multiplication. Then $H$ has an orbit of length 5 on $\Omega$ if and only if $p^{2f} \equiv 1 (\text{mod } 16)$, that is, $G$ is isomorphic to $\text{PSL}(2, p)$ ($p^2 \equiv 1 (\text{mod } 80)$), $\text{PSL}(2, p^2)$ ($p^2 \equiv -1 (\text{mod } 10)$ and $p^4 \equiv 1 (\text{mod } 80)$), $\text{PSL}(2, 5^r)$ ($r$ is an odd prime and $25^r \equiv 1 (\text{mod } 16)$).

**Proof.** Since $H \cong A_5$ is a maximal subgroup of $G = \text{PSL}(2, p^f)$, $G$ must be isomorphic to $\text{PSL}(2, 2^{2r})$ ($r$ is a prime), $\text{PSL}(2, p)$ ($p \equiv \pm 1 (\text{mod } 10)$), $\text{PSL}(2, p^2)$ ($p \equiv \pm 3 (\text{mod } 10)$) or $\text{PSL}(2, 5^r)$ ($r$ is an odd prime) by Lemma 2.2 and Lemma 2.3.

If $H$ has an orbit of length 5, then there exists an element $a \in G \setminus H$ such that $|H : H^a \cap H| = 5$. It follows that $H^a \cap H \cong A_4$. Denote $L = H^a \cap H$. Then $L^a$ and $L$ are two subgroups of $H^a$. Moreover, $L^a$ and $L$ are conjugate to each other in $H^a$ since both of them are isomorphic to $A_4$. So there exists an element $x \in H^a$ such that $L^{ax} = L$, that is, $ax \in N_G(L)$.

We claim $p \neq 2$, namely $G \not\cong \text{PSL}(2, 2^{2r})$. Otherwise, $N_G(L)$ must be contained in a maximal subgroup of $G = \text{PSL}(2, 2^{2r})$ which is isomorphic to $\text{PGL}(2, 4)$ by Lemma 2.2. On the other hand, we know that $L$ is a maximal subgroup of $H \cong \text{PGL}(2, 4)$, thus $N_G(L) = L$. Furthermore, $ax \in N_G(L) = L < H^a$ and then $a \in H$, contradicting to $a \in G \setminus H$.

Assume $p^{2f} \not\equiv 1 (\text{mod } 16)$. Let $K$ be a Sylow 2-subgroup of $L \cong A_4$. Then $K \trianglelefteq N_G(L)$, and hence $L \leq N_G(L) \leq N_G(K)$. On the other hand, we have $|N_G(K)| = 12$ by Lemma 2.1. Therefore $N_G(L) = L$, one has $ax \in N_G(L) = L < H^a$ which implies that $a \in H$, this is a contradiction to $a \in G \setminus H$. Thus $p^{2f} \equiv 1 (\text{mod } 16)$.

Conversely, assume $p^{2f} \equiv 1 (\text{mod } 16)$. Let $K$ be a Sylow 2-subgroup of $H$. Since $L = N_H(K) \cong A_4$ and $|N_G(K)| = 24$ by Lemma 2.1, there exists an element $a \in G \setminus H$ such that $K^a = K$. As $K \leq L$ and $|N_G(K)| = 24$, one has $L \leq N_G(K)$ and $L \leq H \cap H^a$. Note that $N_G(H) = H$ since $H$ is a maximal
subgroup of $G$, it follows that $H \cap H^a = L$. Thus $H$ has an orbit of length 5 on $\Omega$.

In the following, we consider the case when $G \neq \text{PSL}(2, p^f)$. Let $V = V(n, E_q)$ be the $n$-dimensional vector space over the field $GF(q)$. Suppose that $H \cong A_5$ is a maximal subgroup of $G$, according to Aschbacher’s Theorem (Lemma 2.4), $H$ must belong to one of the classes from $C_1$ to $C_9$ of $G$. We first prove that $H$ is not in $C_6$ or $C_9$.

**Theorem 3.4** Suppose that $G \neq \text{PSL}(2, q)$ is a classical simple group and $H \cong A_5$ is a maximal group of $G$. Then $H \not\in C_6$ or $C_9$.

**Proof.** Since $A_5$ has no nontrivial normal subgroup, $H \not\in C_6$. If $H \in C_9$, then

$2A_5 \cong \text{SL}(2, 5)$ acts absolutely irreducible on $V$ by Lemma 2.4 and has no absolutely irreducible representation of degree greater than 6. Furthermore, $H \cong A_5$ is not a maximal subgroup of $G$, contradicting to the condition. Thus $H \not\in C_9$.

Next, we will show that $G$ has no maximal subgroup which is isomorphic to $A_5$ according to the literature [18] and Lemma 2.4.

**Theorem 3.5** Suppose that $H \cong A_5$ is a maximal subgroup of nonabelian simple group $G$, then $G \neq \text{PSL}(n, q) \ (n \geq 3)$.

**Proof.** Assume $G = \text{PSL}(n, q) \ (n \geq 3)$. Since $H \cong A_5$ is a maximal subgroup of $G$, one has $H \in C_1 \ (1 \leq i \leq 8$ and $i \neq 6)$ by Lemma 2.4 and Theorem 3.4.

If $H \in C_1$, then $H$ is the stabilizer of an $r$-dimensional subspace of $V$. It follows that $H$ has a normal subgroup of order $q^{r(n-r)}$ by [18, Proposition 4.1.17], contradicting to $H \cong A_5$. If $H \in C_2$, then $H$ contains a section which is isomorphic to $\text{PSL}(s, q) \rtimes S_r$ with $n = s \cdot r$ and $r \geq 2$, but $H \cong A_5$, this is impossible. If $H \in C_3$, then $H$ contains a section which is isomorphic to $\text{PSL}(n/r, q^r)$ for some prime $r$. We have $n = 4, q^r = 2^2$ and $G \cong \text{PSL}(4, 2) \cong S_6$, this is a contradiction to that $H \cong A_5$ is not a maximal subgroup of $S_6$. If $H \in C_4$ or $C_7$, then $H$ contains a section which is isomorphic to $\text{PSL}(m, q)$ for some $m \geq 3$, which is a contradiction to $H \cong A_5$. If $H \in C_5$, then $H$ contains a section which is isomorphic to $\text{PSL}(n, q^{r^2})$ since $n \geq 3$ and $H \cong A_5$, this is also impossible. Since $H$ is a simple group of the minimal order, one has $H \not\in C_8$. Based on the above arguments, we get $G \neq \text{PSL}(n, q) \ (n \geq 3)$.

**Theorem 3.6** Suppose that $H \cong A_5$ is a maximal subgroup of the nonabelian simple group $G$, then $G \neq \text{PSp}(n, q) \ (n \geq 4, n \text{ is even})$.

**Proof.** Assume $G = \text{PSp}(n, q) \ (n \geq 4, n \text{ is even})$. Since $H \cong A_5$ is a maximal subgroup of $G$, $H \in C_i \ (1 \leq i \leq 8$ and $i \neq 6)$ by Lemma 2.4 and Theorem 3.4.
If $H \in C_1$, then $H$ is the stabilizer of a subspace $W$ of $V$. We have that $H$ either has a section isomorphic to $\text{PSp}(m, q) \times \text{PSp}(n - m, q)$ if $W$ is non-degenerate or contains a nontrivial normal subgroup if $W$ is totally singular, both contradicting to $H \cong A_5$. If $H \in C_2$, since $H$ is non-local, one has either $H \cong \text{PGL}(\frac{n}{2}, 3).2(n \geq 6)$ or $H \cong \text{PSp}(m, 2^k) \cdot \text{S}_t(n = mt, (2^k, m) = (2, 2))$ by [18], contradicting to $H \cong A_5$. If $H \in C_3, C_4$ or $C_7$, then $H$ contains a nontrivial normal subgroup by [18], contradicting to $H \cong A_5$. If $H \in C_5$ or $C_8$, then either $H \cong \text{PSp}(n, q^2).c(c = h.c.f(2, q - 1, r))$ or $H \cong \text{O}^\varepsilon(n, q)$ ($\varepsilon = 0$ or $\pm$, where $q$ is even) by [18]. Note that $n \geq 4$ and $H \cong A_5$, this is impossible. Therefore, we have $G \neq \text{PSp}(n, q)(n \geq 4, n$ is even). □

According to the literature [18], Lemma 2.4 and Theorem 3.4, take similar proof method above, it is not difficult to prove the following conclusion.

**Theorem 3.7** Suppose that $H \cong A_5$ is a maximal subgroup of the nonabelian simple group $G$, then $G$ can’t be $\text{PSU}(n, q)(n \geq 3)$, $\text{PΩ}(n, q)(n \geq 5, nq$ is odd) and $\text{PΩ}^\pm(n, q)(n \geq 8, n$ is even).

By Theorems 3.1–3.7 we have the following theorem.

**Theorem 3.8** Let $G$ be a simple group and primitive acting on a finite set $\Omega$ and have a suborbit $\Delta(\alpha)$ of length 5. If the subconstituent $G^\Delta(\alpha)$ is faithful and isomorphic to $A_5$, then $G$ is a simple group which is isomorphic to one of the following groups: $A_6, \text{PSL}(2, p)(p^2 \equiv 1(\text{mod } 80))$, $\text{PSL}(2, p^2)(p^2 \equiv -1(\text{mod } 10)$ and $p^4 \equiv 1(\text{mod } 80))$, $\text{PSL}(2, 5^r)$ ($r$ is an odd prime and $25^r \equiv 1(\text{mod } 16))$, $J_2, E_i(q)(i = 6, 7, 8)$ or $^2E_6(q)$.

### References


Received: February 27, 2021; Published: March 18, 2021